Anisotropy analysis of 3d point processes

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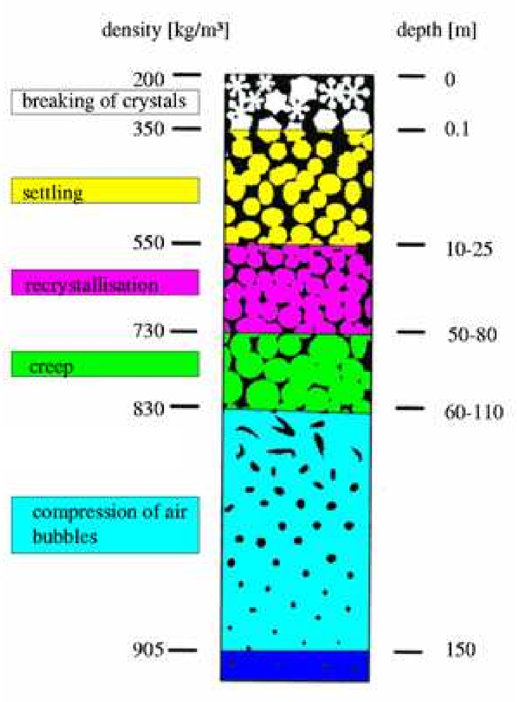
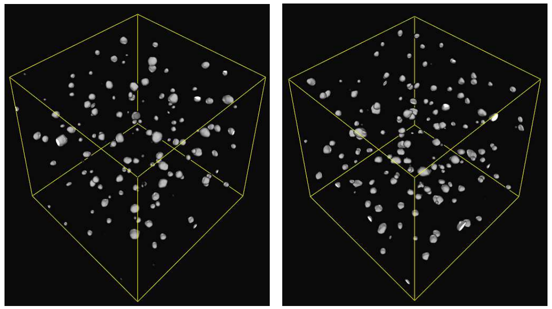
Motivation



Polar ice is compacted (sintered) snow. During the compression air pores are isolated in the ice.

Image: Freitag 1), Kipfstuhl 1), Stauffer 2)
1) Alfred-Wegener-Institute for polar and marine research, Bremerhaven
2) University Bern



Question

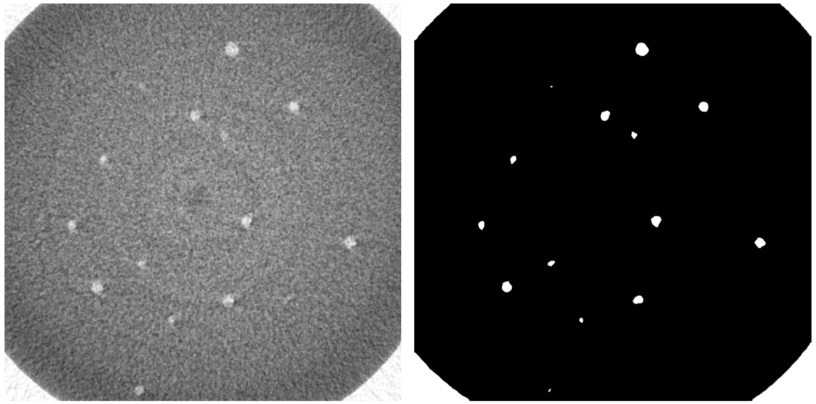
Does the location of the pores tell anything about the movements of the ice?

Data

tomographic images of ice samples
cylinder 15 mm height, 15 mm diameter
imaged inside a cold room at $-15°C$

images from three depths: 153 m, 353 m, 505 m

14 samples per depth



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Model for pressing

shape of the pores not significant -> concentrate on pore centres

Transformation

volume preserving compression

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{x}{\sqrt{c}} \\ \frac{y}{\sqrt{c}} \\ cz \end{pmatrix}, 0 < c < 1$$

Aim

- detection of the distortion
- estimation of the parameter c

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Methods in 2d

– point-pair rose density (Stoyan, Beneš, 1991)

– directional pair correlation function (Stoyan, Stoyan, 1992)

– Fry-method (Fry, 1979)

– directional K-function, 0-contour of the density $\frac{d}{d\varphi}K(r,\varphi)$ (Stoyan, Kendall, Mecke, 1995)

Idea

partition of circle
compare estimates of summary statistics for different directions

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Problems and ideas in 3d

Requirements

- measurement of characteristics easy and robust
- methods for visualisation of results

