The Laplace driven moving average – a non-Gaussian stationary process

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Objectives

Background:

- Gaussian process very convenient in environmental sciences since they allow for covariance/spectral modelling.
- Sometimes not sufficient, does not allow for skewed marginal distributions and has often too light tails.

Goals:

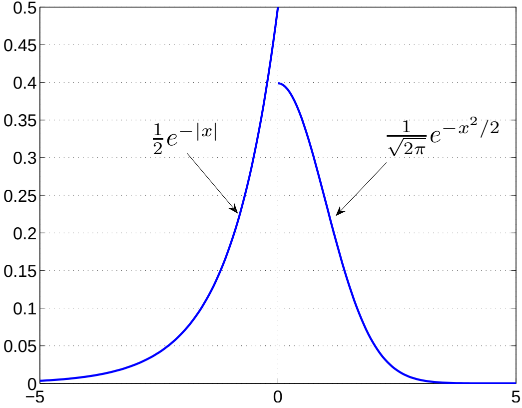
- Construct non-Gaussian stationary process...
- ... possessing a spectrum,
- a skewed marginal distribution,
- and heavier tails than the Gaussian distribution.

Starting point:
the Laplace distributions

The Laplace distribution – from a historical point of view

First and second Laplace law of error.

I. The Laplace distribution. (Laplace, 1774).
II. The normal (Gauss) distribution. (Laplace, 1778).



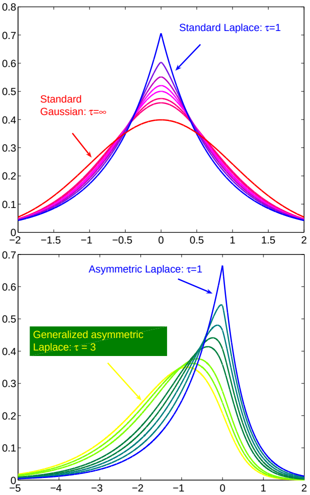
Generalized Laplace distributions

- Laplace distribution:

$$\phi(t) = \left(\frac{1}{1 + \frac{\sigma^2 t^2}{2}} \right)$$

- Generalization:

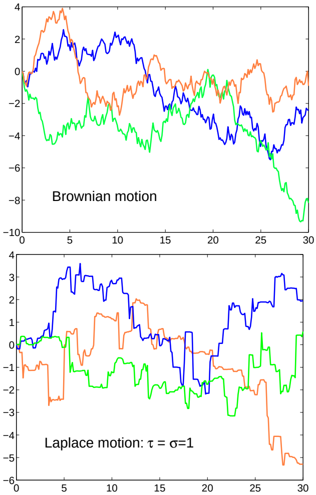
$$\phi(t) = \left(\frac{1}{1 - i\mu t + \frac{\sigma^2 t^2}{2}} \right)^{\tau}$$



Laplace motion - a counterpart to Brownian motion

A stochastic process $\Lambda(t)$ is called asymmetric LM if

1. it starts at the origin
2. it has independent and stationary increments
3. the increments have a generalized asymmetric Laplace distribution



The Laplace driven moving average

Using the Laplace motion $\Lambda(x)$ one can define a Laplace driven moving average by

$$X(t) = \int_{-\infty}^{\infty} f(t-x) d\Lambda(x).$$

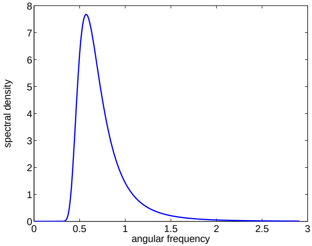
The function f is called a kernel and should satisfy $\int f^2(x)\,dx < \infty$. A similar definition is possible in higher dimensions.

