REFINED OPERATIONS ON K-THEORY BY LIFTING TO THE VIRTUAL CATEGORY

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Abstract: This is the first article in an upcoming series of papers that have arisen through an attempt to answer open questions of Deligne proposed on the determinant of the cohomology. It amounts to functorial and metrized versions of the Grothendieck-Riemann-Roch theorem. In this article we treat various results on virtual categories which are the categories where the Deligne-Riemann-Roch theorem is originally formulated which is of independent interest. In particular we construct functorial versions of Adams operations and prove a rigidity result for endofunctors of the virtual category. Finally we compare our constructions to constructions of Franke on Chern intersection functors and Chow categories.

Contents

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1. Introduction

This is the first article in an upcoming series of papers. They have arisen through an attempt to answer open questions of Deligne proposed in [4]. This is supposed to be understood in the following sense, for which we refer to *loc. cit.* for the best introduction: A special case of the Grothendieck-Riemann-Roch theorem can be understood as the formula

$$\operatorname{ch}(Rf_*E) = f_*(\operatorname{ch}(E)\operatorname{Td}(T_f)) \tag{1}$$

for a projective smooth morphism of smooth varieties $f : X \to Y$ (cf. [13], chapter V, §7 or the book [19] for a precise and more general formulation). The general question on functoriality becomes whether there are categorical replacements of all the objects and homomorphisms involved. This is an approach to obtain secondary information which gets lost when one quotients out with various equivalences. In particular Deligne deduces (cf. see [4], Thorme 9.9) a unique, up to sign, isomorphism of line bundles

$$(\det Rf_*L)^{\otimes 12} \simeq \langle \omega, \omega \rangle \langle L, L\omega^{-1} \rangle^{\otimes 6} \tag{2}$$

for $f : C \to S$ a smooth family of proper curves and L a line bundle on C. This isomorphism is suggested by the same Grothendieck-Riemann-Roch theorem which says that the classes of the two line bundles are the same in the Picard group (if S is regular enough).

Some remarks are in order. The approach using virtual categories as we have done are, as was already noted in [4], most suitable to treat questions on the determinant of the cohomology. This can for example be seen from the observation that the virtual category is a truncation of the K-theory space and the determinant of the cohomology corresponds to weight one Adams eigenspaces which have all their K-theory in K_0 and K_1. In this case the problem on functoriality is to describe the above alluded to homomorphisms as functors and describe a natural transformation of functors in the target category, refining the identity (1).

The approach we have chosen here is more or less K-theoretical in nature, using the decomposition of K-theory into Adams eigenspaces which admits a classical relation to Chow-theory. This article then deals with various fundamental properties of both these decompositions as well as virtual categories in general, which hopefully are of interest independently of the applications to the Riemann-Roch problem envisioned here.

More precisely the article is organized as follows. We first consider determinant functors on Waldhausen categories establishing results on associated virtual categories (cf. Theorem 2.2). This extends the situation of determinant categories on exact categories in [4]. In the next section we consider various natural operations on virtual categories and in particular construct Adams and λ-operations (cf. Proposition 3.2, Proposition 3.4). We then establish certain rigidity results on operations on virtual categories in Theorem 4.5, giving existence of lifts of operations on K_0 to the virtual category, at least after inverting the integers for stacks, or 2 in the case of schemes. Finally in the last section we make comparisons to constructions of Franke in "Chern functors" in [11] of Chern intersection functors. We have also included three appendices on $\mathbb{A}^1$-homotopy theory, algebraic stacks and virtual categories to fix language and recall the necessary utensils.

Finally, it should be noted that the techniques in this article resemble a lot the techniques in [14] in that they use homotopy theory of simplicial sheaves to tackle Riemann-Roch-type problems. Also related results have been obtained in a series of papers [10], "Chern functors" in [11], [9] as well as that of [7]. Part of these results were announced in [8].

2. The virtual category and triangulated categories

Given a small exact category $\mathcal{C}$, we can consider its K-theory. The first case of K_0 can be defined explicitly in terms of the category $\mathcal{C}$, as the Grothendieck group of $\mathcal{C}$. This is the free abelian group on the objects of $\mathcal{C}$, modulo the relationship $B = A + C$ if

$$0 \to A \to B \to C \to 0$$

is an exact sequence in $\mathcal{C}$. A more sophisticated approach was taken by Quillen, [32], where he constructs a certain topological space $BQ\mathcal{C}$ associated to a (small) exact category $\mathcal{C}$ such that

$$K_i(\mathcal{C}) := \pi_{i+1}(BQ\mathcal{C}).$$

Now, let $X \in \mathrm{ob}(\mathrm{Top}_\bullet)$ be a pointed topological space. One defines the fundamental groupoid of X to be the category whose objects are points of X, and morphisms are homotopy-classes of paths relative to the end-points, *i.e.* it is associated to the diagram

$$[PX \rightrightarrows X]$$

where PX is the space of paths of X. Denote the corresponding functor $\mathrm{Top}_\bullet \to \mathrm{Grp}$ by Π_f. In [4] one defines a category of virtual objects of an exact category, which offers a type of abelianization of the derived category and K_0 of the category. Somewhat more precisely, let $\mathcal{C}$ be a small exact category. The category of virtual objects of $\mathcal{C}$, $V(\mathcal{C})$ is the following: Objects are loops in $BQ\mathcal{C}$ around a fixed zero-point, and morphisms are homotopy-classes of homotopies of loops. Recall that $BQ\mathcal{C}$ is the geometrical realization of the Quillen Q-construction of $\mathcal{C}$. Addition is the usual addition of loops. This construction is the fundamental groupoid of the space $\Omega BQ\mathcal{C}$. In case $\mathcal{C}$ is not small we will always consider an equivalent small category, and ignore any purely categorical issues this might cause.

Remark 2.0.1. $V(\mathcal{C})$ is a groupoid, *i.e.* any morphism is an isomorphism, and the set of equivalence-classes is in natural bijection with $K_0(\mathcal{C})$. For any object $c \in obV(\mathcal{C})$, we have $\mathrm{Aut}_{V(\mathcal{C})}(c) = \pi_1(\Omega BQ\mathcal{C}) = K_1(\mathcal{C})$.

Deligne also provides a more algebraic and universal definition of $V(\mathcal{C})$. We will give an additional description in terms of derived categories in the next section (cf. Theorem 2.2 for a precise statement).

2.1. **Algebraic definition.** The above category is a so called universal Picard category with respect to $\mathcal{C}$. We include the precise definition because of the role it will play in the rest of the paper. A (commutative) Picard category is a groupoid $\mathcal{C}$ with an auto-equivalence $P \mapsto P \oplus Q$ for any object Q of $\mathcal{C}$, satisfying certain compatibility-isomorphisms plus some commutativity and associativity-restraints (cf. [3], XVIII, Dfinition 1.4.2 for the definition of a (strictly commutative) Picard category, or [26], 14.4, axiome du pentagone et de l'hexagone): There is an associativity-isomorphism

$$a_{x,y,z} : (x \oplus y) \oplus z \to x \oplus (y \oplus z)$$

such that

$$\begin{tikzcd}
 & (w \oplus x) \oplus (y \oplus z) \arrow[dr, "a_{w,x,y\oplus z}"] & \\
((w \oplus x) \oplus y) \oplus z \arrow[ur, "a_{w\oplus x,y,z}"] \arrow[d, "a_{w,x,y}\oplus 1_z"'] & & w \oplus (x \oplus (y \oplus z)) \\
(w \oplus (x \oplus y)) \oplus z \arrow[rr, "a_{w,x\oplus y,z}"] & & w \oplus ((x \oplus y) \oplus z) \arrow[u, "1_w \oplus a_{x,y,z}"']
\end{tikzcd}$$

commutes. There is a commutativity-isomorphism $c_{x,y} : x \oplus y \to y \oplus x$ such that

$$\begin{tikzcd}
(x \oplus y) \oplus z \arrow[r, "a_{x,y,z}"] & x \oplus (y \oplus z) \arrow[d, "1_x \oplus c_{y,z}"] \\
z \oplus (x \oplus y) \arrow[u, "a_{w\oplus x,y,z}"] & x \oplus (z \oplus y)) \\
(z \oplus x) \oplus y \arrow[u, "a_{z,x,y}"] \arrow[r, "c_{z,x}\oplus 1_y"] & (x \oplus z) \oplus y \arrow[u, "a_{x,z,y}"']
\end{tikzcd}$$

commutes. It follows that a category such as this has a zero-object, has inverses etc. (see [3], XVIII, 1.4.4). In other words, a Picard category is a symmetric monoidal groupoid whose functor $- \oplus Q$ is an equivalence of categories for each object Q in $\mathcal{C}$. It is moreover said to be strictly commutative if $c_{x,x} : x \oplus x \to x \oplus x$ is the identity. In general we denote by $\epsilon(x) = c_{x,x}$. An additive functor between Picard categories is defined to be a monoidal functor between Picard categories.

Observe we merely have isomorphisms $B \oplus (-B) \to 0$, not equality. For any exact category $\mathcal{C}$, the universal Picard category $V(\mathcal{C})$ is a Picard category $\mathcal{C}$ with a functor $[] : (\mathcal{C}, is) \to V(\mathcal{C})$ which is universal with respect to morphisms $T : (\mathcal{C}, is) \to P$ into Picard categories P, satisfying the following compatibility conditions:

(a): For any short exact sequence

$$\mathcal{A} : 0 \to A' \to A \to A'' \to 0,$$

there is an isomorphism, functorial with respect to isomorphisms of exact sequences,

$$T(\mathcal{A}) : T(A) \to T(A') \oplus T(A'').$$

(b): For any 0-object of $\mathcal{C}$, there is an isomorphism $T(0) \simeq 0$.

(c): If $\phi : A \to B$ is an isomorphism, with exact sequence $0 \to A \to B \to 0$, the induced map $T(\phi)$ is the composite

$$T(A) \to T(0) \oplus T(B) \to T(B).$$

(d): The functor T is compatible with filtrations, *i.e.* for an admissible filtration $A \subset B \subset C$, the diagram

$$\begin{tikzcd} T(C) \arrow[r] \arrow[d] & T(A)+T(C/A) \arrow[d] \\ T(B)+T(C/B) \arrow[r] & T(A)+T(B/A)+T(C/B) \end{tikzcd}$$

is commutative (here the quotients are only defined up to unique isomorphism, but are well-defined by the other conditions).

In [4] it is mentionned that the functor $(\mathcal{C}, is) \to V(\mathcal{C})$ factors as

$$(\mathcal{C}, is) \to (D^b(\mathcal{C}), is) \to V(\mathcal{C}). \tag{3}$$

Here $D^b(\mathcal{C})$ is the derived category of $\mathcal{C}$ (supposed to be the full subcategory of a fixed abelian category), formed out of bounded complexes up to homotopy and then localized at the thick subcategory of acyclic complexes. The extra suffix is to denote we consider the category where the objects are the same, but the morphisms are the isomorphisms.

2.2. **Additional descriptions.** In this section we study the virtual category of a Waldhausen category, proving it provides the same construction as that of virtual categories associated to exact categories but extending the construction to the situation where we consider categories with quasi-isomorphisms instead of just isomorphisms. This can be seen as an extension of [23], Definition 4.

Definition 2.0.1. Let $\mathbf{A}$ be a Waldhausen category ([42], Example 1.3.6) with weak equivalences w. By $(\mathbf{A}, w)$ we denote the category having the same objects as $\mathbf{A}$ but the morphisms being weak equivalences. Given a Picard category P, a determinant functor from $\mathbf{A}$ to P is a functor

$$[-] : (\mathbf{A}, w) \to P$$

which satisfies the following constraints:

(a): For any cofibration exact sequence

$$\Sigma : A' \rightarrowtail A \twoheadrightarrow A''$$

an isomorphism $\{\Sigma\} : [A] \to [A'] \oplus [A'']$ functorial with respect to weak equivalences of cofibration sequences.

(b): For any object A, the cofibration sequence $\Sigma : A = A \twoheadrightarrow 0$ decomposes the identity-map as:

$$[A] \stackrel{\{\Sigma\}}{\to} [A] \oplus [0] \stackrel{\delta^R}{\to} [A]$$

where $\delta^R : [A] \oplus [0] \to [A]$ is given by the structure of $[0]$ as a right unit (see Lemma 2.1 for a unicity and existence statement).

(c): Suppose we have a commutative diagram

$$\begin{tikzcd}
\Sigma' : & A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\
\Sigma : & A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\
\Sigma'' : & A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'' \\
& \Sigma_A & \Sigma_B & \Sigma_C
\end{tikzcd}$$

were all the vertical and horizontal lines are cofibration sequences. Then the diagram

$$\begin{tikzcd}
{[B'] \oplus [B'']} \arrow[d, "\{\Sigma'\}\oplus\{\Sigma''\}"] & {[B]} \arrow[l, "\{\Sigma_B\}"'] \arrow[r, "\{\Sigma\}"] & {[A] \oplus [C]} \arrow[d, "\{\Sigma_A\}\oplus\{\Sigma_C\}"] \\
{[A'] \oplus [C'] \oplus [A''] \oplus [C'']} \arrow[rr, "{[C']+[A'']\to[A'']+[C']}"] & & {[A'] \oplus [A''] \oplus [C'] \oplus [C'']}
\end{tikzcd}$$

is commutative whenever also the natural map

$$\mathrm{Pushout}(A \leftarrowtail A' \rightarrowtail B') \to B$$

is a cofibration.

It is furthermore said to be commutative if the following holds:

(d): The triangle

$$\begin{tikzcd}
{[A'] \oplus [A'']} \arrow[rr] \arrow[dr] & & {[A''] \oplus [A']} \arrow[dl] \\
& {[A' \bigcup A'']} &
\end{tikzcd}$$

commutes.

We record the following lemma:

Lemma 2.1. *Suppose* $[] : (\boldsymbol{A}, w) \to P$ *is a determinant functor and* P*. Then for any 0-object of* $\boldsymbol{A}$*,* $[0]$ *has the structure of a unit in* P*, i.e. there are canonical isomorphisms* $\delta^L : [0] \oplus B \simeq B$ *and* $\delta^R : B \oplus [0] \simeq B$*. In particular, there is a canonical isomorphism* $[0] \simeq 0$ *with any unit object 0 of* P*.*

Proof. Applying $[]$ to the cofibration sequence

$$0 \rightarrowtail 0 \twoheadrightarrow 0$$

we obtain an isomorphism $[] : [0] \oplus [0] \simeq [0]$. By [35], 2.2.5.1 $[0]$ has a unique structure of a unit such that $[0] = \delta^R([0]) = \delta^L([0])$. $\square$

We note the following theorem which extends Deligne's categorical description of the virtual category:

Theorem 2.2. *Let* $\boldsymbol{A}$ *be a small Waldhausen category with weak equivalences* w*. Then there is a universal category for determinant functors:*

$$[-] : (\boldsymbol{A}, w) \to V(\boldsymbol{A}).$$

More precisely, for any Picard category P, the category of determinant functors is equivalent to the category of additive functors $V(\boldsymbol{A}) \to P$. Moreover, this category is the fundamental groupoid of the Waldhausen K-theory space of $\boldsymbol{A}$.

Proof. The proof organized as follows. We construct a certain (pointed) bisimplicial set $\mathrm{BiNerve}(P)$, functorial in P, and given a determinant functor $(A, w) \to P$, a commutative diagram

$$\begin{array}{ccc} (A,w) & \longrightarrow & P \\ \downarrow & & \downarrow \\ \pi_f(\Omega|N_\bullet wS_\bullet A|) & \longrightarrow & \pi_f(\Omega|\,\mathrm{BiNerve}(P)|) \end{array}$$

where the lower horizontal map is a canonical map $N_\bullet wS_\bullet A \to \mathrm{BiNerve}(P)$ of bisimplicial set. The right vertical functor is an additive equivalence of Picard categories, whose inverse is unique up to unique natural transformation. We also prove that the left vertical morphism is a determinant functor. This would prove the assertion.

Step 1 - the determinant functor $(A, w) \to \pi_f(\Omega|N_\bullet wS_\bullet A|)$

Recall that the Waldhausen K-theory space is the loop space of the geometric realization of the bisimplicial set $N_\bullet wS_\bullet \mathbf{A}$ where $wS_p\mathbf{A}$ is the category whose objects are, for $0 \leq i \leq j \leq p$, sequences $A_i \rightarrowtail A_j$ of cofibrations with $A_0 = 0$ and with choices of quotients A_j/A_i, and natural compatibility with composition so that $A_i \rightarrowtail A_j \rightarrowtail A_k$ coincides with $A_i \rightarrowtail A_k$ for $i \leq j \leq k$, and whose morphisms between two objects A and A' are given by weak equivalences $A_i \to A'_i$ making all the diagrams commute. $N_p wS_q\mathbf{A}$ is the p-nerve of the category $wS_q\mathbf{A}$. The categories $wS_0\mathbf{A}, wS_1\mathbf{A}, wS_2\mathbf{A}$ are, respectively, the trivial category, the category of objects of $\mathbf{A}$ and weak equivalences as morphisms, and the category of cofibration sequences with weak equivalences of cofibration sequences as morphisms.