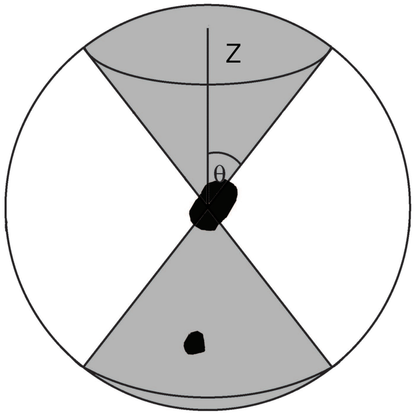
Problems

- more parameters than in 2d
- partition of the ball into suitable equal volume parts?

-> choose cumulative instead of density functions

-> adapt directions of investigation to the problem

Idea



– investigate point pattern in double cones aligned along coordinate axes

– compare observations

– large differences -> anisotropy

here: $\theta = \frac{\pi}{4}$

Advantage of cone

+ easy parametrisation w.r.t. spherical coordinates

+ can be rotated to arbitrary directions -> adaptable to data

Directional summary statistics: directional K

Directional version of the K-function

K_{dir}

Second reduced moment measure of the cone C
expected number of points within the double cone $x_0 + C$

Estimator

$$\lambda^2 \hat{K}_{\text{dir},u,\theta}(r) = \sum_{x \in \psi} \sum_{y \in \psi, y \neq x} \frac{\mathbb{1}_{C_u(r,\theta)}(x-y)}{|W_x \cap W_y|}, \; r \geq 0,$$

W_x translation of the window W by the vector x
$|B|$ volume of a set $B \subset \mathbb{R}^3$

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Directional summary statistics: local G

Directional version of the nearest neighbour distance distribution function G

G_{loc} Distribution function of the distance from the typical point of the process to the closest point in the cone $x_0 + C$.

Estimator

$$\widehat{G}_{\text{loc},u,\theta}(r) = \frac{\sum\limits_{(x,d)\in\psi} \mathbb{1}_{[0,r]}(d)\, \mathbb{1}\{x + C_u(d,\theta) \subset W\}}{\sum\limits_{(x,d)\in\psi} \mathbb{1}\{x + C_u(d,\theta) \subset W\}}, \; r \geq 0.$$

Directional summary statistics: global G

Directional version of the nearest neighbour distance distribution function G

G_{glob} Distribution function of the distance between x_0 and its nearest neighbour y conditioned on $y \in x_0 + C$.

Estimator

$$\widehat{G}_{\text{glob},u,\theta}(r) = \frac{\sum\limits_{(x,y)\in\Psi} \mathbb{1}_{C_u(r,\theta)}(x-y)\, \mathbb{1}_{W\ominus b(0,||x-y||)}(x)}{\sum\limits_{(x,y)\in\Psi} \mathbb{1}_{C_u(\infty,\theta)}(x-y)\, \mathbb{1}_{W\ominus b(0,||x-y||)}(x)}, \; r \geq 0.$$

less points investigated, stronger for higher intensities?

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Isotropy tests

Given n point patterns $\psi_1, \ldots, \psi_n$.

$\widehat{S}_x$, $\widehat{S}_y$, $\widehat{S}_z$ Estimators of one of the summary statistics with respect to the x-, y-, and z-direction.

Test statistics

$$T_{xy} = \int_{r_1}^{r_2} |\widehat{S}_x(r) - \widehat{S}_y(r)| \, dr, \text{ and}$$

$$T_z = \min \left(\int_{r_1}^{r_2} |\widehat{S}_x(r) - \widehat{S}_z(r)| \, dr, \int_{r_1}^{r_2} |\widehat{S}_y(r) - \widehat{S}_z(r)| \, dr \right),$$

where $[r_1, r_2]$ is a given interval.

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Isotropy tests

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Test

reject isotropy hypothesis at level α if $T_{z,i}$ is larger than $100(1-\alpha)\%$ of the estimated T_{xy} values

Alternative

Monte Carlo test, if few replications available
requires model for the data

Simulation study

Simulate regular point patterns in the unit cube.
Compress with pressing factor c.