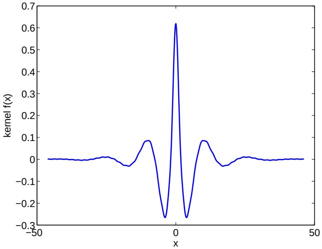
Spectral properties of Laplace moving averages

- The spectrum of $X(t)$ is given by

$$S(\omega) = \frac{\tau(\sigma^2 + \mu^2)}{2\pi} |\mathcal{F}f(\omega)|^2$$

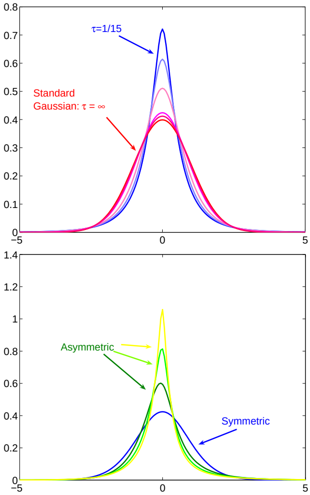
- By requiring that f is a symmetric function one can estimate the kernel from the spectral density.
- By "minimum phase" assumptions one can get causal kernels.





Marginal distribution

- The marginal distribution is given in terms of its characteristic function.
- Moments of the distribution can be computed.
- The method of moments (with first four moments) can be used in fitting the marginal distribution to data.



Simulation

The LMA can be seen as a convolution of Laplace noise with a kernel f. Discrete version:

$$\int f(t-x)d\Lambda(x) \approx \sum f(t-x_i)\Delta\Lambda(x_i)$$

Time domain simulation:

- simulate iid Laplace noise
- convolve the noise with the kernel

Frequency domain simulation:

- simulate iid Laplace noise
- Fourier transform the noise and the kernel f using FFT
- Take product of the Fourier transforms
- Take inverse Fourier transform of the product

Example: Random fields with Matérn covariance

The Matérn family of covariances is commonly used to describe spatial dependence in geostatistics. It has covariance

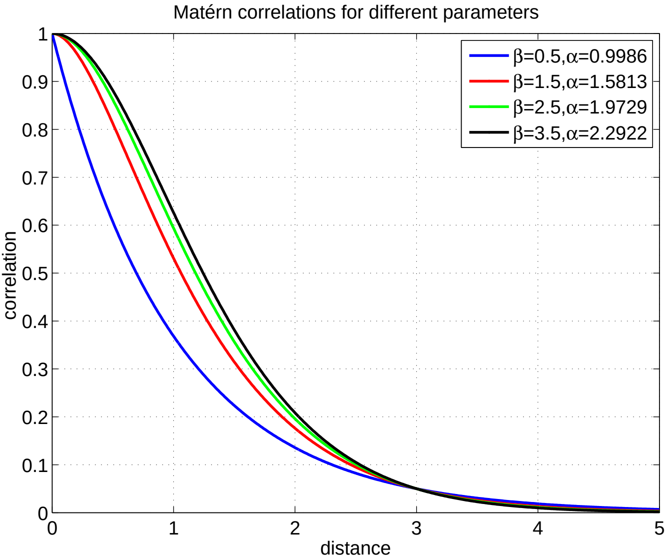
$$r(x) = \frac{\phi}{2^{\beta-1}\Gamma(\beta)}(\alpha|x|)^{\beta}K_{\beta}(\alpha|x|),$$

and spectrum

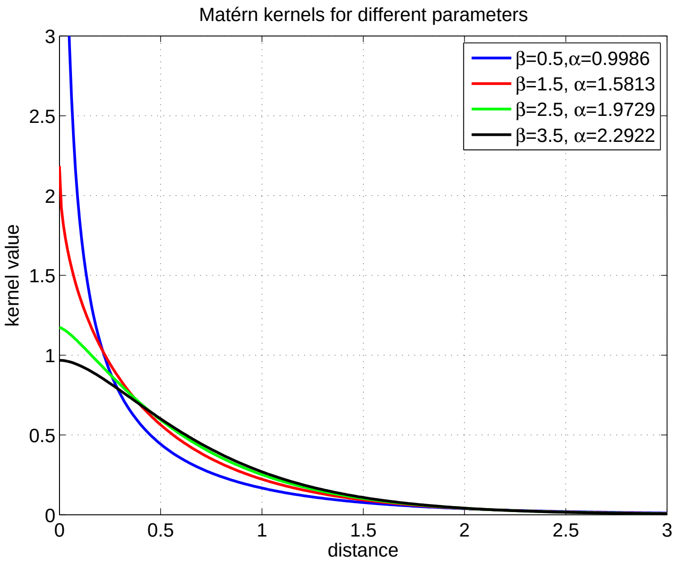
$$S(\omega) = \frac{\Gamma(\beta + \frac{d}{2})\alpha^{2\beta}}{\Gamma(\beta)\pi^{d/2}}\frac{\phi}{(\alpha^2 + |\omega|^2)^{\beta+\frac{d}{2}}}.$$

d is the dimension ϕ is variance, α a range parameter, β a smoothness parameter and K is a modified Bessel function.