The geometric realization in question is the (left-right) realization

$$|q \mapsto |p \mapsto N_p wS_q\mathbf{A}||,$$

which is functorially homeomorphic to the usual diagonal realization (see [32], Lemma, p. 86). Thus we obtain from the above description that the "0-simplices" are reduced to a point and the "1-simplices" in the $S_\bullet$-direction is obtained by adjoining the set

$$|p \mapsto N_p wS_1\mathbf{A}| \times \Delta^1.$$

This defines a canonical map $|wS_1| \wedge S^1 \to |N_\bullet wS_\bullet \mathbf{A}|$, and by adjunction a map $|wS_1| \to \Omega|N_\bullet wS_\bullet\mathbf{A}| = K(\mathbf{A})$. By applying the fundamental groupoid-functor we obtain a functor $[] : \mathbf{A} \to (w^{-1}\mathbf{A}, w) = \Pi_f(|wS_1\mathbf{A}|) \to \Pi_f(K(\mathbf{A}))$, by sending an object to the loop represented by $A \in N_0 wS_1\mathbf{A}$. We verify that this is a determinant functor:

Axiom a: A cofibration sequence

$$\Sigma : A \rightarrowtail B \twoheadrightarrow C$$

defines an element in $N_0 wS_2(\mathbf{A})$, and the face-maps to $N_0 wS_1(\mathbf{A})$ are given by $\partial_0\Sigma = A, \partial_1\Sigma = B, \partial_2\Sigma = C$, thus providing a path from $[B]$ to $[A] + [C]$. A weak equivalence of cofibration sequences defines an element in

$$N_1 wS_2\mathbf{A}$$

whose faces are in $N_1wS_1\mathbf{A}$, which provides the necessary path.
Axiom b: This is just a simplicial identity corresponding to the degeneracy $N_0wS_1(\mathbf{A}) \to N_0wS_2(\mathbf{A}) \to N_0wS_1(\mathbf{A}), A \mapsto [A \to A \to 0] \mapsto A$.

Axiom c: We first show that the commutativity can be rephrased as: if $A \rightarrowtail B \rightarrowtail C$ of cofibrations then

$$\begin{tikzcd} {[C]} \arrow[r] \arrow[d] & {[A]+[C/A]} \arrow[d] \\ {[B]+[C/B]} \arrow[r] & {[A]+[B/A]+[C/A]} \end{tikzcd}$$

commutes. This is clear since

$$\begin{tikzcd} & B \arrow[dr, tail] & \\ A \arrow[ur, tail] \arrow[rr, tail] & & C \end{tikzcd}$$

is a 3-simplex (an object in $N_0wS_3\mathbf{A}$) and provides the necessary relationship between morphisms induced from the 2-simplices in $N_0wS_2\mathbf{A}$ (one also needs to use the commutativity in Axiom d, which is easy). In the general case, as in the corresponding case in [4], Lemme 4.8 we compare the filtrations given by $A' \rightarrowtail B' \rightarrowtail \text{Pushout}(A \leftarrowtail A' \rightarrowtail B') \rightarrowtail B$ and $A' \rightarrowtail A \rightarrowtail \text{Pushout}(A \leftarrowtail A' \rightarrowtail B') \rightarrowtail B$ and we conclude as in loc. cit..
Step 2 - The BiNerve of P.
We need to prove this construction is universal. For this, given a Picard category P, consider the associated bicategory, in fact bigroupoid, ([6]), $\widetilde{P}$. The objects are reduced to a single one, the set of 1-morphisms/morphisms is the set of objects of P, and the set of 2-cells are given by automorphisms $A + B \simeq C$. The nerve of this bicategory is constructed from the following data.

- A_0 is reduced to a single point.
- A_1 is the set of objects of P, with degeneracy maps $A_1 \rightrightarrows A_0$ being trivial. The choice of a zero-object determines a face map $s_0 : A_0 \to A_1$.
- A_2 is the set of $\sigma = (g_0, g_1, g_2, \Sigma)$, where $\Sigma : g_2 + g_1 \simeq g_0$, and comes equipped with three face map, $d_i : A_2 \to A_1$, $d_i(\sigma) = g_i$, and two degeneracy-maps, $s_0(g) = (g, g, 0, 0+g \simeq g)$ and $s_1(g) = (0, g, g, g + 0 \simeq g)$.
- A_3 is the set of $(g_0, g_1, g_2, g_3, g_4, g_5) \in \text{ob}\, P^6$ together with isomorphisms $g_0 + g_1 \overset{f_0}{\simeq} g_2$, $g_5 + g_0 \overset{f_1}{\simeq} g_3$, $g_4 + g_2 \overset{f_2}{\simeq} g_3$, $g_4 + g_1 \overset{f_3}{\simeq} g_5$, such that the induced diagram

$$\begin{tikzcd} & g_3 \arrow[dl] \arrow[dr] & \\ g_4 + g_2 \arrow[d] & & g_5 + g_0 \arrow[d] \\ g_4 + (g_1 + g_0) \arrow[rr, "\text{associativity}"] & & (g_4 + g_1) + g_0 \end{tikzcd}$$

commutes (see [6], section 6.3, pp. 240-242). More concisely it can be written as the set of 2-simplices (f_0, f_1, f_2, f_3) satisfying the above commutativity. The face maps d_i send this 3-simplex to the 2-simplex corresponding to f_i. We also set, for a 2-simplex

σ,

$$s_0(\sigma) = (\sigma, \sigma, s_0(d_1(\sigma)), s_0(d_2(\sigma)))\,,$$
$$s_1(\sigma) = (s_0(d_0(\sigma)), \sigma, \sigma, s_1(d_2(\sigma)))\,,$$
$$s_2(\sigma) = (s_1(d_0(\sigma)), s_1(d_1(\sigma)), \sigma, \sigma)\,.$$

- Nerve(P) is the simplicial complex whose n-simplices A_n are the n-simplices of the coskeleton
$$\mathrm{cosk}^3(A)_\bullet = \varprojlim_{\mathbf{3\geq k\to \bullet}} A_k,$$
where A is the above constructed 3-truncated complex. Further, for $i = 0,1,2,3$ it defines a truncated simplicial category, by defining morphisms to be isomorphisms of all the relevant objects making the diagrams commute, which also equips the coskeleton with the structure of a simplicial category.

- The binerve BiNerve(P) = BiNerve$(P)_{\bullet,\bullet}$ of P is bisimplicial set determined by the nerve of the simplicial category Nerve(P). In particular $N_\bullet A_1$ is just the usual nerve of the Picard category P and $N_k A_2$ is the set of isomorphisms $\Sigma_i : g_1^i + g_2^i \simeq g_3^i$, $i = 0, \ldots k-1$ and "isomorphisms" $\Sigma_i \simeq \Sigma_{i+1}$. The degeneracy and face maps are the obvious ones one would expect from usual nerves. That this actually is a bisimplicial set follows from functoriality of the constructions.

Let now $[] : (\mathcal{A}, w) \to P$ be a determinant functor. We first define a simplicial functor $N_\bullet wS_\bullet A \to \mathrm{BiNerve}(P)_{\bullet,\bullet}$ as follows. The map $wS_\bullet A \to \mathrm{Nerve}_\bullet\, P$ is defined for $i = 0,1,2,3$:

- For $i = 0$, the map sends the single object to the single object of A_0.

- For $i = 1$, it sends $0 \rightarrowtail A$ to $[A] \in A_1$.

- For $i = 2$, it sends $0 \rightarrowtail A_1 \rightarrowtail A_2$ to the 2-simplex $([A_2/A_1], [A_1], [A_2], [A_1] + [A_2/A_1] \simeq [A_2])$.

- For $i = 3$, it sends $0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail A_3$ to the 3-simplex, written in short form,

$$([A_2/A_1]+[A_3/A_2] \simeq [A_3/A_1], [A_2]+[A_3/A_2] \simeq [A_3], [A_1]+[A_3/A_1] \simeq [A_3], [A_1]+[A_2/A_1] \simeq [A_2]).$$

This is really a 3-cell due to compatibility with filtrations for determinant functors. The induced simplicial functor on the coskeleton then induces a map of nerves $N_\bullet wS_\bullet A \to \mathrm{BiNerv}_{\bullet,\bullet}\, P$.

There is an induced simplicial map of nerves and coskeletons, $N_\bullet wS_\bullet A \to \mathrm{BiNerv}(P)_{\bullet,\bullet}$.
Step 3. The homotopy type of the binerve
We now claim that the loop space of the bisimplicial set BiNerv$(P)_{\bullet,\bullet}$ is naturally homotopic to the realization of the nerve of the groupoid P. The "1-skeleton" in the $A_\bullet$-direction is the point $|A_0$ with $|A_1| \times |\Delta^1|$ adjoined, whose natural map to $|\mathrm{BiNerv}(P)_{\bullet,\bullet}|$ factors over $|A_1| \wedge S^1$. By adjointness we obtain a natural map $|A_1| \to \Omega|\mathrm{BiNerv}(P)_{\bullet,\bullet}|$. But the left realization of BiNerv$(P)_{\bullet,\bullet}$ is the natural simplicial space $\Delta^{op} \to |A_\bullet|$ and $|\mathrm{BiNerv}(P)_{\bullet,\bullet}|$ is the realization of this space (up to functorial homeomorphism). The maps s_i in this simplicial space are injective maps (in view of having a right inverse d_i of CW-complexes), and are thus cofibrations. In particular the space $\Delta^{op} \to |A_i|$ is "good" in the sense of Segal and the natural

map $|A_1| \to \Omega|\operatorname{BiNerv}(P)_{\bullet,\bullet}|$ is a homotopy equivalence, if we can show that the natural map $|A_i| \to |A_1|^i$ is a homotopy equivalence. By Whitehead's theorem we need to verify that the map induces isomorphisms on homotopy groups. The cases $i = 0, 1$ being trivial, we treat first $i = 2$ and $i = 3$. The nerve $N_\bullet A_i$ being fibrant, we can compute the homotopy groups simplicially. The group $\pi_0(N_\bullet A_2)$ is represented by pairs $([A], [B - A]) \in \pi_0(A_1)^2$, and $\pi_1(N_\bullet A_2)$ is represented by isomorphisms of the objects $A, B - A$, i.e. two elements of $\pi_1(A_1)^2$. The map on the homotopy groups is the one coming from this representation and induces an isomorphism. Since $|A_2|$ is the realization of the nerve of the groupoid A_2, there are no higher homotopy groups. The same argument applies to $|A_3|$. In the general case, $|A_n|$, $n \geq 4$, is constructed of those "commutative n-simplices" constructed out the simplices of $i = 0, 1, 2, 3$. An arrow-chasing shows that we have the necessary isomorphisms on homotopy-groups.
We leave to the reader to verify that the map $|A_1| \to \Omega|\operatorname{BiNerv}(P)_{\bullet,\bullet}|$ gives rise to an additive functor after applying fundamental groupoids. This is so for the same reasons that the map $(A, w) \to \Pi_f(|wS_1\mathbf{A}|) \to \Pi_f(K(\mathbf{A}))$ was a determinant functor.
Step 4. We have constructed a commutative diagram, associated to a determinant functor $(A, w) \to P$:

$$\begin{tikzcd} |wS_1A| \arrow[r] \arrow[d] & |P| \arrow[d] \\ \Omega|N_\bullet wS_\bullet A| \arrow[r] & \Omega|\operatorname{BiNerve}(P)| \end{tikzcd}$$

and applying fundamental groupoids we obtain a commutative diagram of functors.

$$\begin{tikzcd} (A, w) \arrow[r] \arrow[d] & P \arrow[d] \\ \Pi_f(|wS_1A|) \arrow[r] \arrow[d] & \Pi_f(|P|) \arrow[d] \\ \Pi_f(\Omega|N_\bullet wS_\bullet A|) \arrow[r] & \Pi_f(\Omega|\operatorname{BiNerve}(P)|) \end{tikzcd}$$

which is moreover functorial in P and natural transformations of additive functors. Applying fundamental groupoids we obtain a commutative diagram of Picard categories. The equivalence of categories $P = \Pi_f(|P|) \to \Pi_f(\Omega|\operatorname{BiNerve}(P)|)$ admits a inverse, unique up to unique isomorphism. The functoriality of the above constructions shows that $\operatorname{Hom}_{\text{add}}(\Omega|N_\bullet wS_\bullet A|, P) \to \operatorname{Hom}_{\det}((A, w), P)$ is an equivalence of categories.

$\square$

Notice that the argument "basically" considers the functor $wS_kA \to P^k$, mapping $A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_k \mapsto ([A_0], [A_1/A_0], \ldots, [A_k/A_{k-1}])$ and equipping $\prod_1^k P$ with the bar simplicial resolution. This works well if P is actually a group (see the proof of the cofinality theorem 1.10.1 in [42] for this statement, on which the above theorem was also modeled), but since the associator and the commutator are not identities this does not give a map of simplicial sets and only defines a pseudo-functor of pseudo-simplicial categories. The above argument replaces concretely the almost simplicial category $\prod_1^\bullet P$ with an actual one, which is still homotopic term-wise to $\prod_1^\bullet P$. I do not know if the above argument could have been replaced by a Grothendieck rectification argument. I'm grateful to the referee for pointing out that $\prod_1^\bullet P$ actually is not a simplicial category.

Definition 2.2.1. Given a determinant (additive) functor $F : P \to P'$ of Picard categories, we can define [45], the Kernel and Cokernel of F. We denote them by $\operatorname{Ker} F$ and $\operatorname{Coker} F$ respectively. If $F : P \to P'$ is an additive functor, we can define the cokernel of F, denoted by P'/P as the category whose objects are the same as that of P', and

$$\begin{aligned}\operatorname{Hom}_{P'/P}(A, A') \quad &= \quad \{B, B' \in \operatorname{ob} P', f \in \operatorname{Hom}_{P'}(A + F(B), A' + F(B')) \\ &\qquad \text{s.th. } f \sim f', \text{ if } \exists C, C' \in \operatorname{ob}(P'), f - f' \in F\operatorname{Hom}_P(C, C')\}.\end{aligned}$$

In general the cokernel of F is the cokernel of the inclusion of the image in the target category. For the kernel: if $F : P \to P'$ is an additive functor we define $\ker F$ as the category whose objects are objects of P plus an isomorphism $F(A) \simeq 0$ in P'. An isomorphism of two objects are isomorphisms P respecting the isomorphisms in P'. All these categories have natural structures of Picard categories and the induced functors are additive functors. A more functorial point of view is taken in [45], and we refer the reader there for such a definition.

Remark 2.2.1. Since any exact category can be equipped with the structure of a biWaldhausen category, where the weak equivalences are the isomorphisms and the cofibrations are the admissible monomorphisms, it is clear that the above definition generalizes that of Deligne [4], 4.3. It is a simple exercise to verify that in this case the above axioms for determinant functors are equivalent to those given in loc.cit.

We also have the following stronger assertion:

Proposition 2.3 ([46] Theorem 1.9)**.** *The Waldhausen K-theory spectrum of an exact category $\mathcal{E}$, $K(\mathcal{E})$, is naturally homotopy-equivalent to the K-theory spectrum of Quillen. A fortiori it induces an equivalence of fundamental groupoids and Picard categories.*

Proposition 2.4 ([42], 1.9.6)**.** *Suppose in addition that $\boldsymbol{A}$ is complicial biWaldhausen so that it is a full subcategory of $C(\mathcal{A})$ for an abelian an category $\mathcal{A}$. Furthermore suppose that it is closed under taking exact sequences in $C(\mathcal{A})$, is closed under finite degree shifts and co$(\boldsymbol{A})$. Then $Ho(\boldsymbol{A}) = w^{-1}\boldsymbol{A}$ is a triangulated category and admits a calculus of fractions.*

Remark 2.4.1. By [42], Theorem 1.9.2, we can suppose that cofibrations are the degree-wise admissible monomorphisms with quotients in $\mathbf{A}$.

Corollary 2.5. *[Knudsen, [22]] Let $i : \mathcal{E} \to \mathcal{A}$ be an exact fully faithful embedding of an exact category $\mathcal{E}$ in an abelian category $\mathcal{A}$, such that for any morphism in $\mathcal{E}$ which is an epimorphism in $\mathcal{A}$, is admissible in $\mathcal{E}$. Denote by $C(\mathcal{E})$ the full subcategory of bounded complexes of the category of complexes in $\mathcal{A}$. Then we have a natural equivalence of categories between the virtual category of $(\mathcal{E}, is)$ and the virtual category of $(C(\mathcal{E}), q.i.)$ of complexes in E with quasi-isomorphisms in $\mathcal{A}$.*

Proof. Equip the category $C(\mathcal{E})$ with the structure of a complicial biWaldhausen category where the weak equivalences are given by quasi-isomorphisms and the cofibrations are either of the two following: degree-wise admissible monomorphisms or degree-wise split monomorphisms whose quotients lie in $C(\mathcal{E})$. Denote the corresponding biWaldhausen categories by $\mathbf{E}$ and $\widetilde{\mathbf{E}}$. By [42], Theorem 1.11.7, we have natural homotopy-equivalences

$$K(\mathcal{E}) \simeq K(\mathbf{E}) \simeq K(\widetilde{\mathbf{E}})$$

and hence equivalent virtual categories. Moreover, this does not depend on the choice of $\mathcal{A}$. $\square$

If $i : \mathcal{E} \to \mathcal{A}$ is the fully faithful Gabriel-Quillen embedding reflecting exactness (see [42], Appendix A), or if $\mathcal{E}$ is the category of coherent vector bundles and $\mathcal{A}$ is the category of coherent sheaves respectively on a scheme, i satisfies the above hypothesis.