Investigate different

- degrees of regularity: Matérn hard core point process and packing of balls (force biased algorithm)
- intensities: $\lambda = 500$ and $\lambda = 1000$
- hard core radii: $R = 0.025, 0.05$, and 0.075
- pressing factors: $c = 0.7, 0.8$, and 0.9

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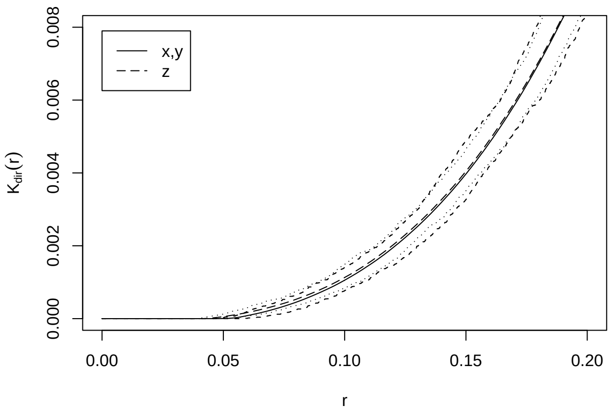
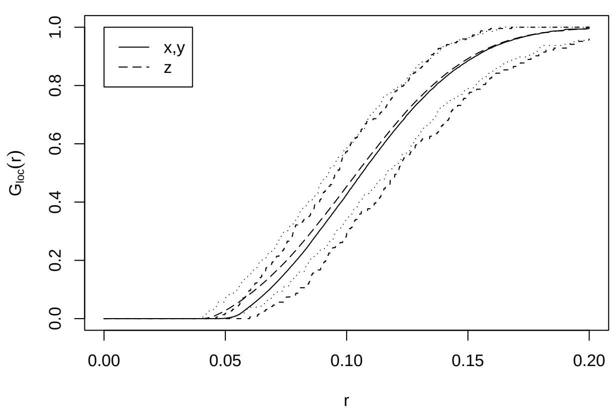
Simulation results: Matérn hard core

Parameters $\lambda = 500$, $R = 0.05$, and $c = 0.8$

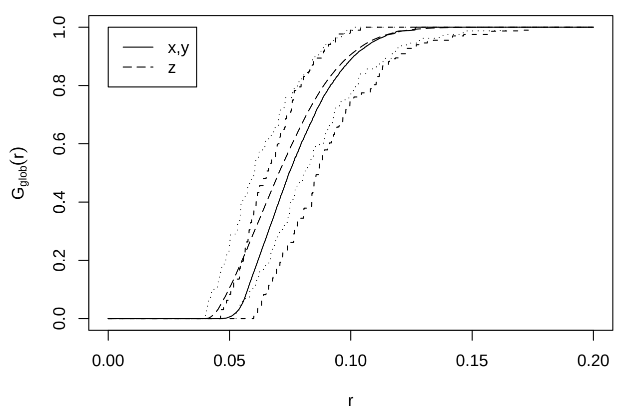
K_{dir}



G_{loc}



G_{glob}



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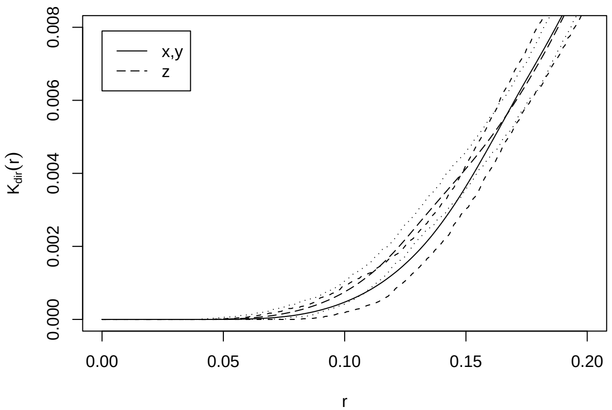
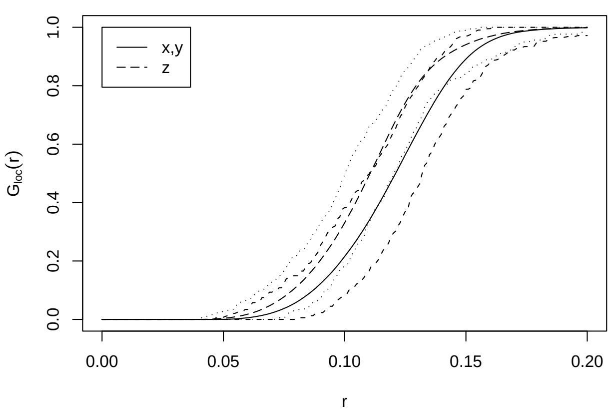
Simulation results: Force biased

