Dynamically evolving Gaussian spatial fields

Anastassia Baxevani

Department of Mathematical Statistics

University of Gothenburg

Chalmers University of Technology

Joint work with K. Podgórski and I. Rychlik

Introduction

Recent technological advances, like aerial laser and satellite scanning result in increasingly complex environmental data over large regions in space and relatively long periods of time. In Baxevani, Caires and Rychlik (2006), we have used non homogeneous spatio-temporal Gaussian fields to model the variation of significant wave height using data from satellites and buoys.

Spatio-Temporal Model in Baxevani et al. (2006)

For $r_S(\mathbf{p}) = \sigma^2 \exp(-|\mathbf{p}|^2/2L^2)$ we introduce time dependence by considering the recursive autoregression

$$Z(\mathbf{p}, t) = \rho Z(\mathbf{p} - \mathbf{v}dt, t - dt) + \sqrt{1 - \rho^2}\Phi_t(\mathbf{p}),$$

with independent (in t) innovations $\Phi_t(\mathbf{p})$ having the covariance r_S. The velocity $\mathbf{v}$ depends on both position $\mathbf{p}$ and time t. The resulting (stationary) covariance is of the form $\mathrm{Cov}(Z(\mathbf{p}, 0), Z(\mathbf{p}, t)) = \rho^t r_S(\mathbf{p} - \mathbf{v}t)$.

Motivation: At each time step the past surface is moving forward to the new location with the velocity $\mathbf{v}$ and is modified by an independent innovation with the prescribed (fixed) spatial covariance structure.

Goal

We want to provide with a fully continuous set up!

Stochastic Integration

First, we establish the integrals that allow to introduce spatio-temporal fields:

$$X(\mathbf{p}, t) = \int f(t, s)\Phi(\mathbf{p}; ds),$$

for deterministic f and $\Phi(\mathbf{p}; ds)$ Gaussian random field measure.

Stochastic Integration I

For each $t \in \mathbb{R}$ let $r_S(\mathbf{p}, \mathbf{p}'; t)$ be a spatial covariance in $\mathbf{p}$ and $\mathbf{p}'$ (non-negative definite). Then, for a measure μ on the real line (most often we consider the Lebesgue measure):

$$r_{(a,b]}(\mathbf{p}, \mathbf{p}') = \int_a^b r_S(\mathbf{p}, \mathbf{p}'; s) d\mu(s)$$

is a well defined covariance for any given $a < b \in \mathbb{R}$.
It follows from the additivity of the covariance function with respect to independent fields and its correspondence to the additivity of the integral with respect to the measure μ, that there exists a family of Gaussian spatial fields $\Phi(\mathbf{p}; (a, b])$ centered at zero so that:

Stochastic Integration II

- For $a < b, c < d \in \mathbb{R}$, we have

$$r_{(a,b]\cap(c,d]}(\mathbf{p}, \mathbf{p}') = \text{Cov}(\Phi(\mathbf{p}; (a, b]), \Phi(\mathbf{p}'; (c, d]))$$

Stochastic Integration II

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- For $[a,b) = \bigcup_{i=1}^{\infty}[a_i, b_i)$ where $(a_i, b_i]$ are pairwise disjoint intervals, then

$$\Phi(\mathbf{p}; [a,b)) = \sum_{i=1}^{\infty} \Phi(\mathbf{p}; [a_i, b_i))$$

with probability one.

Stochastic Integration III

Thus Φ is a σ- additive measure having as values Gaussian random fields and

$$f(t) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{[a_i,b_i)}(t),$$

for $[a_i, b_i), i = 1 \dots n$ disjoint intervals, we write

$$X(\mathbf{p}) := \int f(s)\Phi(\mathbf{p}; ds) = \sum_{i=1}^{n} \alpha_i \Phi(\mathbf{p}; [a_i, b_i)).$$

$X(\mathbf{p})$, is Gaussian centered with covariance $\sum_{i=1}^{n} \alpha_i^2 r_{[a_i,b_i)}(\mathbf{p}, \mathbf{p}') = \int |f(t)|^2 r_S(\mathbf{p}, \mathbf{p}'; s) d\mu(s)$.