3. Some operations on virtual categories

In this section we include some standard operations on the virtual category. Of special interest is a splitting principle and the construction of Adams- and λ-operations. As an application we deduce a relationship between the weight filtration and the Adams filtration on a regular scheme for K_1.

3.1. **A splitting principle.** Below we sketch a criterion for when we can descend a morphism on the level of the complete flag-variety to the base [1]. First, let E be an vector-bundle of rank $e+1$ on a separated algebraic stack $\mathcal{X}$. Then $p^1 : Y_1 = \mathbb{P}(E) \to \mathcal{X}$ is a projective bundle which on which we have a canonical sub-line bundle $\mathcal{O}(-1)$, and a canonical quotient-bundle of ${p^1}^*E$. Repeating this construction with the quotient-bundle, we eventually obtain a map $p : \mathcal{Y} = \mathcal{Y}_e \to \mathcal{Y}_{e-1} \to \ldots \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{X}$, where the top space is the complete flag-variety of E on $\mathcal{X}$, which also comes equipped with a canonical complete flag. Suppose P is a contravariant functor from the category of separated algebraic stacks to the category of Picard categories such that for any $\mathcal{X}$ there is a distributive additive functor $V(-) \times P(-) \stackrel{\otimes}{\to} P(-)$ moreover satisfying the projective bundle axiom; for any $\mathcal{X}$, the functor

$$\times_{i=0}^{e} P(\mathcal{X}) \to P(\mathbb{P}_{\mathcal{X}}(E))$$

given by $(f_i)_{i=0}^e \mapsto \sum_{i=0}^e {p^1}^* f_i \otimes \mathcal{O}(-i)$ is an equivalence of categories. Then the following is a version of an observation of Franke in terms of Chow categories of ordinary schemes (see the article by J. Franke, "Chern Functors" in [11], 1.13.2):

Theorem 3.1. *[Splitting principle] Let $p_1, p_2 : \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ be the two projections, and $r = pp_1 = pp_2$. Then*
(a) $p^* : P(\mathcal{X}) \to P(\mathcal{Y})$ is faithful.
(b) Suppose we have two objects $A, B \in \operatorname{ob} P(X)$, and $f : p^*A \to p^*B$ a morphism in $\operatorname{Hom}_{P(\mathcal{Y})}(p^*A, p^*B)$, then f comes from a (unique) morphism $h : A \to B$ if and only if $p_1^*(f) = p_2^*(f)$ in $\operatorname{Hom}_{P(\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y})}(r^*A, r^*B)$.

Proof. From the projective bundle axiom it follows each ${p^i}^*$ is injective on the level of automorphism-groups, *i.e.* for any object A in $P(\mathcal{Y}_i)$, $\operatorname{Aut}_{P(\mathcal{Y}_i)}(A) \to \operatorname{Aut}_{\mathcal{Y}_{i+1}}({p^i}^*A)$ is injective, so the functor is faithful. For (b), the condition is obviously necessary. To prove that the condition is sufficient we can assume $A = B$. Let $0 = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_e = p^*E$ be the universal flag on $\mathcal{Y}$, and $L_i = E_i/E_{i-1}$, then by the projective bundle axiom we have natural isomorphisms

$$\operatorname{Aut}_{P(\mathcal{Y})}(A) = \bigoplus_{j_1=0}^{e} \ldots \bigoplus_{j_e=0}^{1} L_1^{j_1} \otimes \ldots \otimes L_e^{j_e} \otimes p^* \operatorname{Aut}_{P(\mathcal{X})}(A)$$

[1] recall that a flag is a sequence of sub-vector bundles $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n$ whose successive quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are also vector bundles. It is furthermore complete if each such quotient is a line bundle.

and

$$\mathrm{Aut}_{P(\mathcal{Y}\times_{\mathcal{X}}\mathcal{Y})}(A) = \bigoplus_{j_1,j_1'=0}^{e} \ldots \bigoplus_{j_e=0,j_e'=0}^{1} p_1^* L_1^{j_1} \otimes p_2^* L_1^{j_1'} \otimes \ldots \otimes p_1^* L_e^{j_e} \otimes p_2^* L_e^{j_e'} \otimes r^* \mathrm{Aut}_{P(\mathcal{X})}(A).$$

Representing f in the form suggested above, we see that $p_1^*(f) = p_2^*(f)$ exactly when all components of f are zero except for the one belonging to $(j_1, \ldots, j_e) = (0, \ldots, 0)$, which means exactly that f is equal to p^*h for some morphism $h : A \to A$. Moreover h is unique because of (a). □

By basechange to the flag-variety we can suppose we have nice enough flags. If we define an isomorphism dependent on this flag, the content of the proposition is that this descends to the base whenever this isomorphism isn't dependent on the flag.

3.2. **Adams and λ-operations on the virtual category.** Let S be a scheme, and $\mathcal{X}$ an algebraic stack over S. Recall that we denote by $\mathbf{P}(\mathcal{X})$ the category of vector bundles on $\mathcal{X}$. Denote by $V(\mathcal{X})$ the virtual category thereof. We have the following result which is an adaption of the main result in [17] to our situation;

Proposition 3.2. *There is a unique family of determinant functors $\Psi^k : \mathbf{P}(\mathcal{X}) \to V(\mathcal{X})$, and thus $\Psi^k : V(\mathcal{X}) \to V(\mathcal{X})$, stable under pullback, such that if L is a line bundle, $\Psi^k(L) = L^{\otimes k}$.*

Proof. Unicity of the operations clearly follows from the characterizing properties and the splitting principle (Theorem 3.1). To prove existence, we apply the ideas of loc.cit.. Let N be a complex of vector bundles, and CN be the cone of the identity morphism $\mathrm{id} : N \to N$. Furthermore, let S^k be the k-th symmetric power, so that the p-th term of S^kCN is $S^{k-p}N \otimes \wedge^p N$, whenever N is reduced to a vector bundle in degree 0 (for details, see loc.cit., p. 4). Finally, for a bounded complex $N_\bullet = [\ldots \to N_{i-1} \to N_i \to N_{i+1} \to \ldots]$, define the secondary Euler characteristic $\chi'(N_\bullet) = \sum (-1)^{p+1} p[N_p] \in V(\mathcal{X})$. One of the key ideas of loc.cit. (formula (3.1)) is the formula in $K_0(\mathcal{X})$, for a vector bundle E,

$$\Psi^k(E) = \chi'(S^kCE).$$

We propose the same definition for Adams operations in the virtual category $V(\mathcal{X})$. Clearly $\Psi^k(L) = L^{\otimes k}$ for a line bundle L. Now, given a flag $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n$, define $E_1 \cdot E_2 \ldots \cdot E_n$ to be the image of $E_1 \otimes E_2 \otimes \ldots \otimes E_n$ in $S^n E_n$. Suppose that we have an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, and consider the filtration

$$\begin{aligned} S^kCE' &= CE'.CE'.\ldots.CE'.CE' \\ &\subseteq CE'.CE'.\ldots.CE'.CE \\ &\subseteq CE'.CE'.\ldots.CE.CE \subseteq \ldots \\ &\subseteq CE.CE.\ldots.CE.CE = S^kCE \end{aligned}$$

induces by a certain addivity of the secondary Euler characteristic, isomorphisms

$$\begin{aligned} \chi'(S^kCE) &= \chi'(S^kCE'') + \chi'(S^kCE') + \sum_{i=0}^{k-1} \chi'(S^iCE'' \otimes S^{k-i}CE') \\ &= \chi'(S^kCE'') + \chi'(S^kCE') \end{aligned}$$

since the secondary Euler-characteristic of a product of acyclic complexes is 0 and by the multilinearity-property of loc.cit (formula (2.1)). We need only verify that this operation respects filtrations. Let $F \subseteq H \subseteq E$ be an admissible filtration, and consider the double graded filtrations of S^kCE, $A_{\bullet,\bullet}$, where $A_{i,j} = S^{k-i-j}CE.S^jCF.S^iCH$. Applying secondary Euler characteristics in every direction, we obtain that the diagram of isomorphisms

$$\begin{tikzcd} \chi'(S^kCE) \arrow[r] \arrow[d] & \chi'(S^kCH) + \chi'(S^kCE/H) \arrow[d] \\ \chi'(S^kCF) + \chi'(S^kCE/F) \arrow[r] & \chi'(S^kCF) + \chi'(S^kCH/F) + \chi'(S^kCE/H) \end{tikzcd}$$

constructed above commutes. We leave as an exercise to provide the isomorphism, a simple manipulation of symbols, $\chi'(A \otimes B) = \chi(A) \otimes \chi'(B) + \chi'(A) \otimes \chi(B)$. Condition "b)" of Definition 2.0.1 is trivial. Also everything is clearly stable under pullback. $\square$

Remark 3.2.1. In the next chapter we will show that whenever we restrict ourselves to regular schemes, the constructed Adams-operations are actually unique liftings of the usual Adams-operations on the level of K_0, at least whenever one inverts 2 or more primes in the virtual category.

We record the following corollary (of the splitting principle applied to the above case):

Corollary 3.3. *There is a mutliplicativity isomorphism*

$$\Psi^k \circ \Psi^{k'} \simeq \Psi^{kk'}$$

and $\Psi^k : V(\mathcal{X}) \to V(\mathcal{X})$ is a ring-homomorphism in the sense that there are natural isomorphisms, for $A, B \in \mathrm{obV(X)}$,

$$\Psi^k(A \otimes B) \simeq \Psi^k(A) \otimes \Psi^k(B)$$

compatible with the above sum-operation and compatible with basechange.

Proof. The first point follows by unicity and the splitting principle. For the second, we only need to verify the multiplicative operation. It suffices to show that for any $A \in V(\mathcal{X}), B \in \mathbf{P}(\mathcal{X})$, $\Psi^k(A \otimes B) = \Psi^k(A) \otimes \Psi^k(B)$ naturally. Or, by the splitting principle since Ψ^k is already an additive determinant functor, that if B is a line bundle, then $\Psi^k(A \otimes L) = \Psi^k(A) \otimes L^{\otimes k}$ naturally. For this we can assume that A is also a line-bundle M, in which case we have $\Psi^k(M \otimes L) = (M \otimes L)^{\otimes k} = M^{\otimes k} \otimes L^{\otimes k} = \Psi^k(M) \otimes \Psi^k(L)$. $\square$

Now define $V^{\mathcal{Z}}(\mathcal{X})$ to be the virtual category associated to the category of complexes on $\mathcal{X}$ exact off $\mathcal{Z}$. Then the method of [15] provides us with λ-operations:

Proposition 3.4. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{Z}$ a closed substack thereof and let k be a positive integer. There are naturally defined functors $\lambda^k : V^{\mathcal{Z}}(\mathcal{X}) \to V^{\mathcal{Z}}(\mathcal{X})$ satisfying the following compatibilities:*

(a) $\lambda^1 = \mathrm{id}$.
(b) *They are stable under basechange.*
(c) *If $\mathcal{Z} = \mathcal{X}$ and E is a vector bundle, $\lambda^k E = \wedge^k E$.*
(d) *If*

$$0 \to E'_\bullet \to E_\bullet \to E''_\bullet \to 0$$

is a short exact sequence of complexes of vector bundles exact off $\mathcal{Z}$ there is a canonical isomorphism, where we set $\lambda^0 = 1$:

$$\lambda^k(E_\bullet) = \sum_{i+j=n} \lambda^i E'_\bullet \otimes \lambda^j E''_\bullet.$$

Proof. The method of *loc. cit.* provides us with a functor, via the Dold-Puppe construction, a functor $\lambda^k : \mathbf{P}_\mathcal{Z}(\mathcal{X}) \to \mathbf{P}_\mathcal{Z}(\mathcal{X})$ where $\mathbf{P}_\mathcal{Z}(\mathcal{X})$ denotes the category of complexes of vector bundles on $\mathcal{X}$ exact off $\mathcal{Z}$. Thus a functor $\lambda^k : (\mathbf{P}_\mathcal{Z}(\mathcal{X}), q.i.) \to V^\mathcal{Z}(\mathcal{X})$. Writing $V^\mathcal{Z}(\mathcal{X})[[t]] := \coprod_{k\geq 0} V^\mathcal{Z}(\mathcal{X})t^k$ it is naturally a Picard category with respect to multiplication with unit element $1 \otimes t^0 + \sum 0 \otimes t^k$. The usual proof shows that $\lambda_t = \sum_{k\geq 0} \lambda^k t^k$ is a determinant functor $\lambda_t : (\mathbf{P}_\mathcal{Z}(\mathcal{X}), q.i.) \to V^\mathcal{Z}(\mathcal{X})[[t]]$ and hence the searched for functor. Notice that we use Theorem 2.2 since the category in question is not an exact category with isomorphisms but a Waldhausen category with quasi-isomorphisms. □

Corollary 3.5. *Suppose that $Z = X$. Then, possibly up to sign, there is a canonical isomorphism*

$$\Psi^k = \lambda^1 \otimes \Psi^{k-1} - \lambda^2 \otimes \Psi^{k-2} + \ldots + (-1)^k \lambda^{k-1} \otimes \Psi^1 + (-1)^{k+1} k\lambda^k$$

where Ψ^* and λ^* denotes the functors constructed in Proposition 3.2 and Proposition 3.4.

Proof. The left hand side is already an endofunctor on the virtual category of vector bundles. We need to verify that the right hand side is additive since by the splitting principle we can then reduce to the case of line bundles, for which the statement is clear. This follows by induction on k and the property (d) in the above proposition. □

This also defines inductively Adams operations via λ operations (cf. Proposition 3.7 below) to virtual categories with support. The following Corollary is a consequence of the calculation on the level of complexes in *loc. cit.* :

Corollary 3.6. *Let R be a ring with a a non-zero divisor of R. Denote by $K(a)$ the complex $[R \xrightarrow{a} R]$ with R placed in degree 0 and 1 and $X = \operatorname{Spec} R, Y = \operatorname{Spec} R/a$. Then there is a canonical isomorphism $\Psi^k(K(a)) \simeq kK(a)$ in $V^Y(X)$.*

Proposition 3.7. *Suppose that X is regular. The Adams operations uniquely extend via Corollary 3.5 to operations on virtual categories with support such that they are stable under pullback and compatible with the functor $i_* : V^Y(X) \to V(X)$ for a scheme X with $i : Y \hookrightarrow X$ a closed subscheme.*

Proof. By the general arguments of [21], any complex $E_\bullet$ on a scheme X which is acyclic outside of a closed subscheme Y pulls back from a universal complex $C_\bullet$ on a classifying-type scheme $\pi : G \to X$ acyclic outside of G_Y, such that the support of $C_\bullet$ maps to the support of $E_\bullet$. Moreover, this scheme has the property that the induced maps $\pi_i(V^{G_Y}(G)) \to \pi_i(V(G))$ are injective, using the localization exact sequence for K-theory in [42] and homotopy invariance for regular schemes. By construction there is an equivalence of functors $i_*\lambda_t(\mathcal{E}_\bullet) \to \lambda_t(i_*\mathcal{E}_\bullet)$, where $E_\bullet$ is any complex of vector bundles on G with support on G_Y and λ_t is the functor in the proof of Proposition 3.4. Restricting this isomorphism to $G \setminus G_Y$ gives both sides canonically isomorphic to zero compatible with the identity map of the zero-object. By the exact sequence $0 \to \pi_1(V^{G_Y}(G)) \to \pi_1(V(G)) \to \pi_1(V(G \setminus G_Y)) \to 0$ this restriction comes from a canonical element in $\pi_1(V^{G_Y}(G))$. This argument proves that any functor λ_t on $V^Y(X)$ compatible with pullback and Φ^X_Y is unique up to unique isomorphism. □

4. Rigidity and operations on virtual categories

In this section we exhibit certain rigidity-properties of the virtual categories we are working on, and also the main technical results of this part of the article. As such, it rests heavily on the results obtained in [29], [34] (largely published in [33]) and [30]. The formulation in terms of K-cohomology was inspired from [43], and seems interesting if one wants to obtain functorial Lefschetz-Riemann-Roch formulas.