Matérn correlations



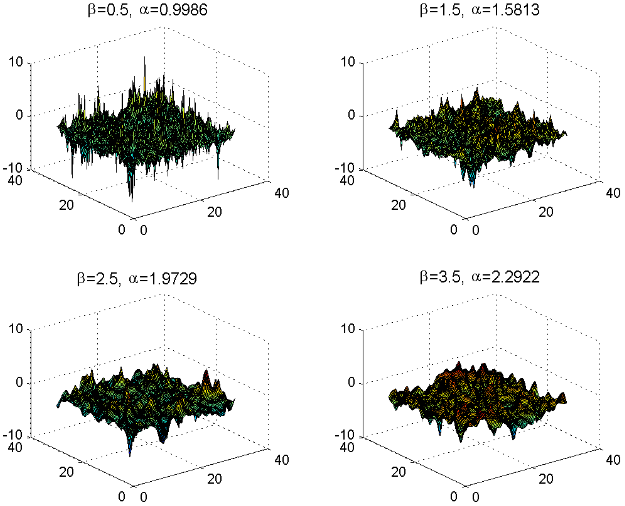
... and kernels



Matérn kernels for different parameters
β=0.5,α=0.9986
β=1.5, α=1.5813
β=2.5, α=1.9729
β=3.5, α=2.2922
kernel value
distance
0
0.5
1
1.5
2
2.5
3

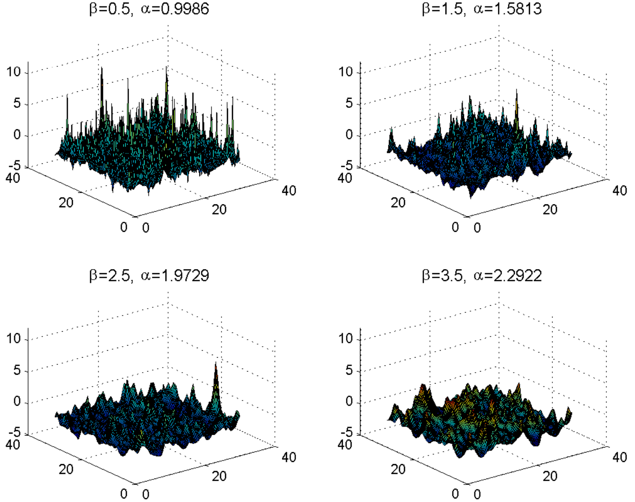
Symmetric fields

Laplace parameters: $[\tau, \sigma, \mu, c] = [1, 1, 0, 0]$.

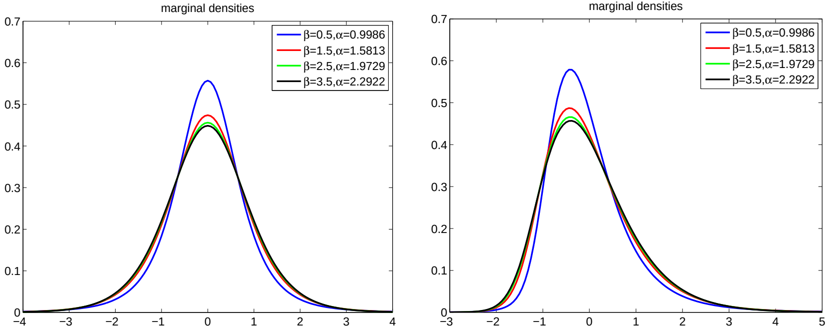


Asymmetric fields

Laplace parameters: $[\tau, \sigma, \mu, c] = [1, 1/\sqrt{2}, 1/\sqrt{2}, -1/\sqrt{2}]$.



Marginal densities



All distributions have mean zero and variance one

Relation to Gaussian processes

A generalized Laplace distributed random variable Λ can be represented using a $\Gamma(\tau, 1)$-distributed random variable Γ and standard Gaussian variable B:

$$\Lambda \stackrel{D}{=} c + \mu\Gamma + \sigma\sqrt{\Gamma}B.$$

Similarly, the Laplace motion can be represented as

$$\Lambda(t) \stackrel{D}{=} c \cdot t + \mu\Gamma(t) + \sigma B(\Gamma(t)),$$

where $\Gamma(t)$ is a Gamma-process with parameter τ and $B(t)$ is standard Brownian motion.