Stochastic Integration IV

The remainder off the construction involves extending this mapping to be valid for any complex-valued function f that satisfies

$$\int |f(s)|^2 r_S(\mathbf{p}, \mathbf{p}; s) d\mu(s) < \infty,$$

which can be done using standard measure theoretic arguments. In particular, for any $f, g \in \mathcal{L}^2(\mu)$

$$r_{X,Y}(\mathbf{p}, \mathbf{p}') = \mathsf{Cov}(X(\mathbf{p}), Y(\mathbf{p}')) = \int f(s)\overline{g(s)} r_S(\mathbf{p}, \mathbf{p}'; s) d\mu(s)$$

Example I

A simple version of the previous covariance is for r_S independent of time, $r_S(\mathbf{p}, \mathbf{p'}; s) \equiv r_S(\mathbf{p}, \mathbf{p'})$. Then,

$$\text{Cov}(X(\mathbf{p}), X(\mathbf{p'})) = r_S(\mathbf{p}, \mathbf{p'}) \int |f(s)|^2 d\mu(s)$$

and the role of f is reduced to multiplication by a constant.

Temporal Moving Averages I

The previous construction allows to build a very general temporal dependence by considering:

$$X(\mathbf{p}, t) = \int f(t, s)\Phi(\mathbf{p}; ds), f \in \mathcal{L}^2(\mu).$$

A special case is: for r_S independent of time and

$f(t, s) = f(t - s)$,

$$X(\mathbf{p}, t) = \int f(t - s)\Phi(\mathbf{p}; ds)$$

is called a temporal moving average of Φ.

Temporal Moving Averages

Lemma: The temporal moving average of Φ is Gaussian and stationary in time. Indeed, Φ has covariance

$$\begin{aligned}\text{Cov}(X(\mathbf{p},t),X(\mathbf{p}',t')) &= r_S(\mathbf{p},\mathbf{p}')\int f(t-s)f(t'-s)d\mu(s)\\ &= r_S(\mathbf{p},\mathbf{p}')f*\tilde{f}(t-t') =\\ &= r_S(\mathbf{p},\mathbf{p}')\rho(t-t')\end{aligned}$$

where $f*g(u)=\int f(u-s)g(s)d\mu(s)$ and $\tilde{f}(u)=f(-u)$. Also, by the stochastic integral, we have $X(\mathbf{p},t)=\Phi(\mathbf{p};f(t-))$ is real valued Gaussian random variable for every $(\mathbf{p},t)$.

Discrete Temporal Moving Averages

To see the relation to moving averages in time series analysis, let $s = i\Delta t, i = -M, \ldots, M$ for some large M and $t = n\Delta t$. Then,

$$X(\mathbf{p}, n\Delta t) = X_n(\mathbf{p}) \approx \sum_{i=-M}^{M} \sqrt{\Delta t} f((n-i)\Delta t)\epsilon_i(\mathbf{p}) = \sum_{k=-N}^{N} \alpha_k \epsilon_{n-k}(\mathbf{p}$$

where $\epsilon_i(\mathbf{p})$ are independent (in time) Gaussian fields with $\text{Cov}(\epsilon_i(\mathbf{p}), \epsilon_i(\mathbf{p}')) = r_S(\mathbf{p}, \mathbf{p}')$ that are given by

$$\epsilon_i(\mathbf{p}) = \frac{\Phi(\mathbf{p}; [i\Delta t, (i+1)\Delta t))}{\sqrt{\Delta t}} = \frac{\Phi(\mathbf{p}; (i+1)\Delta t) - \Phi(\mathbf{p}; i\Delta t)}{\sqrt{\Delta t}}.$$

Example II

For $f(t) = e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t)$, the spatio-temporal is defined by

$$X(\mathbf{p},t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}\Phi(\mathbf{p};ds)$$

and is has covariance

$$\text{Cov}(X(\mathbf{p},t), X(\mathbf{p}',t')) = r_S(\mathbf{p},\mathbf{p}')\frac{1}{2\lambda}e^{-\lambda|t-t'|}$$

Static fields? (I)

Fields defined by

$$X(\mathbf{p}, t) = \int f(t-s)\Phi(\mathbf{p}; ds)$$

do not move.
Indeed, the distribution of the velocity defined by

$$\mathbf{V} = \left(-\frac{X_t}{X_x}, -\frac{X_t}{X_y} \right)$$

is centered around zero.

Static fields? (II)

For this, notice that the covariance of the derivatives of the field $X(\mathbf{p},t)$, which are shown to exist as stochastic integrals, are given by

$$\mathrm{Cov}(X(\mathbf{p},t),X_{x'}(\mathbf{p}',t')) = \int f(t-s)f(t'-s)r_S^{x'}(\mathbf{p},\mathbf{p}';s)d\mu(s)$$

$$\mathrm{Cov}(X(\mathbf{p},t),X_{t'}(\mathbf{p}',t')) = \int f(t-s)f_{t'}(t'-s)r_S(\mathbf{p},\mathbf{p}';s)d\mu(s)$$

$$\mathrm{Cov}(X_x(\mathbf{p},t),X_{t'}(\mathbf{p}',t')) = \int f(t-s)f_{t'}(t'-s)r_S^{x}(\mathbf{p},\mathbf{p}';s)d\mu(s)$$

So, in order to define spatio-temporal fields that move in time we need something more!!