4.1. **The main result on rigidity.** The main result of this section (Theorem 4.5) can be phrased, in a certain situation, that there is a certain commutative diagram

$$\begin{tikzcd} \hom_{\mathcal{H}(\mathfrak{R})}(\mathcal{K}_{\mathbb{Q}}, \mathcal{K}_{\mathbb{Q}}) \arrow[rr, equal] \arrow[dr] & & \hom_{\mathfrak{R}^{op}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}}) \arrow[dl, bend right] \\ & \hom(V_{\mathbb{Q}}, V_{\mathbb{Q}}) \arrow[ur, bend right] & \end{tikzcd} .$$

Here $\hom_{\mathfrak{R}^{op}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}})$ is a set of natural endo-transformations of the presheaf $K_0(-)$ on the category of regular schemes, and $\hom(V_{\mathbb{Q}}, V_{\mathbb{Q}})$ is the set of endo-functors of the virtual category of algebraic vector bundles strictly stable under pullback. Finally, $\mathcal{K}_{\mathbb{Q}}$ is a simplicial sheaf representing (rational) algebraic K-theory. This allows us to associate functorial operations on $V_{\mathbb{Q}}$ via the corresponding operations on K_0. We refer to the theorem for a precise formulation.

Let X be a separated regular Noetherian scheme of finite Krull dimension d. Then it is well known (see for example [13], chapter V, Corollary 3.10, [19], chapter VI, Thorme 6.9 or use [42], Theorem 7.6 and (10.3.2)) then any element x of $K_0(X)$ of virtual rank 0 is nilpotent, and moreover we have $x^{d+1} = 0$. One can prove this in several ways, but one of the most natural ways is to construct a certain filtration on $K_0(X)$ which can be compared to other filtrations in terms of dimension of supports, a filtration that will terminate for natural reasons (see loc.cit.). One such filtration is the γ-filtration F_γ^p, built out of the λ-ring structure on $K_0(X)$ (see [13], chapter III, p. 48 or [19], chapter V, 3.10). We wish to incarnate this kind of nilpotence in the virtual category of X. Obviously, if x is a virtual vector-bundle of rank 0, then we know that a high enough power of it is isomorphic to a zero-object, but only non-canonically. A naive idea is to search for a decreasing filtration Fil^i of $V(X)$ which has the property that the functors $\mathrm{Fil}^i \to \mathrm{Fil}^{i-1}$ are faithful additive functors, and for big enough p, Fil^p is a category with exactly one morphism between any two objects.

The approach we have chosen to the problem is to construct the filtration already on the level of classifying spaces of the $\mathbb{P}^1$-spectrum representing rational algebraic K-theory in $\mathcal{SH}(S)$, and then use simplicial realizations to obtain a canonical filtration of $BQ\mathbf{P}(X)$ which eventually becomes trivial. For the notation used in this section we refer the reader to the Appendix A. It should however be noted that we can still introduce some of the main results without reference to $\mathbb{A}^1$-homotopy theory, but the formulation seems to suit us because of the strong assertions it makes on existence of lifting of functors.

In [18] the author proposes that there should be a multiplicative filtration W^i of a space $K(X)$ representing the K-theory of X;

$$\ldots \to W^2 \to W^1 \to W^0 = K(X)$$

such that the two following properties are satisfied:

(a) For any t, the quotient W^i/W^{i+1} is the simplicial realization of a simplicial abelian group.
(b) The Adams operations Ψ^k act by multiplication by k^i on W^i/W^{i+1}.

Such a filtration would immediately give an exact couple and thus give rise to an Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

relating "motivic cohomology" (that is, cohomology of $\mathbb{Z}(i) = \mathbb{Z}(i)^W := \Omega^{2i}(W^i/W^{i+1})$, in the sense of spectra with negative homotopygroups) on the left with algebraic K-theory on the right. In *loc. cit.* it is noted that the Postnikoff filtration satisfies the first but not the second property. For smooth schemes over a field [28] constructed a coniveau-filtration which gives the correct spectral sequence for smooth varieties over a field.
The starting point of this section is the following theorem, which states that if we tensor with $\mathbb{Q}$ we can construct a Grayson-like filtration with various functorial properties. The author ignores if the filtration of [28] coincides with the one considered in this section, both considered as objects of the appropriate homotopy category of schemes.

Theorem 4.1. *There are H-groups $\{\mathrm{Fil}^{(i)}\}_{i=0}^{\infty}$ (i.e. group-objects) and $\{\mathbb{H}^{(i)}\}_{i=0}^{\infty}$ of $\mathcal{H}(\mathfrak{R}_S)_\bullet$, determined up to unique isomorphism, satisfying the following properties:*

(a) $\mathrm{Fil}^0 = (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ *and for any $i \geq 0$, there are morphisms* $\mathrm{Fil}^{(i+1)} \to \mathrm{Fil}^{(i)}$.
(b) *For any i, j, there are natural pairings* $\mathrm{Fil}^{(i)} \wedge \mathrm{Fil}^{(j)} \to \mathrm{Fil}^{(i+j)}$ *making, for $i' \leq i, j' \leq j$, the following diagram commutes*

$$\begin{tikzcd} \mathrm{Fil}^{(i)} \wedge \mathrm{Fil}^{(j)} \arrow[r] \arrow[d] & \mathrm{Fil}^{(i+j)} \arrow[d] \\ \mathrm{Fil}^{(i')} \wedge \mathrm{Fil}^{(j')} \arrow[r] & \mathrm{Fil}^{(i'+j')} \end{tikzcd} .$$

(c) *For any i, j, there are natural pairings* $\mathbb{H}^{(i)} \wedge \mathbb{H}^{(j)} \to \mathbb{H}^{(i+j)}$.
(d) *There is a factorization* $\mathrm{Fil}^{(i+1)} \to \mathrm{Fil}^{(i+1)} \times \mathbb{H}^{(i)} \overset{\Phi_i}{\approx} \mathrm{Fil}^{(i)}$ *which is compatible with the two above products. The pairings are also associative in the obvious sense.*
(e) *The Adams operations Ψ^k act on all the above objects and morphisms and acts purely by multiplication by k^i on $\mathbb{H}^{(i)}$.*

Proof. It follows from Theorem A.8 that we have a filtration of $\mathbf{BGL}_{\mathbb{Q}}$ in $\mathcal{SH}(\mathfrak{R}_S)$ given by $\mathrm{Fil}^p = \bigoplus_{n \geq p} \mathbb{H}^n$. By definition there is an evaluation-functor $\mathrm{ev}_n : \mathcal{SH}(\mathfrak{R}_S) \to \mathcal{H}(\mathfrak{R}_S)_\bullet$ sending a spectra (cf. Appendix A) $\mathbf{E}$ to $\mathbf{E}_n$. Evaluating at 0 we obtain a canonical filtration of $\mathrm{ev}_0(\mathbf{BGL}_{\mathbb{Q}}) \simeq (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$, a filtration $\{\mathrm{Fil}^{(i)}\}_{i=0}^{\infty}$ in $\mathcal{H}(\mathfrak{R}_S)_\bullet$. We similarly define $\mathbb{H}^{(i)} = \mathrm{ev}_0(\mathbb{H}^i)$ so that $\mathrm{Fil}^{(i)} = \mathbb{H}^{(i)} \oplus \mathrm{Fil}^{(i-1)}$. They are the 0-th space of a $\mathbb{P}^1$-spectrum and automatically H-groups. We similarly define Adams operations Ψ^k on the various objects via the same functor ev_0. We need to verify the other claimed properties.

Let $\mathcal{X}$ be a pointed simplicial sheaf, and define $\Omega^j X = \underline{Hom}_{\Delta^{op}\,\mathrm{Shv}_\bullet(\mathfrak{R}_S)}(S^j, \mathcal{X})$, the right adjoint to $S^j \wedge -$. Also denote by $R\Omega^j$ the total derived functor of Ω^j in $\mathcal{H}(\mathfrak{R}_S)_\bullet$.

Denote by $\mathfrak{R}_S^c$ the category whose objects are open inclusions of regular schemes $j : U \to X$. Recall that the Yoneda functor Φ is defined by

$$\Phi : \mathfrak{R}_S \to \Delta^{op} \operatorname{Shv}(\mathfrak{R}_{S,\mathrm{sm}}) \to \mathcal{H}(\mathfrak{R}_S)$$

and for any object $G \in \mathcal{H}(\mathfrak{R}_S)$ we denote by $\phi(G)$ the presheaf

$$\mathfrak{R}_S \ni U \mapsto \operatorname{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\Phi U, G)$$

as follows. We define the quotient $\Phi(U \to X) := \Phi X / \Phi(X - U)$ in $\mathcal{H}(\mathfrak{R}_S)$ and we set $\phi(G)(U \to X) = \operatorname{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\Phi(U \to X), G)$. The following follows from the general theory in [42]:

Proposition 4.2. *Let $U \to X$ be an open inclusion of regular schemes. The homotopy fiber of*

$$\underline{Hom}_{\Delta^{op}\operatorname{Shv}_\bullet(\mathfrak{R}_S)}(X, \mathbb{Z} \times \mathrm{Gr}) \to \underline{Hom}_{\Delta^{op}\operatorname{Shv}_\bullet(\mathfrak{R}_S)}(U, \mathbb{Z} \times \mathrm{Gr})$$

identifies with the K-theory space of finite complexes of vector bundles on X exact on U.

We then have:

Lemma 4.3. *Let $j \geq 0$. We have the following natural isomorphisms of presheaves on $\mathfrak{R}_S^c$, where we set $Z := X \setminus U$, in the following cases:*

- $\phi(R\Omega^j(\mathbb{Z} \times \mathrm{Gr}))(U \to X) = K_j^Z(X)$. *This also holds for localizations by integers n and $\mathbb{Q}$.*
- $\phi(R\Omega^j \mathbb{H}^{(i)})(U \to X) = K_j^Z(X)^{(i)}$, *the presheaf of sections of $(U \to X) \mapsto K_j^Z(X)_{\mathbb{Q}}$ with Ψ^k-eigenvalue k^i (which is independent of $k \geq 2$).*
- $\phi(\mathrm{Fil}^{(i)})(U \to X) = F^i K_0^Z(X)_{\mathbb{Q}} = \bigoplus_{p \geq i} K_0^Z(X)^{(p)}$, *where*

  $$F^i K_0^Z(X)_{\mathbb{Q}} = \bigcup_{Y \subset X} \operatorname{im}[K_0^{Z \cap Y}(X)_{\mathbb{Q}} \to K_0^Z(X)_{\mathbb{Q}}]$$

  and the union is over all closed subschemes $Z \subset X$ of codimension at least i.
- *Suppose $U = \emptyset$. Let $\mathbb{P}^\infty = \operatorname{colim}_n \mathbb{P}^n$, then $\phi(\mathbb{P}^\infty)(-) = \operatorname{Pic}(-)$, $\phi(R\Omega\mathbb{P}^\infty) = \mathbb{G}_m$ and $\phi(R\Omega^j \mathbb{P}^\infty) = 0$ otherwise.*

Proof. In view of how the Adams-operations act on the various objects involved, using Theorem A.5 the first non-trivial part is the equality

$$\bigoplus_{p \geq i} K_0(-)^{(p)} = F^i K_0(-)_{\mathbb{Q}}$$

which is [37], chapter I, Lemma 6 . The last point is established as in [30], Section 4, Proposition 3.8. $\square$

Lemma 4.4. *[[34], proof of Thorme III.29 + 31,] Suppose S is a regular scheme, and $\mathcal{X}$ and $\mathcal{Y}$ are objects of $\mathcal{H}(\mathfrak{R}_S)_\bullet$. Then the natural maps*

$$\operatorname{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(\mathcal{X}, \mathcal{Y}) \to \operatorname{Hom}_{\bullet, \mathfrak{R}_S^{op}\operatorname{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

and

$$\operatorname{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\mathcal{X}, \mathcal{Y}) \to \operatorname{Hom}_{\mathfrak{R}_S^{op}\operatorname{Set}}(\phi\mathcal{X}, \phi\mathcal{Y})$$

are bijective in the case $\mathcal{Y}$ and $\mathcal{X}$ are products of any of the following:

- $(\mathbb{Z} \times \mathrm{Gr})$ *or any localization thereof by natural integers n or $\mathbb{Q}$.*

- $\mathrm{Fil}^{(i)}$.
- $\mathbb{H}^{(i)}$.
- $\mathbb{P}^\infty$.

Proof. The cited proof goes through with the following remarks. By [34], Lemme III.19, for any objects X and E in $\mathcal{H}(\mathfrak{R}_S)_\bullet$ with E an H-group, there is an injection $\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(X,E) \to \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(X,E)$ whose image is that of morphisms $X \xrightarrow{f} E$ such that

$$f^*(\bullet) = \bullet \in \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(S,E).$$

Thus one reduces to the non-pointed case. The objects in question are retracts of $(\mathbb{Z}\times \mathrm{Gr})_\mathbb{Q}$ or equal to $\mathbb{P}^\infty$, which are the cases treated in the reference and one concludes. $\square$

We are now ready to complete the proof of Theorem 4.1. Using the above two lemmas we deduce morphisms $\mathrm{Fil}^{(i)}\times \mathrm{Fil}^{(j)} \to \mathrm{Fil}^{(i+j)}$ from the morphisms $F^iK_0(-)\times F^jK_0(-) \to F^{i+j}K_0(-)$ and similarly for $\mathbb{H}^{(i)}\times\mathbb{H}^{(j)} \to \mathbb{H}^{(i+j)}$. As in ibid, Lemma III.33 we have the following proposition: for an H-group E and objects A, B of $\mathcal{H}(\mathfrak{R}_S)_\bullet$, the map

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(A\wedge B, E) \to \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)_\bullet}(A\times B, E)$$

is injective and its image consists of morphisms $A\times B\to E$ such that the restriction to $\bullet\times B$ and $A\times\bullet$ is zero. It follows that both of the two morphisms factor as $\mathrm{Fil}^{(i)}\wedge\mathrm{Fil}^{(j)}\to\mathrm{Fil}^{(i+j)}$ and $\mathbb{H}^{(i)}\wedge\mathbb{H}^{(j)}\to\mathbb{H}^{(i+j)}$. The same argument shows the necessary diagrams are commutative. The Adams-operations act appropriately for the same reason. $\square$

For the below, recall that $V_\mathcal{X}$ is the associated virtual category to $\mathcal{X}$ as in Proposition A.9. The main theorem of this section is now the following:

Theorem 4.5 (Rigidity). *[proof of* [34], *Section III.10] Suppose* $S = \mathrm{Spec}\,\mathbb{Z}$. *In the cases considered in the above lemma, except whenever* $\mathcal{Y}$ *involves a factor of* $\mathbb{P}^\infty$, *the morphisms*

$$\mathrm{Hom}_f(V_\mathcal{X}, V_\mathcal{Y}) \to \mathrm{Hom}_{\mathfrak{R}^{op}\,\mathrm{Set}}(\phi\mathcal{X},\phi\mathcal{Y})$$

and

$$\mathrm{Hom}_{f,\bullet}(V_\mathcal{X}, V_\mathcal{Y}) \to \mathrm{Hom}_{\bullet,\mathfrak{R}^{op}\,\mathrm{Set}}(\phi\mathcal{X},\phi\mathcal{Y})$$

have natural sections (and are thus surjective), which are canonical up to unique isomorphism (see A.8.1 for a definition of the functor V*). Moreover, any natural transformation of two functors* $V_\mathcal{X}\to V_\mathcal{Y}$ *is unique up to unique natural transformation.*

Proof. This follows directly from Lemma 4.4 and by considering the composition

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\mathcal{X},\mathcal{Y}) \to \mathrm{Hom}_f(V_\mathcal{X}, V_\mathcal{Y}) \to \mathrm{Hom}_{\mathfrak{R}^{op}\,\mathrm{Set}}(\phi\mathcal{X},\phi\mathcal{Y})$$

obtained from pre-rigidity in Proposition A.9. The essential point is that $\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_S)}(\mathcal{X},\Omega\mathcal{Y})$ disappears since they can all be related to $K_1(\mathbb{Z}) = \mathbb{Z}/2$-modules, and all the groups in question are 2-divisible by construction. Now, suppose we have $\phi\in\mathrm{Hom}_f(V_\mathcal{X},V_\mathcal{Y})$ and an automorphism ϕ, *i.e.* a functorial isomorphism of functors $\phi\simeq\phi$. Suppose for simplicity that $\mathcal{X}=\mathcal{Y}=(\mathbb{Z}\times\mathrm{Gr})$. It is easy to see it determines an element in $\mathrm{Hom}_{\mathfrak{R}^{op}\,\mathrm{Set}}(K_0(-)_\mathbb{Q}, K_1(-)_\mathbb{Q})$, and moreover that any such element determines an automorphism of ϕ. The latter group is zero by Theorem A.5 and an argument analogous to the proof in the previous lemma. $\square$

Under the conclusion of the above theorem we say that the functor $V_{\mathcal{X}} \to V_{\mathcal{Y}}$ lifts that of $\phi\mathcal{X} \to \phi\mathcal{Y}$. We say the lifting given by the section of the theorem is given by "rigidity". To state the next proposition, denote by $\mathfrak{Pic}_{\frac{1}{n}}(\mathcal{X})$ and $\mathfrak{Pic}_{\mathbb{Q}}(\mathcal{X})$ the Picard category of line bundles on $\mathcal{X}$ localized at an integer n or $\mathbb{Q}$ respectively.