Parameters $\lambda = 500$, $R = 0.05$, and $c = 0.8$

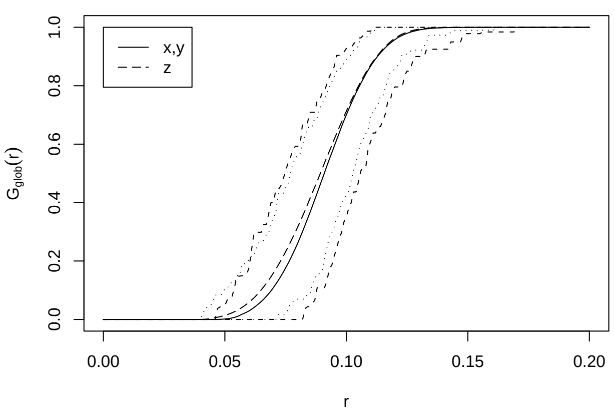
K_{dir}



G_{loc}



G_{glob}



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Powers of tests

Investigate influence of

- degree of regularity, intensity, hard core radius, pressing factor
- interval of observation

test intervals

$[0, 1.1R]$, $[0, 4/3\,R]$, $[0, 0.1]$ ($[0, 0.2]$, resp.)

R	0.05	0.05	0.05	0.075	0.075	0.075
r_2	0.1	0.067	0.055	0.2	0.1	0.0825
G_{loc}	17.2	73.8	97.2	91.7	99.8	**100**
G_{glob}	**21.3**	68.2	96.6	**93.9**	98.6	**100**
K_{dir}	19.9	**79.7**	**98.8**	36.9	**100**	**100**

powers of the tests on a 5% significance level, based on 1000 realisations

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Existence of outliers

In real data the existence of outliers, i.e. points which violate the hard core condition, is likely. How does this influence the results?

Therefore:

- use point patterns from previous simulation
- choose 5 random points $x_1, \ldots, x_5$ from each pattern
- include a further point y_i in balls of radius R centred in x_i

Result

- decreasing powers of the tests
- better results for K_{dir} and G_{loc} than for G_{glob}.
- larger integration intervals should be chosen.

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Alternative: Investigate direction to nearest neighbour

Given set of unit vectors $v_i = (x_i, y_i, z_i), i = 1, \ldots, n$

Orientation matrix

$$A = \begin{pmatrix} \sum x_i^2 & \sum x_i y_i & \sum x_i z_i \\ \sum x_i y_i & \sum y_i^2 & \sum y_i z_i \\ \sum x_i z_i & \sum y_i z_i & \sum z_i^2 \end{pmatrix}$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3$ eigenvalues of A

Test statistic largest eigenvalue λ_1

Significance points at 5% level: $\frac{1}{3} + \frac{0.873}{\sqrt{n}}$ for $n > 100$, Anderson, Stephens (1972)

Results weaker performance than tests based on summary statistics

Summary of results

- tests using summary statistics better than analysis of direction to nearest neighbour

Higher power for

- higher regularity, larger intensities, stronger pressing
- integration interval should be chosen suitably
- in most cases: K_{dir} yields best results
- test with G functions more stable when changing interval
- tests also work for compressed clustered patterns

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Estimation of the pressing factor

- rescale by $(\sqrt{d}, \sqrt{d}, \frac{1}{d})$ with $d \in [0.6, 1.1]$

- compute statistics

$$T_{\sum,d} = \int_{r_1}^{r_2} \left(|\widehat{S}_{x,d}(r) - \widehat{S}_{y,d}(r)| + |\widehat{S}_{y,d}(r) - \widehat{S}_{z,d}(r)| + |\widehat{S}_{z,d}(r) - \widehat{S}_{x,d}(r)|\right) \, dr,$$