Flow of diffeomorphisms

Consider a velocity field $\mathbf{v} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ and $0 \leq t_0 \leq s \leq 1$. Then the motion of a point is modelled by means of a flow of diffeomorphisms

$$\phi : \mathbb{R}^2 \times [0,1]^2 \to \mathbb{R}^2$$

$\phi(\mathbf{p}, 0, 1) = \phi(\mathbf{p}), \phi(\mathbf{p}, s, s) = \mathbf{p}, \phi(\cdot, t_0, s) = \phi(\cdot, u, s) \circ \phi(\cdot, t_0, u)$
that are the solution to the transport equation

$$\phi(\mathbf{p}, t_0, s) = \mathbf{p} + \int_{t_0}^{s} \mathbf{v}(\phi(\mathbf{p}, t_0, u), u) du, \quad t < s.$$

For simplicity consider $\phi_s(\mathbf{p}) = \phi(\mathbf{p}, 0, s), \ t < s$ and $\psi_s(\mathbf{p}) = \phi_s^{-1}(\mathbf{p})$ for the point that after time s is located at $\mathbf{p}$.

Spatio-temporal dynamic models I

Now define

$$X(\mathbf{p}) := \int f(s)\Phi(\phi_{-s}(\mathbf{p}); ds) = \sum_{i=1}^{n} \alpha_i \Phi(\psi_{-s}(\mathbf{p}); a_i, b_i).$$

Then we can get a general spatio-temporal model by taking

$$X(\mathbf{p}, t) := \int f(s,t)\Phi(\psi_{-s}(\mathbf{p}); ds)$$

Spatio-temporal dynamic models II

Finally, define the following new dynamical model

$$Y(\mathbf{p}, t) = X(\psi_t(\mathbf{p}), t) = \int f(t-s)\Phi(\psi_{t-s}(\mathbf{p}); ds)$$

with covariance function

$$\text{Cov}(Y(\mathbf{p}, t), Y(\mathbf{p}', t')) = \int f(t-s) f(t'-s) r_S(\psi_{t-s}(\mathbf{p}), \psi_{t'-s}(\mathbf{p}'); s) d\mu$$

Spatio-temporal dynamic models III

Theorem: Consider a spatio-temporal centered Gaussian field, $Z(\mathbf{p}, t)$ defined by the recursive formula

$$Z(\mathbf{p}, t) = \rho Z(\psi_{dt}(\mathbf{p}), t - dt) + \sqrt{1 - \rho^2}\epsilon_t(\mathbf{p}),$$

where $\rho = \rho(dt) = e^{-\lambda dt}$ for some $\lambda \geq 0$, a suitably chosen time lag, dt, and independent in time random fields $\epsilon_t(\mathbf{p})$ with the spatial covariance $\text{Cov}(\epsilon_t(\mathbf{p}), \epsilon_t(\mathbf{p}')) = r_S(\mathbf{p}, \mathbf{p}'; t)$. Let also $\psi_t(\mathbf{p})$ be a flow of diffeomorhisms as defined before. Then, the Gaussian field $Z(\mathbf{p}, t)$ has a covariance function that converges with time to the covariance function of the field $Y(\mathbf{p}, t)$. Moreover the field $Z(\mathbf{p}, t)$ moves with velocity the $\mathbf{v}$ that governs the diffeomorphism.

Conclusions I

- We have constructed dynamically evolving stationary fields that locally represent the data, and the long range and long time variability are represented by location and time dependent spectra.

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Conclusions I

- We have constructed dynamically evolving stationary fields that locally represent the data, and the long range and long time variability are represented by location and time dependent spectra.
- Starting from a spatial only covariance we introduce temporal dependence following a classical time series analysis but with independent innovations having the assume spatial structure.
- For any covariance in space $r_S(\mathbf{p})$ and a fairly general family of temporal correlations $\rho(t)$, we show how to effectively construct Gaussian fields having covariance $r_S(\mathbf{p})\rho(t)$.

Conclusions II

- For the so-obtained fields, properly defined velocities when sampled randomly from the surface are centered at zero, indicating that these surfaces are dynamically inactive.

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- We introduce dynamics by a velocity field representing the motion of the surface.