Proposition 4.6. *Let $\mathfrak{RCh}/S$ be the category of regular algebraic stacks over S, and $\mathfrak{RCh}^c/S$ the category of inclusions of regular algebraic stacks $U \to X$. Put $\Phi' : \mathfrak{RCh}/S \to \mathcal{H}(\mathfrak{R})$ be the functor determined by the extended Yoneda-functor (see Definition B.1.1) and for an object $\mathcal{X}$ of $\mathcal{H}(\mathfrak{R})$, denote by $\phi'(\mathcal{X})$ the functor $\mathfrak{RCh}/S \to \mathrm{Grp}$ such that $\phi'(\mathcal{X})(\mathcal{U} \to \mathcal{Y}) = V_{\mathcal{X}}(\Phi'(\mathcal{Y})/\Phi'(\mathcal{U}))$. Then we have the following equivalences of functors:*

- *$\phi'((\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}})$ = the fibered Picard category over $\mathfrak{RCh}^c/S$ that is the fundamental groupoid of K-cohomology with support.*
- *Let n be an integer. Then $\phi'(\mathbb{P}^\infty[\frac{1}{n}]) = \mathfrak{Pic}[\frac{1}{n}]$ and $\phi'(\mathbb{P}^\infty_{\mathbb{Q}}) = \mathfrak{Pic}_{\mathbb{Q}}$, the fibered category of line bundles localized at n or $\mathbb{Q}$ over $\mathfrak{RCh}/S$, associating to any object of $\mathfrak{RCh}/S$ the category of localized linebundles thereupon.*

Proof. The first statement is essentially by definition. Consider the second statement. For a simplicial sheaf $\mathcal{X}$ and a sheaf of groups G, a G-torsor is a simplicial sheaf $\mathcal{Y} \to \mathcal{X}$ with a free action of G such that $\mathcal{Y}/G = \mathcal{X}$. In other words, a collection of G-torsors $\mathcal{Y}[n]$ on $\mathcal{X}[n]$ such that for a morphism $\phi : [n] \to [m]$ there are induced morphisms ϕ^* interchanging the data in the obvious manner. Now, it follows from [30], Section 4, Proposition 3.8 that for a simplicial sheaf $\mathcal{X}$, $\phi'(\mathbb{P}^\infty)(\mathcal{X})$ is the category of $\mathbb{G}_m$-torsors on $\mathcal{X}$. Thus, for a regular algebraic space U it is clear that this is the category of line bundles on U. Let U be an regular algebraic stack with smooth presentation $X \to U$ with X an algebraic space. Then U identifies with the simplicial sheaf whose n-simplices are given by $X \times_U X \times_U \ldots \times_U X$, n-time, and face and edge-maps by repeated diagonals and projections as face and edge-maps. Since a surjective morphism of line bundles is necessarily injective one verifies that a $\mathbb{G}_m$-torsor on U necessarily has isomorphisms as transition-morphisms, and we conclude by smooth descent.

□

Remark 4.6.1. By [25], Lemma 3.2, it follows that a Deligne-Mumford stack $\mathcal{M}$, separated and of finite type over a Noetherian base scheme with coarse moduli space M, then $\mathfrak{Pic}(M)_{\mathbb{Q}} \to \mathfrak{Pic}(\mathcal{M})_{\mathbb{Q}}$ is an equivalence of categories.

A priori the operations given by rigidity are abstract and one might want to relate them to other operations. One standard way of doing so is as follows. Restricting any "virtual" operation given by a morphism $(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ along $\mathbb{P}^\infty \to (\{0\} \times \mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ gives us the behavior of the operation on an actual line bundle where we can often write down explicitly what it does. Then by the splitting principle one can often compare this to other operations. In the case of determinant functors we can say even more:

Proposition 4.7. *If in the above theorem we restrict ourselves to determinant (i.e. additive) functors between virtual categories of schemes, then all the arrows are isomorphisms and are isomorphic to $\mathbb{Q}[[T]]$ where T^k corresponds to the k-th Adams operation. The Adams operations are thus dense in the ring of endofunctors. Moreover, any additive functor is unique up to unique isomorphism.*

Proof. Given any determinant endofunctor, it is determined by the splitting principle by its valuation on line bundles. For any regular X, by Jouanolou-Thomason, [47], Proposition 4.4, there is a torsor $\xi \to X$ under some vector bundle E such that ξ is affine. Then any line bundle is affine and thus the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^n_{\mathbb{Z}}$ for some n. In this situation everything is rigidified since the virtual category of $\mathbb{P}^n_{\mathbb{Z}}$ doesn't have any non-trivial automorphisms after tensoring with $\mathbb{Q}$. It is thus a rigid functor because it can be reduced to this situation. A Segre embedding argument shows that the chosen isomorphism is independent of the morphism to $\mathbb{P}^n_{\mathbb{Z}}$ and n. The second statement is now Proposition 5.1.1 in [33]. □

Clearly, for the same reason, any functor of virtual categories stable under pullback is uniquely determined on vector bundles by considering maps to Grassmannians instead of simply $\mathbb{P}^n$.

4.2. Some consequences of rigidity.

Definition 4.7.1. We denote by $F^iW(-)$ (resp. $W^{(i)}$) the category fibered in groupoids $\phi'(\mathrm{Fil}^{(i)})$ (resp. $\phi'(\mathbb{H}^{(i)})$) over $\mathfrak{RCh}^c := \mathfrak{RCh}^c/\operatorname{Spec}\mathbb{Z}$. We write furthermore $F^iW(\mathcal{U} \to \mathcal{X}) = F^iW^Z(X)$ and $W(\mathcal{U} \to \mathcal{X})^{(i)} = W^{\mathcal{Z}}(X)^{(i)}$. Notice that for an algebraic stack $\mathcal{X}$ that is not an algebraic space $F^0W(\mathcal{X}) = W(\mathcal{X})$ is in general not the virtual category $V(\mathcal{X})_{\mathbb{Q}}$ of $\mathcal{X}$.

We record the following.

Theorem 4.8. *The functors $F^iW(-)$ have the following properties.*

(a) *The functors $F^{i-1}W(-) \to F^iW(-)$ are faithful additive functors of Picard categories.*
(b) *There are pairings, unique up to unique isomorphism,*
$$F^iW(-) \times F^jW(-) \to F^{i+j}W(-)$$
lifting the pairings $F^iK_0(X)_{\mathbf{Q}} \times F^jK_0(X)_{\mathbf{Q}} \to F^{i+j}K_0(X)_{\mathbf{Q}}$ on regular schemes, such that for $i' \leq i, j' \leq j$, we have a commutative diagram
$$\begin{tikzcd} F^iW(-) \times F^jW(-) \arrow[r] \arrow[d] & F^{i+j}W(-) \arrow[d] \\ F^{i'}W(-) \times F^{j'}W(-) \arrow[r] & F^{i'+j'}W(-) \end{tikzcd} .$$
In particular there are pairings
$$F^iW^{\mathcal{Z}}(\mathcal{X}) \times F^jW^{\mathcal{Z}'}(\mathcal{X}) \to F^{i+j}W^{\mathcal{Z}\cap\mathcal{Z}'}(\mathcal{X}).$$
(c) *There are unique pairings $W^{(i)} \times W^{(j)} \to W^{(i+j)}$, extending the usual pairings $K_0(X)^{(i)} \times K_0(X)^{(j)} \to K_0(X)^{(i+j)}$ on regular schemes.*
(d) *The pairings are compatible with the isomorphism*
$$F^iW(-) = F^{i-1}W(-) \times W^{(i)}$$
and they all satisfy the obvious associativity constraints.
(e) *The above is compatible with zero-objects in that a zero-object in one variable maps to a zero-object in the second.*
(f) *The Adams-operations act on all the objects and functors involved, and these operations are moreover, up to unique isomorphism, uniquely defined as liftings of the usual Adams operations.*

(g) *Let $\mathcal{X}$ be a regular algebraic stack of dimension d with finite affine stabilizers and $\mathcal{U} \to \mathcal{X}$ an open substack such that $\mathcal{X} \setminus \mathcal{U}$ is of codimension m. Then $F^{d+2}W(\mathcal{U} \to \mathcal{X})$ is equivalent to the trivial category with one object and the identity as only morphism and we have an equivalence of categories: $W^{\mathcal{Z}}(\mathcal{X}) = \bigoplus_{i=m}^{d+1} W^{\mathcal{Z}}(\mathcal{X})^{(i)}$.*

Proof. For simplicity we treat the case when $\mathcal{U} = \emptyset$, the other cases are analogous. First, (b),(c),(d) and (f) are clear from rigidity. For (a), it is enough to show that for any $\mathcal{X}$ and object x of $F^{i-1}W(\mathcal{X})$, $\mathrm{Aut}_{F^{i-1}W(\mathcal{X})}(x) \to \mathrm{Aut}_{F^iW(\mathcal{X})}(x)$ is injective. But this is clear since this map identifies with the injection $F^{i-1}K_1^{sm}(\mathcal{X}) \to F^iK_1^{sm}(\mathcal{X})$. Now (e) follows from the description of the pairing in Theorem 4.1. Since we will only be concerned with (g) for a scheme we give the proof in this case. We give a proof along classical lines when $\mathcal{U} = \emptyset$:

Lemma 4.9. *Let X be a regular scheme with $d = \dim X$, and $i = 0, 1$. Recall that $F^jK_i(X)_{\mathbb{Q}}$ is the filtration on $K_i(X)_{\mathbb{Q}}$ determined by*

$$F^jK_i(X)_{\mathbb{Q}} = \bigoplus_{p\geq j} K_i(X)^{(p)}.$$

Then $F^{d+i+1}K_i(X)_{\mathbb{Q}} = 0$.

Proof. Consider the Quillen coniveau spectral sequence

$$E_1^{p,q}(X) = \bigoplus_{x\in X^{(p)}} K_{-p-q}(k(x)) \Longrightarrow K_{-p-q}(X)$$

where $X^{(p)}$ denotes the codimension p-points of X and $k(x)$ is the residue field of x. By [36] Thorme 4, iv), we have that, for $i = 0, 1$,

$$K_i(X)_{\mathbb{Q}} = \bigoplus_{p=0}^{d} E_2^{p,-p-i}(X)_{\mathbb{Q}}.$$

Furthermore, it is remarked in [15], proof of Theorem 8.2, that the Adams operations Ψ^k act on the spectral sequence and in particular on $E_r^{p,-p-i}(X)$ by k^{p+i} ($i = 0, 1$ and $r \geq 1$). Also, $E_r^{p,-p-i}(\Psi^k)$ converge towards the Adams operations on $K_i(X)$. Thus, for $i = 0, 1$, it follows that $K_i(X)^{(d+i+1+k)} = 0$ for $k \geq 0$ so that $F^{d+i+1}K_i(X)_{\mathbb{Q}} = 0$. □

We immediately deduce that the categories $F^iW(X)$ are trivial, *i.e.* all objects are uniquely isomorphic, for $i \geq \dim X + 2$. □

Remark 4.9.1. The proof of property (g) in the case of a regular algebraic space goes through verbatim. The general case is obtained in a similar way, but one has to work instead with the spectral sequence $E_1^{p,q}(\mathcal{X}) = \bigoplus_{\xi\in X^{(p)}} K_{-p-q}^{sm}(G_{\xi,\mathrm{red}}) \Longrightarrow K_{-p-q}^{sm}(\mathcal{X})$ which exists by a Brown-Gersten argument applied to the flabby S^1-spectrum representing cohomological K-theory and by virtue of $G_p^{sm} = K_p^{sm}$ by Poincar duality for the cohomology of the K-theory for regular algebraic stacks (see Theorem B.4). Then each $G_{\xi,\mathrm{red}}$ is a gerbe banded by some reduced algebraic group H, which is in fact necessarily an abstract finite group over the algebraic closure of the moduli space. To understand the Adams operations we can by tale descent moreover suppose that the moduli space $\mathrm{Spec}\, k(x)$ of $G_{\xi,\mathrm{red}}$ is separably closed so that the gerbe is trivial and $G_{\xi,\mathrm{red}} = [\mathrm{Spec}\, k(x)/H]$. By the arguments of [39], 2.3 there is a spectral sequence

$$E_1^{p,q} = K_q(\prod^p H) \to K_{q-p}^{sm}(G_{\xi,\mathrm{red}}).$$

Then $K_i^{sm}(G_{\xi,\mathrm{red}})_{\mathbb{Q}} = K_i(\operatorname{Spec} k(x))^H_{\mathbb{Q}}$ for all i. The Adams operations Ψ^k act on $K_i^{sm}(G_{\xi,\mathrm{red}})$ via the restriction of $K_i(\operatorname{Spec} k(x))$ to the H-invariant part and thus by k^i. The rest is similiar but skipped.

Remark 4.9.2. From [27], Theorem 11.5 it follows that if R is a Dedekind domain, and X is a regular finite type $\operatorname{Spec} R$-scheme, then the γ-filtration on $K_n(X)$ for any integer n terminates after $d+n+1$ steps.

We harvest some obvious corollaries:

Corollary 4.10. *Let $\mathcal{X}$ be a regular algebraic stack with closed substack $\mathcal{Z}$. The Adams operations on $W^{\mathcal{Z}}(\mathcal{X})$ are compatible with the Adams operations constructed on $V^{\mathcal{Z}}(\mathcal{X})$ in Proposition 3.2 under the functor $V^{\mathcal{Z}}(\mathcal{X}) \to W^{\mathcal{Z}}(\mathcal{X})$. Moreover, there is a determinant functor* $\det : W(\mathcal{X}) \to \mathfrak{Pic}(\mathcal{X})_{\mathbb{Q}}$ *such that the diagram*

$$\begin{tikzcd} V(\mathcal{X}) \arrow[r] \arrow[d, "\Psi^k"] & W(\mathcal{X}) \arrow[d, "\Psi^k"] \\ V(\mathcal{X}) \arrow[r] \arrow[d, "\det"] & W(\mathcal{X}) \arrow[d, "\det"] \\ \mathfrak{Pic}(\mathcal{X}) \arrow[r] & \mathfrak{Pic}(\mathcal{X})_{\mathbb{Q}} \end{tikzcd}$$

commutes up to canonical natural transformation.

Proof. By rigidity the two Adams-operations coincide on line bundles and we conclude by the splitting principle. Moreover, $\mathfrak{Pic}(\mathcal{X})_{\mathbb{Q}}$ clearly satisfies coherent descent since it is a localization of the category $\mathfrak{Pic}(\mathcal{X})$ which does. The determinant functor then exists by cohomological descent and the diagram commutes again by rigidity and the splitting principle. ☐

Corollary 4.11. *There are unique λ-operations on $F^iW(-)$ λ-operations satisfying, for a regular algebraic stack $\mathcal{X}$,*

$$\lambda^k(x+y) = \sum_{j=0}^{k} \lambda^j(x) \otimes \lambda^{k-j}(y)$$

and moreover for a bounded complex of vector bundles $E_\bullet$ one has $t_{E_\bullet} : \lambda^k E_\bullet \simeq \wedge^k E_\bullet$, where $\wedge$ refers to the already constructed operations in Proposition 3.4. For an exact sequence of bounded complexes of vector bundles $0 \to E'_\bullet \to E_\bullet \to E''_\bullet \to 0$ a commutative diagram of isomorphisms

$$\begin{array}{ccc} \lambda^k(E_\bullet) & = & \sum_{j=0}^k \lambda^j(E'_\bullet) \otimes \lambda^{k-j}(E''_\bullet) \\ \Big\| t & & \Big\| t \\ \wedge^k(E_\bullet) & = & \sum_{j=0}^k \wedge^j(E'_\bullet) \otimes \wedge^{k-j}(E''_\bullet) \end{array}$$

with the lower row defined as in Proposition 3.4.

Proof. We assume again $\mathcal{U} = \emptyset$, the general result follows from an argument of the type in Proposition 3.7. We might also suppose that the complexes of vector bundles are actually vector bundles, by the equivalence of the virtual category of complexes of vector bundles and vector bundles and the equivalence of the constructed $\wedge$-operations (cf. Proposition 2.5 and Proposition 3.4 (c)).

Unicity is then clear by the splitting principle. Existence of the λ-operations are given by rigidity, and we suppose for simplicity that $i = 0$. For a line bundle rigidity provides us with an isomorphism $\lambda^k L \simeq \wedge^k L = L$ if $k = 1$ and an isomorphism with 0 if $k > 1$. The existence of t and the commutativity now follows from Corollary 3.5, Corollary 4.10 and the argument of Proposition 4.7 applied to the additive functor $\lambda_t = \sum \lambda^k t^k$. □

Corollary 4.12. *We restrict ourselves to the category of regular schemes. There are γ-operations on the virtual category $W(-)$, $j \geq 2$, γ^j, inducing the natural operations on $K_{0,\mathbb{Q}}$. For a virtual bundle of rank 0,*

$$\gamma^j(v) \in F^j W.$$

Furthermore, for any two virtual objects x of rank 0*, we have a family of isomorphisms in $F^k W$, functorial in x and y;*

$$\gamma^k(x+y) \simeq \sum \gamma^j(x) \otimes \gamma^{k-j}(y). \tag{4}$$

Since for a line bundle L, $1 - L$ identifies with an object of $F^1 W$, we have that $(1-L)^i$ is an object of $F^i W$. We then have canonical isomorphisms in $F^i W$:

$$\gamma^i(1-L) \simeq (1-L)^i$$

and

$$\gamma^i(L-1) = 0 \text{ for } i \geq 2.$$

Moreover, for $i \geq 1$, the trivialization

$$\gamma^{e+i}(E-e) \simeq 0$$

for $e = \operatorname{rk} E$ determined by the splitting principle coincides with the trivialization determined by

$$\gamma^{e+i}(E-e) \simeq \wedge^{e+i}(E+i-1) \simeq 0$$

determined by Corollary 4.11 and the fact that the exterior power vanishes for high enough degrees.