Estimator for c $\qquad \widehat{c} = \mathrm{argmin}_d \, T_{\sum,d}$

Results for simulated data

Parameters: $\lambda = 500$, $R = 0.05$

r_2	c	$\bar{\hat{c}}_{\text{loc}}$	MSE	$\bar{\hat{c}}_{\text{glob}}$	MSE	$\bar{\hat{c}}_K$	MSE
0.055	1.0	0.9828	**3.944e-3**	0.9795	6.763e-3	0.9875	4.850e-3
0.055	0.9	0.8810	5.813e-3	0.8855	5.138e-3	0.8933	**3.631e-3**
0.055	0.8	0.7780	5.363e-3	0.7818	4.544e-3	0.7923	**2.844e-3**
0.055	0.7	0.6880	**1.850e-3**	0.6913	1.981e-3	0.6860	1.988e-3
0.15	1.0	1.0005	2.625e-3	0.9385	2.081e-2	0.9955	**1.813e-3**
0.15	0.9	0.8953	2.806e-3	0.9060	1.819e-2	0.8993	**1.669e-3**
0.15	0.8	0.7968	2.306e-3	0.8655	2.511e-2	0.8030	**1.013e-3**
0.15	0.7	0.6958	1.631e-3	0.8128	3.222e-2	0.7015	**7.125e-4**

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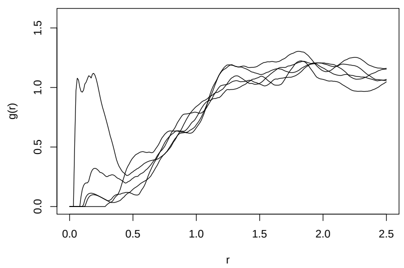
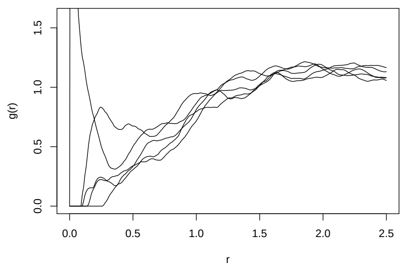
Ice: Choice of parameters





– between 329 and 733 pores per sample

– study degree of regularity using isotropic pair-correlation function

– investigation of distances to nearest neighbours shows existence of outliers

-> choose larger interval of observation

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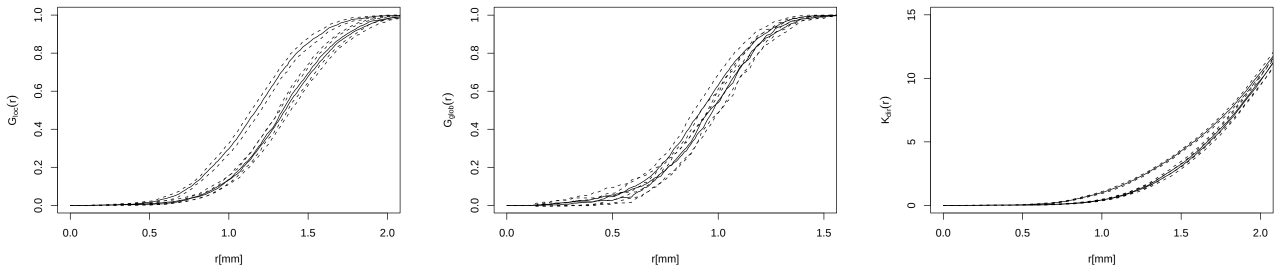
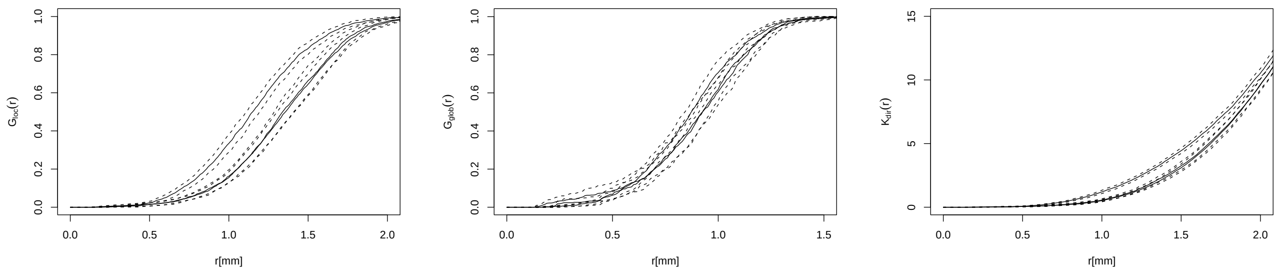
Ice: Isotropy test

Means and confidence bands of directional summary statistics

353 m



505 m



Pooling of the data as in Baddeley et. al. (1993)