Conditioning on the Gamma process – a smart trick

Conditional on a specific realisation γ of the gamma-process the Laplace moving average becomes a non-stationary Gaussian process! It will have mean

$$m_1(t) = E[X(t) \mid \Gamma(x) = \gamma(x)] = c \int f(t-x)\, dx + \mu \int f(t-x)\, d\gamma(x)$$

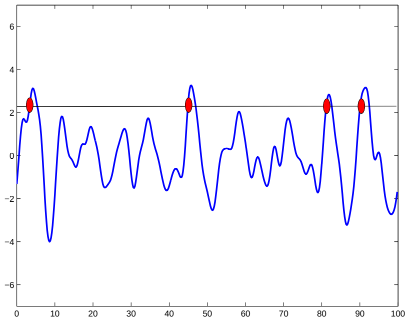
and variance

$$\sigma_{11}^2(t) = Var[X(t) \mid \Gamma(x) = \gamma(x)] = \sigma^2 \int f^2(t-x)\, d\gamma(x)$$

both depending on time t.

Example: Rice's formula

- Formula for computing the expected number of level crossings
- Very important in reliability applications
- For Gaussian stationary processes there is a closed form solution



Rice's formula - non-stationary case

For a non-stationary process

$$E[N_T^+(u)] = \int_0^T \int_0^\infty z f_{Y(t),Y'(t)}(u,z)\, dz\, dt$$

- $N_T^+(u)$ - number of upcrossings of level u during $[0, T]$.
- For a Gaussian process the innermost integral can be evaluated.
- The outer integral can be computed numerically

Monte-Carlo approach to Rice's formula

- $N_T^+(u)$- number of upcrossings of level u in time interval $[0, T]$
- Condition on $\Gamma(x) = \gamma(x)$:

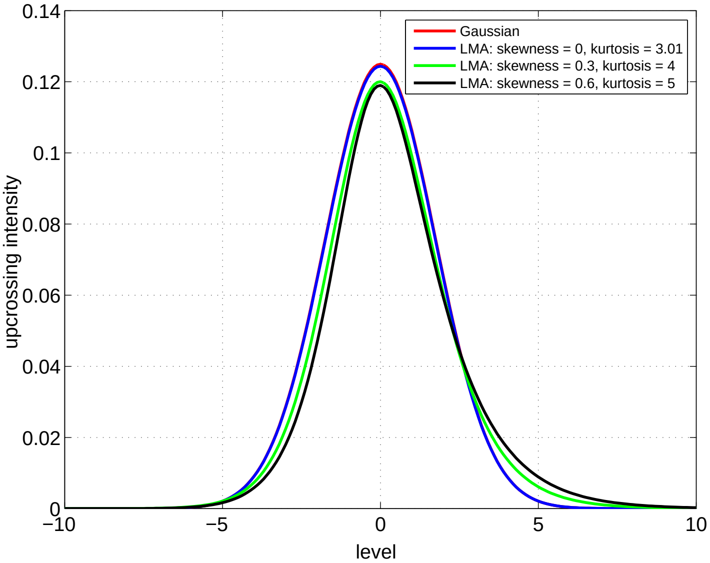
$$\mu^+(u) = E[N_1^+(u)] = E[E[N_1^+(u) \mid \Gamma(x) = \gamma(x)]]$$

- Approximate by forming a Monte-Carlo average:

$$\mu^+(u) \approx \frac{1}{n} \sum_{k=1}^{n} E[N_1^+(u) \mid \Gamma(x) = \gamma_k(x)]$$

- The terms in the sum are level-crossing intensities for non-stationary Gaussian processes.

Upcrossing intensity



Summary

- The Laplace driven moving average can be used to model second order stationary loads with skewed marginal distribution.
- The model can be fitted to data using a moment matching approach.
- Simulation can either be done in time or in frequency domain.
- Conditional on a realisation of a gamma-process the LMA becomes a non-stationary Gaussian process.
- Rice's formula can be evaluated by a Monte-Carlo method.