Proof. All statement except the last one are direct consequences of rigidity. By definition (cf. [13], chapter III) and rigidity we are given a relationship of γ and λ-operations $\sum_{i=0} \gamma^i(u) t^i = \sum_i \lambda^i(u) \left(\frac{t}{1-t}\right)^i$ so that, in view of that $1/(1-t)^{r+1} = \sum \binom{j+r}{j} t^j$, the relationship, for $k > 0$, in $W(\mathcal{X})$

$$\gamma^k(u) = \sum_{i=0}^{k} \lambda^i(u) \binom{k-1}{k-i} = \lambda^k(u+k-1).$$

If u is a virtual bundle of rank 0 we deduce the equality in $F^k W$ compatible with sums and the product on the filtration. We obtain a trivialization $\gamma^{k+n}(E-n) = \lambda^{k+n}(E+k-1) = \wedge^{k+n}(E+k-1) = 0$ and we want to prove that it is compatible with the trivialization given by splittings of E into line bundles. As before we can suppose that X is affine. Given an exact sequence of vector bundles we can suppose it is split: the variety of splittings of such a sequence is a torsor under a vector bundle and thus passage to this variety is an equivalence of categories by the homotopy invariance in K-theory. Finally, by considering surjections of global sections we obtain maps to Grassmannians where the two definitions necessarily coincide because they pull back from the arithmetic situation where there are no automorphisms. A

Segre embedding argument shows that this is not dependent on the choice of surjections and by the splitting principle we conclude.

□

Corollary 4.13. *Let X be a regular scheme of dimension d. Then for any virtual bundle v in $W(X)$ of rank 0, $k > 1$,*

$$\gamma^{d+k}(v) \simeq 0$$

canonically.

Proof. This follows from Theorem 4.8 and Corollary 4.12. □

Remark 4.13.1. It is not difficult to see that the above is really the Adams-eigenspace in functorial form; we could also have taken this as the definition without any reference to $\mathbb{A}^1$-homotopy theory of schemes: Let X be a scheme and Z a closed subscheme of X. Then for any k, and a virtual bundle v, an isomorphism $\Psi^k(v) = k^i v$ in $W^Z(X)$ defines a projection of v into $W^{Z,(i)}(X)$. Thus it implies that there is a canonical isomorphism $\Psi^k(v) = k^i v$ for any i. An object of $W^{Z,(i)}$ is an element v of $W^Z(X)$ together with an isomorphism $\Psi^k(v) = k^i v$ and the morphisms are described by isomorphisms of respecting this automorphism.

5. Comparison with constructions via algebraic cycles

Warning: This section is being re-written, the dimension theory as written is not the correct one, but should be the relative dimension with respect to a regular base scheme.

In [10] and [11] there is a careful outline of his notion of Chow categories and Chern functors. This section provides a comparison of the constructions therein and the constructions above.

5.1. **Rational Chow categories.** In this section we define homological Chow categories. Also, a scheme will be a scheme Z which is separated and of finite type over a regular scheme S, so that particular Z is universally catenary. We will also suppose that Z admits an ample family of line bundles. By supposing that Z has an ample family of line bundles, by Jouanolou-Thomason (cf. [47], Proposition 4.4) Z has the $\mathbb{A}^1$-homotopy type of an affine scheme and by Corollary 5.3 below we can often suppose that Z is affine and admits an embedding into a scheme smooth over S. We will also suppose that any regular scheme is connected. The dimension theory for the relevant schemes will be the relative dimension theory as given in [12], chapter 20.

Definition 5.0.1. Suppose $Z \subseteq X$ is closed immersion of a scheme into a regular scheme X. Define the (rational) cohomological Chow category with support, denoted $\mathcal{CH}^i_Z(X)$, as the category $W^Z(X)^{(i)}$ considered in the last section.

Recall also the construction of Chow categories of J. Franke. For a scheme X, write $X^{(p)}$ for the codimension p-points of X. Suppose X is regular and connected and that Z is a closed subscheme of X. Consider the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^p \cap Z} K_{-p-q}(k(x)) \Longrightarrow K^Z_{-p-q}(X)$$

deduced from the filtration by codimension, giving rise to

$$\bigoplus_{z \in X^{(p-1)} \cap Z} K_{1-p-q}(k(z)) \xrightarrow{d_1} \bigoplus_{y \in X^{(p)} \cap Z} K_{-p-q}(k(y)) \xrightarrow{d_0} \bigoplus_{x \in X^{(p+1)} \cap Z} K_{-p-q-1}(k(x)).$$

Notice that since X is catenary and equidimensional, $Z \cap X^{(p)} = Z_{(\dim X - p)}$, where $Z_{(d)}$ denotes the dimension d-points of Z. Without the assumption that X is regular and taking $Z = X$, this is the classical coniveau spectral sequence converging to G-theory. We have the following from [10]:

Definition 5.0.2. The category $\mathbf{CH}^i(X)$ is defined as follows. The objects are given by codimension i-cycles on X, and homomorphisms between two cycles z and z' are described by

$$\operatorname{Hom}(z, z') := \{f \in E_1^{i-1,-i}(X), d_1(f) = z' - z\}/d_1 E_1^{i-2,-i}(X).$$

The category $\mathbf{CH}^i(X)_{\mathbb{Q}}$ is then the category $\mathbf{CH}^i(X)$ localized at $\mathbb{Q}$. They both have natural structures of Picard categories.

Definition 5.0.3. Let X be regular and Z be a closed subscheme of X. The $\dim X - i$-th (homological) Chow category is the following: The objects are elements of $Z_{\dim X - i}(Z) = Z^i_Z(X)$ and whose homomorphisms are given as $\operatorname{Hom}(z, z') := \{f \in E_{1,Z}^{i-1,-i}(X), d_1(f) = z' - z\}/d_1 E_{1,Z}^{i-2,-i}(X)$. Here $Z^i_Z(X)$ denotes the cycles of codimension i on X with support on Z. It is moreover clear from the description of the coniveau spectral sequence that the categories are independent of X, since it reduces to the niveau spectral sequence for Z and $X^{(i)} \cap Z = Z_{(\dim X - i)}$ since X is catenary and equidimensional. The category localized at $\mathbb{Q}$ is the rational (homological) Chow category and we denote it by $\mathcal{CH}_{\dim X - i}(Z)$. In view of that for a field k, $K_2(k) = k^* \otimes_{\mathbb{Z}} k^*$ modulo symbols of the form $(x, 1-x), x \in k \setminus \{0,1\}, K_1(k) = k^*$ and $K_0(k) = \mathbb{Z}$, it is thus a Picard category related to a complex (in the above sense)

$$\bigoplus_{z \in X^{(i-2)} \cap Z} k(z)^* \otimes_{\mathbb{Z}} k(z)^* \xrightarrow{d_1} \bigoplus_{y \in X^{(i-1)} \cap Z} k(y)^* \xrightarrow{d_0} \bigoplus_{x \in X^{(i)} \cap Z} \mathbb{Z}$$

defined elementary in terms of fields and where d_0 is the associated divisor to a rational function and d_1 is the (generalized) tame symbol (cf. [14], Theorem 7.21).

We included X to emphasize that it is clearly equivalent to the category $\mathbf{CH}^i(Z)_{\mathbb{Q}}$, as noted in the next proposition, but it also has an obvious definition in terms of the niveau filtration as the Picard category associated to the resulting exact sequence

$$E^1_{k+2,-k}(X) \to E^1_{k+1,-k}(X) \to E^1_{k,-k}(X)$$

where $E^1_{p,q}(X) = \oplus_{x \in X, \dim \overline{\{x\}} = p} K_{p+q}(k(x))$ and the maps are the analogues of the above.

Proposition 5.1 (Poincar duality). *Keep the above notation and assumptions. Then:*

(a) *The category $\mathcal{CH}^i_Z(X)$ be identified with the cycle-groupoid giving a Poincar duality-type equivalence:*

$$\mathcal{CH}^i_Z(X) \simeq \mathcal{CH}_{\dim X - i}(Z).$$

(b) *Suppose moreover Z is equidimenional of dimension d, then there is a natural equivalence of Picard categories*

$$\Psi : \boldsymbol{CH}^i(Z)_{\mathbb{Q}} \to \mathcal{CH}_{d-i}(Z)$$

.

Proof. We already mentioned the second part is true. For the first, given a cycle z in the latter category, associate to it the natural edge in G-theory. $E_{2,Z}^{k-1,-k}(X)$ in turn admits a canonical map to the G-theory space by sending an element $f \in k(x)^*, x \in X^{(k-1)} \cap Z$ to the associated map of linebundles $\mathcal{O} \simeq \mathcal{O}(\operatorname{div} f)$ on $\overline{\{x\}}$. By [15] the projection onto the proper Adams eigenspace in $V^Z(X)$ induces an isomorphism on the level of π_0. The statement for π_1 is an analogous result which is not made explicit therein but which follows analogously. Thus the map induces an equivalence of categories.

$\square$

Remark 5.1.1. If Z is normal and locally factorial, then there is an equivalence of categories $\mathbf{CH}^1(Z) \simeq \mathfrak{Pic}(Z)$, the category of linebundles with isomorphisms (cf. [10], section 2). If furthermore Z is equidimensional of dimension d, by (b) above $\mathcal{CH}_{d-1}(Z) \simeq \mathfrak{Pic}(Z)_{\mathbb{Q}}$.

We go on to note that basically all the properties described in [14] of a "Chow homology theory" carry over to this situation. At certain points we emphasize the K-theoretical approach by working with the Adams filtration (cf. Definition 5.5.1 below) since it will be closer to some of our aims.

Proposition 5.2 (Flat pullback)**.** *Suppose there is a flat morphism of schemes $f : X \to Y$ of relative dimension d. Then there is a natural functor $f^* : \mathcal{CH}_i(Y) \to \mathcal{CH}_{i+d}(X)$ compatible with composition. It maps an i-dimensional cycle V to the closure of $f^{-1}V$ in X.*

The proof is obvious from niveau spectral sequence and the description of the map f^* which is part of the definition of pullback of cycles.

Remark 5.2.1. In the language of [10], section 3.6, this makes the association $Z \mapsto \mathcal{CH}^i(Z) = \mathcal{CH}_{\dim Z - i}(Z)$ with flat pullbacks into a fibered Picard category over the schemes for which these are defined, and moreover satisfy Poincar duality. In other words, a contravariant assignment of Picard categories $Z \mapsto P_Z$ for appropriate Z together with associativity and composition constraints for the class of pullback morphisms.

Corollary 5.3 (Homotopy invariance)**.** *Let $Y \to Z$ be a torsor under a vector bundle on Z, and Z of finite type over a regular scheme S. Then the flat pullback $f^* : \mathcal{CH}_i(Z) \to \mathcal{CH}_{i+d}(Y)$ induces an equivalence of categories.*

Proof. This follows from the homotopy invariance on the second page of the weight spectral sequence by [14], Theorem 8.3, and thus induces isomorphisms on π_0 and π_1 and thus equivalences of categories. □

The following is obvious:

Proposition 5.4 (Topological invariance)**.** *Let Z be a closed subscheme of a regular scheme X. If Z_{red} denotes the associated reduced scheme the natural map gives an equivalence of categories*

$$\mathcal{CH}_i(Z) = \mathcal{CH}_i(Z_{red}).$$

Proposition 5.5 (Proper pushforward)**.** *Suppose $f : Z \to Z'$ is a proper morphism of schemes. There is an induced pushforward $f_* : \mathcal{CH}_k(Z) \to \mathcal{CH}_k(Z')$.*

Proof. The proof of [14], Theorem 7.22 (iii) shows that there is a push-forward on the complexes given by the first page of the niveau spectral sequence. It follows that there is a pushforward $f_* : \mathcal{CH}_i(X) \to \mathcal{CH}_i(Y)$ on the cyclelevel, already without rational coefficients, given by the classical formulas on the level of objects (also cf. [12], section 1.4). □

Definition 5.5.1. Define $\mathrm{Fil}_k(Z) = \coprod_{i \leq k} \mathcal{CH}_i(Z)$. By the proof of Proposition 5.1, if X is a regular containing Z, this is equivalent to $F_Z^{\dim X - k}(X)$ considered in Theorem 4.8. Moreover, we have

$$\mathrm{Fil}_k(Z) / \mathrm{Fil}_{k-1}(Z) \simeq \mathcal{CH}_k(Z)$$

where the quotient denotes cokernel of an additive functor of Picard categories (cf. Definition 2.2.1). In general we can consider it as a faithful (but not full) subcategory of $C(Z)_{\mathbb{Q}}$ (the virtual category of coherent sheaves on Z, cf. Definition C.0.1) since the weight spectral sequence degenerates modulo $\mathbb{Q}$ at the second page (cf. the proof of Proposition 5.1. We can always reduce to this case by Jouanolou-Thomason as follows. If Z is defined over S, we can find an affine S with a morphism $S' \to S$ which is a torsor under a vector bundle, and the basechange $Z \times_S S' \to Z$ induces an equivalence of categories on the revelent categories. Using the same argument, we can assume Z itself is affine, having gained that it is affine over an affine scheme. If it is of finite type, we can find an embedding into $\mathbb{A}^n_S$ for some n, which is smooth over S. This fact will be often be used implicitly hereafter).

Notice that if $f : Z \to Z'$ is a proper morphism of schemes, the functor $Rf_* : C(Z) \to C(Z')$ restricted to $\mathrm{Fil}_k(Z)$ has essential image in $\mathrm{Fil}_k(Z')$. Indeed, we know from the above that any object in $\mathrm{Fil}_k(Z)$ is represented by sheaves of the form $\mathcal{O}_V$ for closed subschemes V of Z of dimension at most k. It is obvious that $Rf_*\mathcal{O}_V$ is a sum of sheaves with support on sheaves with support on points with dimension at most k on Z'. We need to prove that the induced map on $Rf_* : G_1(Z)_{\mathbb{Q}} \to G_1(Y)_{\mathbb{Q}}$ has the same properties with respect to the niveau filtration. Now Z' is defined over a regular scheme S, and by Jouanolou-Thomason we can assume first that S and then that Z' is affine. By Chow's lemma ([5], Lemme 5.6.1)we can find a morphism $p : \widetilde{Z} \to Z$ such p and fp are projective and that $Rp_*(\mathcal{O}_{\widetilde{Z}}) = \mathcal{O}_Z$ and hence we can replace, if we can show that $Rp_* : G_1(\widetilde{Z})_{\mathbb{Q}} \to G_1(Z)_{\mathbb{Q}}$ is surjective on every step of the niveau filtration, Z by $\widetilde{Z}$ and thus that f itself is projective. To prove this we can also replace Z by an affine scheme so there is a closed embedding $Z \subseteq \mathbb{A}^n_S$ for suitable n and since p is projective it fits into a natural compatible diagram (cf. Proposition 5.8). Then the statement follows from the already cited [14], Theorem 7.22 (iii) which gives the map on the first page of the niveau spectral sequence where surjectivity is clear, and [36], Thorme 4, iv) from which it follows that the isomorphic coniveau spectral sequence with supports in question degenerates at the second page when passing to rational coefficients. In the case of a projective morphism $f : Z \to Z'$ factoring as $Z \to \mathbb{P}^n_{Z'} \to Z'$, embedding Z' into a regular scheme X' and Z into $\mathbb{P}^n_{X'}$ reduces us to the same considerations as above. With this pushforward we have the following corollary:

Corollary 5.6. *Let $f : Z \to Y$ be a proper morphism of schemes. The pushforward Rf_* : $C(X)_{\mathbb{Q}} \to C(Y)_{\mathbb{Q}}$ restricted to $\mathrm{Fil}_k X$ has essential image in $\mathrm{Fil}_k(Y)$. The induced functor f_* : $\mathrm{Fil}_k(Z)/\mathrm{Fil}_{k-1}(Z) \to \mathrm{Fil}_k(Y)/\mathrm{Fil}_{k-1}(Y)$ is equivalent to the pushforward on Chow categories.*

Proof. For the second one, we note that the comparison is trivial in the case of close immersions and to compare the two pushforwards $f_*(\mathcal{O}_V)$ and $Rf_*(\mathcal{O}_V)$, we can then assume $V = Z$ and $Y = f(Z)$. If $f : Z \to Y$ is not generically finite, $f_*\mathcal{O}_Z = 0$ and $Rf_*\mathcal{O}_Z$ is in $\mathrm{Fil}_{k-1}(Y)$. Generally, pick an open U of Y such that f is generically finite of degree d and $\mathcal{O}_Z$ free over $\mathcal{O}_Y$. In this case, the pushforward is described as $\mathcal{O}_U^d$. In general, by [13], Ch. IV, Lemma 3.7 there is a coherent sheaf $\mathcal{G}$ on Y, surjective morphisms $\mathcal{G} \to \mathcal{O}_Y^d$ and $\mathcal{G} \to R^0 f_*\mathcal{O}_Z$ such that the restriction to U is an isomorphism and is compatible with the isomorphism $Rf_*\mathcal{O}_Z|_U = R^0 f_*\mathcal{O}_Z|_U \simeq \mathcal{O}_U^d$. This defines an isomorphism of the two pushforwards, and it doesn't depend on $\mathcal{G}$. Indeed, we only need to prove that if we have a surjective morphism of coherent sheaves $g : \mathcal{G}' \to \mathcal{G}$ on a scheme Z which is an isomorphism over some dense open U whose complement is of dimension $k-1$ or less, then it defines an isomorphism in $\mathcal{CH}_k(Z)$. In this case $\mathrm{Fil}_k(Z) = V(Z)_{\mathbb{Q}}$ and $\mathcal{CH}_k(Z)$ don't have any automorphisms and is equivalent to the group of zero-cycles of dimension k on Z, so the natural determinant functor

$C(Z) \to \mathrm{Fil}_k(Z) \to \mathcal{CH}_k(Z)$ is realized by taking lengths of coherent sheaves along subschemes of dimension k and any exact sequence defines an isomorphism of an object of dimension less than k with zero. □

Proposition 5.7. *Let $f : Z \to Z'$ be a proper morphism. Suppose that V is a virtual vector bundle of Z Then there is a projection formula isomorphism*

$$V \otimes Rf_*(\mathcal{F}) \simeq Rf_*(Lf^*V \otimes \mathcal{F})$$

in $\mathrm{Fil}_k(Z')$ where $\mathcal{F}$ is an element of $\mathrm{Fil}_k(Z)$. It is also stable under composition of proper morphisms in the naive way and if furthermore V is of rank 0, this is an isomorphism in $\mathrm{Fil}_{k-1}(Z')$.