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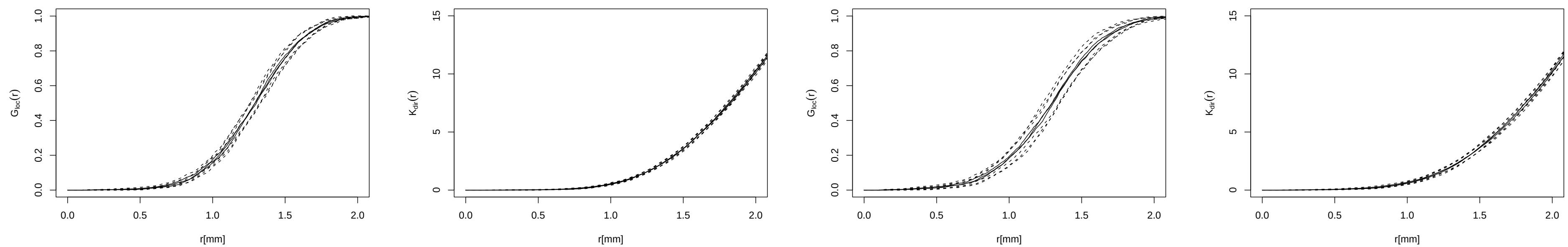
Ice: Estimation of pressing factors

sample	153 m	353 m	505 m
N_V	380.29	454.79	516.71
λ	0.2528	0.3403	0.3241
$\widehat{c}_G$	0.807	0.630	0.534
$\widehat{c}_K$	0.821	0.641	0.545



Summary

We have

– detected the anisotropy within the ice

– estimated the pressing factor

Nye formula

– simplified ice flow model

-> same trend but absolute values higher in our estimates (approx. 0.1)

However

bedrock conditions not taken into account
unknown so far due to incomplete drilling
could shift expected pressing factors to higher values

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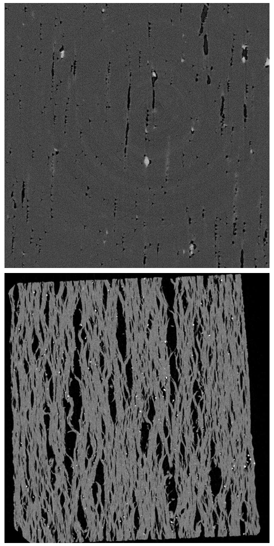
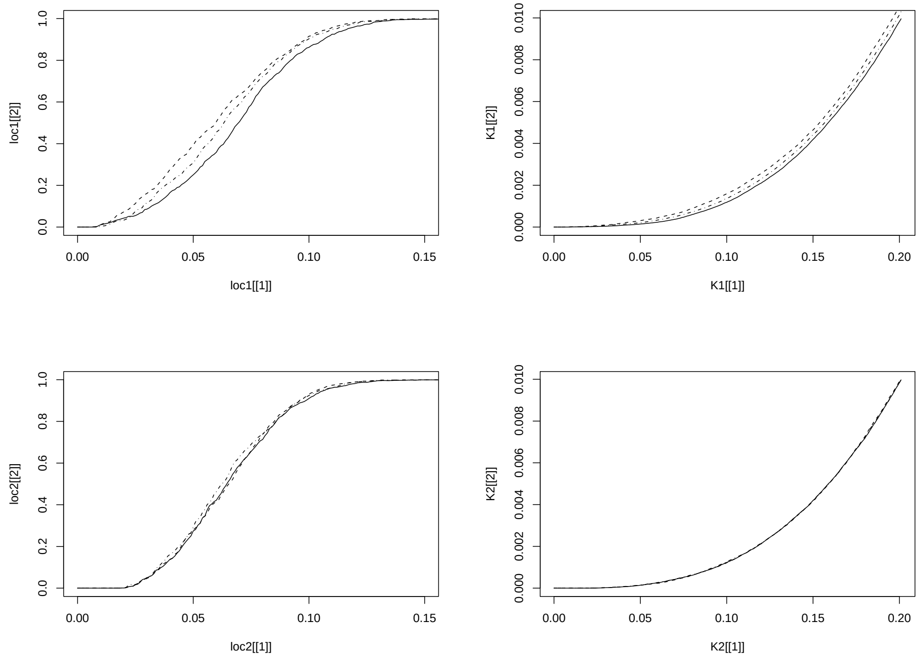
Further example: Foam

aluminium foam: pressed before foaming process
degree of isotropy grows during foaming



grey: aluminium
white: foaming agent
black: pore



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