# A class of non-Gaussian second order spatio-temporal models

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- Spatio-temporal characteristics be studied by the means of generalized Rice's formula.
- The potential for stochastic modeling has been demonstrated.

# Outline

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- Distributional structure coincides with second order structure

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- the right hand side represents the biased sampling distribution when sampling is made over the 0-level contour $\mathcal{C}_0 = \{\tau : X(\tau) = 0\}$

# Are Gaussian models good?

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- They are good!
- Spectral theory or frequency domain analysis is at the center of stochastic modeling in engineering sciences
- Elegant mathematical properties, for which the relation between the frequency and time domain is well understood
- The ability to model spatio-temporal phenomena through essentially the same framework as for time only dependent data contributed the popularity in geostatistics

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- Empirical evidence that the Gaussian models often do not fit properly the phenomena they are intended to describe
- Discrepancies amplified additionally by non-linearity of deterministic physical models behind the data
- Asymmetry and heavy tails – features that cannot be modeled by Gaussian distributions
- Examples: skewness of sea levels data (Åberg (2007)), highly skewed measuremnts of soil properties in geotechnical engineering problems and seismic ground motion (Lagaros et al. (2005)), heavier than Gaussian tails were reported from such spatial phenomena as topographic data, temperature (Palacios and Steel (2006)), or well log data in petroleum application (Røislien and Omre (2006)), Gurley et al. (1996) – critical discussion of various approaches to handling the non-Gaussian loads

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- A need for relatively simple and convieniently parameterized second order non-Gaussian models
- Among candidates are processes linked to the Laplace distributions that allow for simultaneous match of both spectra and higher order moments of the data

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- Lévy motion generated by Laplace distribution

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$$\begin{aligned} X(t) &= \int \exp(it\lambda)\,\zeta(\lambda) \\ &= \int_0^\infty \cos(\lambda t)\,dB_F(\lambda) + \int_0^\infty \sin(\lambda t)\,d\tilde{B}_F(\lambda) \end{aligned}$$

where $F$ is a non-decreasing function.

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- In discretization, this leads to $\sigma^2(\lambda_j) = F(\lambda_j + d\lambda_j) - F(\lambda_j)$.

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### Theorem

Harmonizable Laplace process has a generalized Laplace mariginal distribution, defined by the characteristic function

$$\mathbb{E}\left[e^{i\xi X_t}\right] = \left(\frac{1}{1+\frac{\xi^2}{2}}\right)^{\lambda_0}$$

where $\lambda_0 = F(\infty) - F(0)$ with the density given by

$$f_{X_t}(x) = \frac{\sqrt{2}}{\Gamma(\lambda_0)\sqrt{\pi}}\left(\frac{|x|}{\sqrt{2}}\right)^{\lambda_0 - 1/2} K_{\lambda_0-1/2}(\sqrt{2}|x|), \quad x \neq 0,$$

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The finite-dimensional distributions are defined by the following characteristic function

$$\mathbb{E}\left[e^{i\sum_{j=1}^{n}\xi_j X(t_j)}\right] = \exp\left\{-\int_0^\infty \ln\left(1+\frac{1}{2}\xi^T A\xi\right) dF(\lambda)\right\},$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ and $A$ is a matrix with entries $A_{jk} = \cos(\lambda(t_k - t_j))$.

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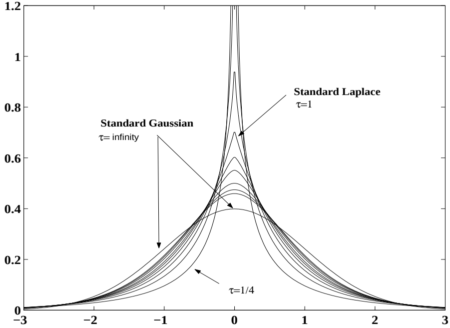
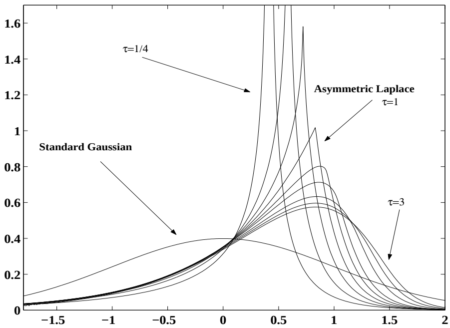
- The construction extends easily to non-symmetric case by considering Brownian motion with shift in place of the regular Brownian motion
- Asymmetric case has explicit one dimensional densities in the terms of Bessel functions
- The construction extends to fields by replacing $F$ on real line by measures $\mathbb{R}^n$

# Examples of the densities




Symmetric cases

Asymmetric cases

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- These random variances are independent
- This can serve as a method of simulation harmonizable Laplace processes

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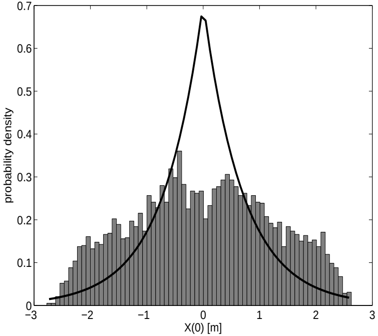
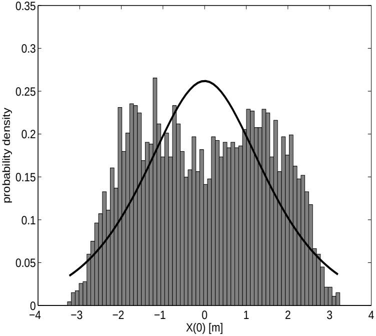
- The samples has been generated for harmonizable processes and their sampling distribution compared with the marginal