Proof. We prove the formula by noting it is true if we replace the filtration by the virtual category of coherent sheaves and then proving that $f_*(f^*V \otimes \mathcal{F}) \simeq V \otimes \mathcal{F}$ in $\mathcal{CH}_k$. The formula then follows from the above since the pushforward and pullbacks in question are compatible with the ones on the level of virtual filtration by dimension by the previous corollary and functoriality of the kernel of an additive morphism. By functoriality of pushforward we can suppose that Z is a k-dimensional integral scheme, that A is $\mathcal{O}_Z$ and f is finite of degree d and by the splitting principle (or a slight modification thereof since the projective bundle formula for Chow groups is slightly different from that of K-theory in terms of indices) we can assume that V is a line bundle. In this case the formula reduces to show that for a rational section s of L, there is an isomorphism $f_* \operatorname{div} f^*s = f_*f^*(\operatorname{div} s) = d(\operatorname{div} s)$ and the projection formula on the level of G_1. These are actual equalities that already holds on the level of cycles (cf. Example 1.7.4 of [12]) so that the "isomorphism" is actuality the identity-map. For the second part, we can suppose that V is an actual difference $E - e$ where E is a vector bundle and $e = \operatorname{rk} E$. By the splitting principle we can furthermore suppose E is a sum of line bundles so we can suppose $E = L$ is a line bundle. By the previous argument, it is enough to verify that the two sides are naturally equivalent to the identity equivalence of the 0-functor on $\mathcal{CH}_k(Z') = \mathrm{Fil}_k(Z')/\mathrm{Fil}_{k-1}(Z')$ and to prove this we start by proving that for any line bundle L of Z, the functor $(L-1) \cap - : \mathrm{Fil}_k(Z) \to C(Z)$ has essential image in $\mathrm{Fil}_{k-1}(Z)$. Any object in the source category $\mathrm{Fil}_k(Z)$ can be written as a sum of objects $\sum i^y_* \mathcal{F}$ where i^y_* is the pushforward associated to a point y in Z such that $Y := \overline{z}$ has $\dim Y \leq k$ and $\mathcal{F}$ is a coherent sheaf (with rational coefficients). By already established projection formula $(L-1) \otimes i^z_*(\mathcal{F}) \simeq i^z_*(i^{z*}(L-1) \otimes \mathcal{F})$ and so we see that we must show that $i^{z*}(L-1) \otimes \mathcal{F}$ has essential image in $\mathrm{Fil}_{k-1}(Z)$. We can thus assume $Z = Y$ and $\mathcal{F} = \mathcal{O}_Z$. Given a rational section s of L the divisor defines a sheaf whose support is of dimension $k-1$ and thus an element of $\mathrm{Fil}_{k-1}(Z)$. Another choice of rational section defines an isomorphism of the two sheaves and it is obvious the data glues together to an object $i^{y*}(L-1) \otimes \mathcal{F}$ in $\mathrm{Fil}_{k-1}(Z)$. Moreover, by the arguments of the proof of [20], Theorem 4, ii), there natural pairing of $K-$ and G-theory restricts to a pairing $F^1K_0(X) \times F_k^{\dim} G_1(X) \to F_{k-1}^{\dim} G_1(X)$ where $F^1K_0(X)$ denotes the virtual bundles of $K_0(X)$ of rank 0 and $F_{k-1}^{\dim} G_1(X)$ denotes the filtration by dimension of $G_1(X)$. Thus $F_k^{\dim} G_1(X)$ map into the $F_{k-1}^{\dim} G_1(X)$. By passing to quotients we obtain the result. Also compare [14] Theorem 7.24.

□

Remark 5.7.1. Since the diagram

$$\begin{tikzcd} V'\otimes V\otimes Rf_*(\mathcal{F}) \arrow[r] \arrow[d] & V'\otimes Rf_*(Lf^*V\otimes\mathcal{F}) \arrow[d] \\ Rf_*(Lf^*(V'\otimes V)\otimes\mathcal{F}) \arrow[r] & Rf_*(Lf^*V'\otimes Lf^*V\otimes\mathcal{F}) \end{tikzcd}$$

is commutative any virtual bundle of the form $(E_1-e_1)(E_2-e_2)\ldots(E_m-e_m)$ where each E_i is a vector bundle of rank e_i defines by multiplication a functor $\mathrm{Fil}_k(Z)\to\mathrm{Fil}_{k-m}(Z)$. In the next chapter we establish further properties of this functor.

Proposition 5.8 (Gysin-type functors)**.** *Suppose that $Z\to X$ and $Z'\to X'$ are closed embeddings of schemes into regular schemes X and X' and that there is a commutative square (which we call a compatible diagram)*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f"] & X \arrow[d, "F"] \\ Z' \arrow[r] & X' \end{tikzcd}$$

with f and F any two morphisms of schemes such that F induces a morphism $X\setminus Z\to X'\setminus Z'$. Then there is a functor

$$F^!:\mathcal{CH}^i_{Z'}(X')\to\mathcal{CH}^i_Z(X)$$

compatible with composition of these types of diagrams. If f is a projective local complete intersection morphism of relative dimension d, then there is also a functor $f^*:\mathcal{CH}_i(Z)\to\mathcal{CH}_{i+d}(Z')$, stable under composition and independent of X and X' and which coincides the functor $F^!$ in case $X'=Z', X=Z$.

Proof. Given the above data there is clearly a functor $Lf^*:\mathbb{H}_{Z'}(X')^{(i)}\to\mathbb{H}_Z(X)^{(i)}$ which induces the map in the first part of the proposition. For the construction of the other map, one uses the flat map to define the pullback along projective bundle projections. Standard techniques reduces us to consider the case of a regular closed immersion f. We then need to prove that the functor $Lf^*:\mathcal{CH}_i(Z')\to C(Z)$ induced from $Lf^*:C(Z')\to C(Z)$ has essential image in $\mathcal{CH}_{i-d}(Z)$ and define the sought for functor in that manner. Instead, we follow the approach of [12] and define for the zero-section s of the projection $p:E\to X$ for a vector bundle E on X of rank d, $s^*:\mathcal{CH}_i(E)\to\mathcal{CH}_{i-d}(X)$ to be the inverse equivalence of Corollary 5.3. For a given subscheme V of dimension i on Z, consider the normal cone $C_{V\cap Z'}(V)$ in the normal bundle $N=N_{Z'}(Z)$ of the embedding $Z\to Z'$. This defines an object of $\mathcal{CH}_i(N)\overset{s^*}{\simeq}\mathcal{CH}_{k-d}(Z')$. We verify that it defines a functor and that it is compatible with the pullback on the level of $Lf^*:C(Z')\to C(Z)$. The specialization $\sigma:\mathcal{CH}_i(Z)\to\mathcal{CH}_i(N)$ can also be defined as the composition $\mathcal{CH}_k(Z)\to\mathcal{CH}_{k+1}(\mathbb{A}^1_Z)\to\mathcal{CH}_k(C)$ where the first functor is the pullback morphism, and the second functor is given by functoriality of the cokernel of i_* and the diagram

$$\begin{tikzcd} \mathcal{CH}_{k+1}(N) \arrow[r, "i_*"] \arrow[d] & \mathcal{CH}_{k+1}(M) \arrow[r, "j^*"] \arrow[d, "i^*"] & \mathcal{CH}_{k+1}(\mathbb{A}^1_Z) \arrow[dl, dashed] \\ 0 \arrow[r] & \mathcal{CH}_k(N) & \end{tikzcd}$$

where $i : N \to M$ is the embedding of N in the scheme for the deformation to the normal cone, and $j : \mathbb{A}^1_Z \to M$ is the open immersion. To see that $\operatorname{coker} i_* = \mathcal{CH}_{k+1}(\mathbb{A}^1_Z)$ it suffices to consider the homotopy sequence induced from the functors $\mathbb{H}^*$ after a choice of embedding of M into a regular scheme. This can be accomplished either directly or by applying Jouanolou-Thomason to find such an embedding. The functor i^* is defined here since it is the embedding of a Cartier divisor and can thus be defined via the argument of the proof of Proposition 5.7. The composition $i^* i_*$ is also naturally the zero functor. The rest follows the verification of [12], Proposition 5.2. Finally, if V is any subvariety of X, then $s^*[p^{-1}V] = V$ naturally for any of the two definitions of pullback functors along the zero-section of a vector bundle projection $p : E \to X$ which is again verified as in [12] so the definitions are compatible. To verify that these two constructions coincide in the case $X' = Z', X = Z$, apply the reasoning in [12], the proof of step 10 of Theorem 8.2.

□

The same construction also proves that $f^* : \mathcal{CH}_i(Z) \to \mathcal{CH}_{i+d}(Z')$ coincides with the pullback $Lf^* : C(Z') \to C(Z)$ after passing to suitable quotients as everything can be represented by explicit sheaves and one can again follow [12], the proof of step 10 of Theorem 8.2. It also proves that $F^! = f^*$ whenever F (and thus f) is flat.

5.2. **Chern intersection classes.** In this section we define Chern intersection functors on the rational Chow categories in two fashions; via rigidity and via using the filtration Fil_k introduced in the previous section. Finally we compare it to the already constructed Chern intersection functors given by Franke in [11]. We will suppose Z has the same properties as in previous section, *i.e.* that Z is a separated scheme of finite type over a regular scheme moreover admitting an ample family of line bundles, for example any quasi-projective scheme over a regular scheme.

Recall that, by the projection formula the space $((\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}})^n$ represents the functor sending a regular X to $K_0(\mathbb{P}^n_X)_{\mathbb{Q}}$. Indeed, if $p : \mathbb{P}^n_X \to X$ is the projection there is the natural isomorphism

$$K_0(\mathbb{P}^n_X) \simeq \bigoplus_{i=0}^{n-1} \mathcal{O}(-i) \otimes Lp^* K_0(X).$$

Denote now by $c_1(\mathcal{O}(j))_i \cap - = c_1(\mathcal{O}(j)) \cap$ the homomorphism projecting multiplication by $\mathcal{O}(j)$ on $K_0(X)^{(i)}$ to $K_0(X)^{(i+1)}$ and $c_1(\mathcal{O}(j))^k$ the iterated homomorphism k times. Then there is a decomposition of Adams-eigenspaces

$$K_0(\mathbb{P}^n_X)^{(i)} = \bigoplus_{j=0}^{n-1} c_1(\mathcal{O}(-1))^j \cap Lp^* K_0(X)^{(i-j)}.$$

This is a consequence of [13], Chapter III, Theorem 1.2 and a comparison with the usual first Chern class is well-known.

Definition 5.8.1. Thus by rigidity we obtain an intersection functor

$$\mathbf{c}'_1(\mathcal{O}_{\mathbb{P}^n}(1)) \cap - : \mathcal{CH}^i_{\mathbb{P}^n_Z}(\mathbb{P}^n_X) \to \mathcal{CH}^{i+1}_{\mathbb{P}^n_Z}(\mathbb{P}^n_X).$$

Given a scheme Z in a regular scheme X and a surjection $\mathcal{O}^{n+1} \xrightarrow{\phi} L$, where L is a line bundle on Z, there is an induced functor, where we use that $\mathcal{CH}_{k+n}(\mathbb{P}^n_Z) = \mathcal{CH}^{\dim X - k}_{\mathbb{P}^n_Z}(\mathbb{P}^n_X)$ and

$\mathcal{CH}_{k+n-1}(\mathbb{P}^n_Z) = \mathcal{CH}^{\dim X-k+1}_{\mathbb{P}^n_Z}(\mathbb{P}^n_X)$,

$$\mathbf{c}_1'(L,\phi)\cap - : \mathcal{CH}_k(Z) \xrightarrow{p^*} \mathcal{CH}_{k+n}(\mathbb{P}^n_Z) \overset{\mathbf{c}_1'(\mathcal{O}_{\mathbb{P}^n}(1))\cap -}{\longrightarrow} \mathcal{CH}_{k+n-1}(\mathbb{P}^n_Z) \xrightarrow{s^*} \mathcal{CH}_{k-1}(Z)$$

where s is the section to $p : \mathbb{P}^n_Z \to Z$ induced by ϕ. We call it the rigid first Chern class-functor, induced by ϕ.

Definition 5.8.2. The functor $\mathbf{c}_1(-)\cap - : \mathrm{Vect}_Z \times \mathcal{CH}_i(Z) \to \mathcal{CH}_{i-1}(Z)$ is defined as follows. For a fixed vector bundle E, $\mathbf{c}_1(E)\cap -$ is the functor

$$\mathcal{CH}_i(Z) \xrightarrow{\pi_{n-i}} G(Z) \overset{(E-\mathrm{rk}\,E)\otimes -}{\longrightarrow} G(Z) \overset{\pi_{n-i+1}}{\longrightarrow} \mathcal{CH}_{i-1}(Z)$$

where the non-trivial functors are given by the natural inclusions and projections. We call it the first Chern class-functor.

Proposition 5.9. *The rigid first Chern class-functor does not depend on the particular choice of X and ϕ and when E is a line bundle L both can be identified with the induced intersection functor*

$$(L-1)\cap - : \mathcal{CH}_i(Z) \to \mathcal{CH}_{i-1}(Z)$$

from the proof of Proposition 5.7.