Uniform spectrum




Pierson-Moskovitz specturm

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- In fact, it has been shown that any second order harmonizable non-Gaussian process can not be ergodic – Wright (1978)
- That maybe not such a bad news as long as invariant sets for the process will be identified
- Non-ergodic models can be used for modeling phenomena that vary from sample to sample
- Finally non-ergodicity in space can be mixed in time and in spatio-temporal models sample distribution collected over time maybe still converging to constant values

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- $X = \int_{\mathcal{X}} f(x)\, d\Lambda(x)$ – the isometry of $L_2(\mathcal{X}, \mathcal{B}, m)$ into $L_2(\Omega, \mathcal{F}, \mathbb{P})$ that relates the indicator functions $\mathbf{1}_A(x)$ with the variables $\Lambda(A)$

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- The characteristic function, first two moments, skeweness and kurtosis

$$
\begin{aligned}
\phi_X(u) &= \exp\left(-\int_{\mathcal{X}} \log\left(1 - i\mu u f(x) + \frac{\sigma^2 f^2(x) u^2}{2}\right) dm(x)\right). \\
\mathbb{E}X &= \mu \cdot \int f dm \\
\mathbb{E}(X - \mathbb{E}X)^2 &= (\mu^2 + \sigma^2) \cdot \int f^2 dm \\
s &= \operatorname{sgn}(\mu) \frac{2\mu^2 + 3\sigma^2}{(\mu^2 + \sigma^2)^{\frac{3}{2}}} \cdot \frac{\int f^3 dm}{\left(\int f^2 dm\right)^{3/2}} \\
k_e &= 3\left(2 - \frac{\sigma^4}{(\mu^2 + \sigma^2)^2}\right) \cdot \frac{\int f^4 dm}{\left(\int f^2 dm\right)^2}.
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- explicit formulas for the moments, skeweness and kurtosis
- large $\alpha$, the kernel is more like the one for the generalized Laplace, i.e. approximately constant on the compact support, while for small $\alpha$ it correspond more to averaging that leads to Gaussian-like distributions

# Laplace integrals – densities

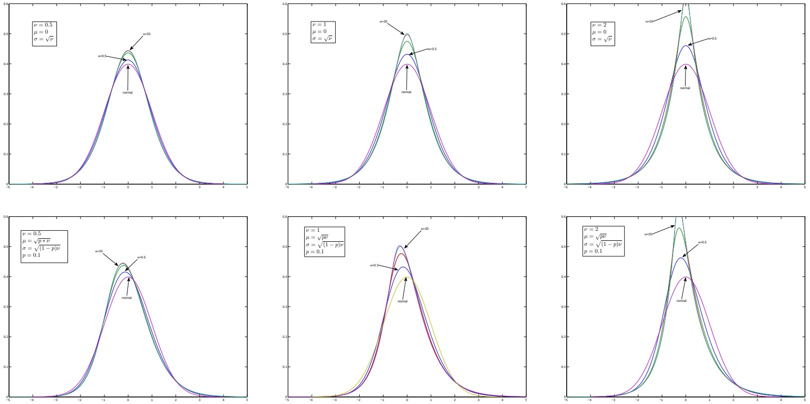

Figure: Examples of densities and their dependence on the parameters. In the top row we see symmetric densities with $\alpha = 0.5, 1, 2, 20$ on each graph. From left to right $\nu = 0.5, 1, 2$, respectively. The top rows deals with the symmetric case ($\mu = 0$) while the bottom one with the asymmetric one in which the asymmetry parameter $\mu = \sqrt{p * \nu}$, with $p = 0.1$.

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- the parameters of the Laplace motion can be fitted using method of moments

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# Laplace moving averages

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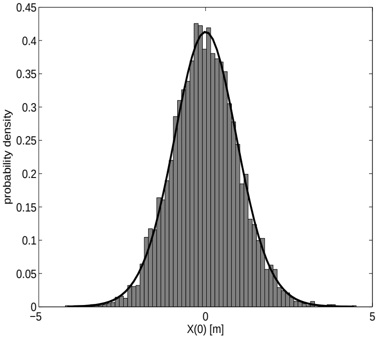
- the parameters of the Laplace motion can be fitted using method of moments
- if the kernel is assumed from a parametric family one can use fit the parameters using correlation function
- a non-parametric approach an estimate $\widehat{f}$ is given by

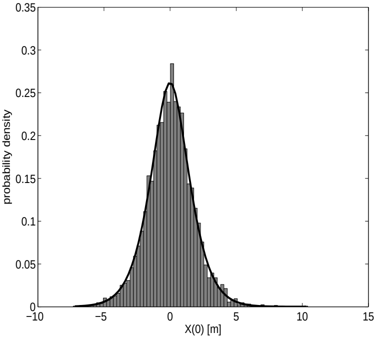
$$\widehat{f}(x) = \mathcal{F}^{-1}\sqrt{\widehat{R}(\omega)},$$

where $\widehat{R}(\omega)$ is an estimate of spectrum

# Ergodicity




Uniform spectrum




Pierson-Moskovitz specturm

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- the crossing intensity can be computed by the integral

$$\mu^+(u) = \frac{1}{(2\pi)^2}\int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z e^{-i(\xi_1 u+\xi_2 z)}\phi_{X(0),X'(0)}(\xi_1,\xi_2)\, d\xi_1\, d\xi_2\, dz.$$

# Outline

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- Fitting the model is fairly straightforward
- Ergodic properties for harmonizable processes should be investigated

# Final Slide

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## Quotation by Pierre-Simon Laplace

*"Nature laughs at the difficulties of integration."*