Proof. To prove that the functor is not dependent on the choice of X and ϕ, it is enough to prove that the two functors coincide for $\mathcal{O}(i)$ over $\mathbb{P}^n_Z$ since the first Chern class-functor does not involve either and is compatible with projection formula together with that $p\circ s = \mathrm{id}$ where s is the section induced by ϕ in Definition 5.8.1. Now, in this case we can describe both intersection functors explicitly and for this we can suppose that $Z = X$ itself is regular. The rigid functor is characterized by compatibility with pullback and acting on the class $[\mathbb{P}^k_Z]$, where $\mathbb{P}^k_Z \subseteq \mathbb{P}^n_Z$ is a linear subspace, by restricting it to $\mathbb{P}^{k-1}_Z = \mathbb{P}^k_Z \setminus \mathbb{A}^k_Z$ on the level of Chow groups. But the same is true for multiplication by $(L-1)$ by the description in the proof of Proposition 5.7 and it is easily seen to be functorial with respect to basechange. This obviously also holds when restricting to the case $Z = X$ and X a Grassmannian and thus by rigidity they must coincide in general. □

Let X now be a noetherian scheme of finite Krull dimension of finite type over a regular scheme S. For the following proposition, consider the presheaf of $U \mapsto G_k(U)_{\mathbb{Q}}$, for $G_k(U)$ being the middle cohomology of the exact sequence defining the Chow categories, on the Grothendieck site where an open set U is an open set such that if x is a point of dimension $k' < k$, then $x \in U$ or $x \notin \overline{U}$. A pretopology is given by requiring that a covering of U is a covering of such open sets U_i with the condition that $U \setminus U_i$ has dimension less than k. Then G_k is easily seen to be a sheaf for this topology. Then the category of principal homogenous $G_{k,\mathbb{Q}}$-sheaves is equivalent to the rational Chow category $\mathcal{CH}_k(X)$ (cf. [10], section 3.1). An equivalence is set up as follows: If z is a cycle of dimension k, it is mapped to the G_k-principal homogenous sheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{CH}_k(U)}(0,z) = \{f \in E_1^{k+1,-k}(U), df = z|_U\}/E_1^{k+2,-k}(U).$$

An inverse to this functor is given by picking a rational section a_A of a G_k-sheaf A with "divisor" $\mathbf{c}(a_A)$ (cf. loc. cit. section 3.3). One can verify that this is well behaved on the level of isomorphisms as well and that it sets up an equivalence of categories. Because of the topology the construction of an intersection product is a purely local question. If f is a local section of one such principal homogenous sheaf A, and if ℓ is a local trivializing section of L, we define a local section in one dimension lower by the product

$$K_i(X) \otimes E^1_{p,q}(U) \to E^1_{p,q+i}(U).$$

Concretely, for $i = 0, 1$, $K_i(X)$ just multiplies just multiplies the groups by the virtual rank and the determinant ($\det : K_1(X) \to \mathcal{O}_X(X)^*$) along the respective points. By a Čech cohomology argument this data glues together to an object of $\mathcal{CH}_{k-1}(X)$ (cf. [11], section 1.2 and also the proposition below) which he also denotes $\mathbf{c}_1(L) \cap A$. We will see that this leads to no confusion:

Proposition 5.10. *The two above constructions define canonically isomorphic functors.*

Proof. Let K^* be a sheaf of complexes and let $\mathcal{U}$ be an open cover of X. Then there is a Čech type resolution sheaf such that the k-degree global sections have $\check{C}^k(\mathcal{U}, K^*)(V) = \oplus_{i+j=k}\check{C}^i(\mathcal{U} \cap V, K^j)$ where the parts of the right hand side denote the usual Čech resolution of K^j. The differential is given by the usual total complex differential, $d = (-1)^j d_0 + d_1$ where d_0 is the Čech differential. If there are products $K^* \otimes G \to F^*$, for K^* and F^* (resp. G) sheaves of complexes (resp. presheaf of groups) we obtain products on the Čech resolutions compatible with the products on the complexes K^* and G, denoted by $\{,\}$. In our case, let us take the complex $K^* = E^1_{*,-k}$, $G = K_1$, and $F^* = E^1_{*,-k+1}$. If L is a line bundle on X, let $\mathcal{U} = \{U_i\}$ be a covering of X with a choice of trivializing sections ℓ_i for ever cover U_i and denote by φ the collection of $\varphi_{ij} = \ell_j/\ell_i \in K_1(U_i \cap U_j)$. The element $\{a, \varphi\}$ then take values in the Čech cocycles of $\check{C}^k(E^1_{*,-k+1}, \mathcal{U})$, which is $(-1)^k$ times the naive definition. Since the sheaves $E^1_{p,q}$ are flabby the obvious spectral sequence argument shows that the Čech resolution complex actually calculate the cohomology of the complexes defining $\mathcal{CH}_k(X)$. Keeping track of the differentials, the inclusion $E^1_{j,-k}(X) \to \check{C}^j(\mathcal{U}, E^1_{*,-k})$ must be multiplied with $(-1)^j$ to obtain an isomorphism compatible with localization. In a completely analogous way we obtain a pairing $\mathrm{Fil}^k(X) \times V(X)^0 \to \mathrm{Fil}^{k-1}(X)$ where $V(X)$ denotes the full subcategory of rank 0 virtual bundles of $V(X)$ and thus a pairing $\check{C}^k(F^*, \mathcal{U}) \times \check{C}(V(-)^0, \mathcal{U}) \to \check{C}^{k-1}(F^*, \mathcal{U})$. If u is a virtual object which is trivialized over each U_i, the second Čeck differential of u in the component $V(U_{ij})$ is $u|_{U_{ij}} - u|_{U_{ij}} = 0$, which, if we trivialize u on each U_i, defines an automorphism of the zero-object which we denote by φ_{ij}. Its clear that in the above case $u = L - 1$ for a line bundle L the trivialization is given by invertible sections ℓ_i of L on U_i. Both of the products coincide with the product in [14], p. 277, because the product of a vector bundle on the complexes defining $\mathcal{CH}_*$ and the product on G-theory coincide under the natural identifications and thus the two constructions coincide for the cover $\mathcal{U}$. This also clearly respects refining the cover $\mathcal{U}$ and changing the trivializations, and thus simultaneously glue together to the same object $\mathbf{c}_1(L) \cap A$ in $\mathcal{CH}_{k-1}(X)$.

□

Theorem 5.11. *Let Z be a scheme as in the beginning of this section and A an object of $\mathcal{CH}_k(Z)$. The Chern intersection functors are additive in A and satisfy the following properties.*

- **(i):** *[Projection formula] Suppose $f : Z \to Z'$ is a proper morphism of schemes embeddable into regular schemes, and L is a line bundle on Z', there is a functorial isomorphism of functors in $\mathcal{CH}_{k-1}(Z')$*

$$f_*(\boldsymbol{c}_1(f^*L) \cap A) \simeq \boldsymbol{c}_1(L) \cap f_*A.$$

(ii): *[Additivity] If L and M are line bundles on Z, there is a canonical isomorphism*

$$\boldsymbol{c}_1(L \otimes M) \cap A \simeq \boldsymbol{c}_1(L) \cap A + \boldsymbol{c}_1(M) \cap A$$

which is commutative.

(iii): *[Commutativity] If L and M are line bundles on Z, then there is a canonical isomorphism*

$$\boldsymbol{c}_1(L) \cap (\boldsymbol{c}_1(M) \cap A) \simeq \boldsymbol{c}_1(M) \cap (\boldsymbol{c}_1(L) \cap A)$$

functorial in A.

(iv): *[Flat basechange] Let $f : Z \to Z'$ be flat. Then there is a canonical isomorphism*

$$f^*(\boldsymbol{c}_1(L) \cap A) \simeq \boldsymbol{c}_1(f^*L) \cap f^*A$$

Proof. For (i), it follows by Proposition 5.7. Thus by (i) we can assume that A is the unit object for the other properties. For (ii), the isomorphism follows from the the fact that $(L-1)\otimes(M-1)\cap-$ defines a functor $\mathrm{Fil}_k(Z) \to \mathrm{Fil}_{k-2}(Z)$ and the isomorphism $LM-L-M = (LM-1)-(L-1)-(M-1) = (L-1)\otimes(M-1)$ which is independent of choices by the fact that we have inverted two (cf. [4], 9.7.4). The commutativity follows from the isomorphism $(L-1)\otimes(M-1) \simeq (M-1)\otimes(L-1)$ which is canonical for the same reason. For (iii), suppose first that we are given two rational section l (resp. m) of L (resp. M) so that L (resp. M) is isomorphic to $\mathcal{O}(\operatorname{div} l)$ (resp. $\mathcal{O}(\operatorname{div} m)$). If they are given by irreducible effective divisors which meet properly so that their scheme-theoretic intersection is of codimension 2 they define the same element in $\mathcal{CH}_{k-2}(Z)$ by the calculation in [12], Theorem 2.4 and the general case done in ibid using blowups and induction on the excess intersection. Given another set of rational sections l' and m' the quotients l/l' and m/m' define rational functions and we obtain an isomorphism using the argument of the main result of [24] (basically Weil reciprocity) so that the data glues together. The final point follows from the basechange formula in (i) which reduces to the case $A = \mathcal{O}_Z$ where the isomorphism is obvious. □

Lemma 5.12. *For any virtual bundle V, there is a canonical isomorphism $\boldsymbol{c}_1(V) \cap - \simeq \boldsymbol{c}_1(\det V)\cap-$ compatible with, for an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$,*

$$\boldsymbol{c}_1(\det E' \otimes \det E'') \cap - = \boldsymbol{c}_1(\det E') \cap - \oplus \boldsymbol{c}_1(\det E'') \cap -$$

induced by the canonical isomorphism $\det E \simeq \det E' \otimes \det E''$.

Proof. The proof of the corresponding theorem in "Chern Functors" in [11], 1.13.2 carries over to this situation.

□

The method of "Chern Functors" in *ibid*, 1.13.2, reminiscent of Grothendieck's construction of Chern classes using the projective bundle formula for Chow groups, provides the construction of functors $\mathbf{c}_j(E) \cap - : \mathcal{CH}_i(Z) \to \mathcal{CH}_{i-j}(Z)$. Alternatively (or analogously), one defines the Segre classes by

$$\mathbf{s}_k(E) \cap A = p_*(\mathbf{c}_1(\mathcal{O}(1))^{e+j} \cap p^*A)$$

where $\operatorname{rk} E = e+1$ and define $\mathbf{c}_k(E) \cap A$ inductively via $\mathbf{c}_0(E) \cap A = A$ and

$$\sum_{i+j=k} \mathbf{c}_i(E) \cap \mathbf{s}_j(E) \cap A = 0.$$

For this we notice that there is an isomorphism $\mathbf{s}_0(E) \cap A \simeq A$.

Proposition 5.13. *Let $p : X \to S$ be a projective flat local complete intersection morphism of constant relative dimension d. Then there is a canonical isomorphism*

$$p_*(\boldsymbol{c}_1(L_1) \cap \boldsymbol{c}_1(L_2) \cap \ldots \cap \boldsymbol{c}_1(L_{d+1}) \cap p^*A) \simeq \boldsymbol{c}_1(\langle L_1, \ldots, L_{d+1}\rangle) \cap A$$

stable under the projection formula in A . Here $\langle L_1, \ldots, L_{d+1}\rangle$ is the line bundle introduced in [4] *and* [7].

Proof. By the projection formula we can suppose that $A = \mathcal{O}_S$ so that we are reduced to constructing a natural isomorphism

$$\det Rf_*((L_1 - 1) \otimes \ldots \otimes (L_{d+1} - 1)) \simeq \langle L_1, \ldots, L_{d+1}\rangle$$

of rational line bundles (i.e. for a big enough power of both sides there exists an isomorphism). The necessary work has been effectuated in [4], Construction 7.2 after the considerations in [7] where this was not effectuated because of signproblems (cf. loc. cit. "Parenthse" pp. 213-214) which we by construction ignore here. □

Remark 5.13.1. Also recall that that the remark in [11], p. 151 already implies that the construction by Franke coincides with the constructions in [7]. Since our construction coincides with that of Franke (in the equidimensional cases) this would be enough. This assertion is however not carefully written down.

The same argumentation provides an isomorphism

$$p_*(P(m) \cap p^*A) \simeq \mathbf{c}_1(\langle P(m)\rangle) \cap A$$

where $P(m)$ is a homogenous polynomial in Chern classes of degree $d+1$ and $\langle P(m)\rangle$ is the corresponding bundle considered in [7].

Proposition 5.14 (Whitney sum isomorphism)**.** *There are natural Chern intersection-functors, for any vector bundle E,*

$$\boldsymbol{c}_j(E) \cap - : \mathcal{CH}_i(Z) \to \mathcal{CH}_{i-j}(Z)$$

satisfying, for a short exact sequence $0 \to E' \to E \to E'' \to 0$ we have an equivalence of functors

$$\boldsymbol{c}_j(E) \cap - \simeq \oplus_{i=0}^{j} \boldsymbol{c}_i(E') \cap \boldsymbol{c}_{j-i}(E'') \cap -$$

and they are isomorphic to the functors in "Chern Functors" in [11]*, 1.13.2.*

Proof. We don't provide a detailed proof, but notice instead that by Proposition 5.10 the construction is basically equivalent to that of [11] and thus the same reasoning applies. □

It follows that the total Chern class $\mathbf{c}(E) \cap - = \sum \mathbf{c}_i(E) \cap -$ is a determinant functor from the category of vector bundles to the Picard category of natural transformations of $\mathcal{CH}_*$. By the universal property of virtual categories it factors over the virtual category and we can define the Chern intersection functor for any virtual bundle. In the next proposition we want to compare the above Chern intersection functors with γ-operations. Recall that we have introduced λ-operations in Proposition 3.4, and to these we can relate γ-operations via $\gamma^k(u) = \lambda^k(u + k - 1)$ as in Corollary 4.12. By the same reasoning as in the last mentioned corollary, the functors $\gamma^j(E - \operatorname{rk} E) \cap - : \operatorname{Fil}_i(Z) \to C(Z)$ factor through $\operatorname{Fil}_{i-j}(Z)$ and have the same distributive property as in the Whitney sum isomorphism. With this we have:

Corollary 5.15. *Let V be a virtual vector bundle on Z. The Chern intersection functors coincide with the functors determined by γ-operations, i.e. the functor*

$$\boldsymbol{c}_j(V) \cap - : \mathcal{CH}_i(Z) \to \mathcal{CH}_{i-j}(Z)$$

is canonically equivalent to the functor

$$\gamma^j(V - \operatorname{rk} V) \cap - : F_i(Z)/F_{i-1}(Z) \to \operatorname{Fil}_{i-j}(Z)/\operatorname{Fil}_{i-1-j}(Z).$$

Proof. One needs to compare the two determinant functors $V(Z) \to \operatorname{Hom}(\mathcal{CH}_*(Z), \mathcal{CH}_*(Z))$ determined by the total Chern class and the corresponding "total γ"-class, defined as $\sum_{i=0}^{\infty} \gamma^i(V - \operatorname{rk} V) \cap -$. This follows from the splitting principle and the fact that that they coincide for line bundles. □

Appendix A. $\mathbb{A}^1$-homotopy theory of schemes

This section is to recall some necessary results and and to fix some notation regarding . In what follows we have but slight extensions of the theorems in the reference-list, and we hope the reader agrees that not spelling out the proofs does not cause any harm. One word of warning though, we have almost completely ignored issues related to smallness of categories. This can be amended by inserting the word "universe" at the appropriate places.

Denote by Δ the category of totally ordered finite sets and monotonic maps. Hence, the objects are the finite sets $[n] = \{0 < 1 < 2 < \ldots < n\}$ and the morphisms of Δ are generated by the face and degeneracy maps

$$\delta_i : [n-1] \to [n], \text{ defined by } \delta_i(j) = \begin{cases} j, & \text{if } j < i \\ j+1, & \text{if } j \geq i \end{cases}$$

and

$$\sigma_i : [n] \to [n-1], \text{ defined by } \sigma_i(j) = \begin{cases} j, & \text{if } j \leq i \\ j-1, & \text{if } j > i \end{cases}$$

which satisfy the usual simplicial relationships ([16], chapter 1). If $\mathcal{C}$ is any category, we denote by $s\mathcal{C}$ or $\Delta^{\mathrm{op}}\mathcal{C}$ the category of simplicial objects of $\mathcal{C}$, *i.e.* the category whose objects are functors $\Delta^{\mathrm{op}} \to \mathcal{C}$, and morphisms are natural transformations of functors.
Let T be a site, and denote by $\mathbf{Shv}(T)$ the category of sheaves of sets on T, and $\Delta^{\mathrm{op}} \mathbf{Shv}(T)$ the category of simplicial sheaves. Note that if we are given a simplicial set E, we can associate to it the constant simplicial sheaf, which we also denote by E, and thus we obtain a functor

$$\Delta^{\mathrm{op}} \operatorname{Set} \xrightarrow{\text{constant}} \Delta^{\mathrm{op}}\mathbf{Shv}(T).$$

The standard n-simplices Δ^n define thus by the Yoneda lemma a cosimplicial object

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^\bullet} & \Delta^{\mathrm{op}} \mathbf{Shv}(T) \\ & & \\ n & \mapsto & \Delta^n \end{array}$$

and we give the category $\Delta^{\mathrm{op}} \mathbf{Shv}(T)$ the structure of a simplicial category with a simplicial function object $\mathbf{hom}(-,-)$ given by

$$\mathbf{hom}(\mathcal{X}, \mathcal{Y}) := \operatorname{Hom}_{\Delta^{\mathrm{op}} \mathbf{Shv}(T)}(\mathcal{X} \times \Delta^\bullet, \mathcal{Y}).$$

Before continuing, we recall the fundamental lemma of homotopical algebra

Theorem A.1. *[*[16]*, II.3.10] Let $\mathcal{C}$ be a closed simplicial model category with associated homotopy-category $\mathcal{H}$, and $\mathcal{X}, \mathcal{Y} \in \mathrm{ob}(\mathcal{C})$. Suppose furthermore that $\mathcal{X}' \to X$ is a trivial fibration with $\mathcal{X}'$ cofibrant and $\mathcal{Y} \to \mathcal{Y}'$ is a trivial cofibration with $\mathcal{Y}'$ fibrant. Then we have a natural identification*

$$\hom_{\mathcal{H}}(\mathcal{X}, \mathcal{Y}) = \pi_0(\mathbf{hom}(\mathcal{X}', \mathcal{Y}')).$$

An adjoint to the functor $\Delta^{\mathrm{op}}\,\mathrm{Set} \to \Delta^{\mathrm{op}}\mathbf{Shv}(T)$ given by $X \mapsto \mathbf{hom}(*, X)$, which we sometimes write as $X \mapsto |X|$.

Definition A.1.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial pre-sheaves. Then

(a) f is said to be a (simplicial) weak equivalence if, for any conservative family $\{x : \mathbf{Shv}(T) \to \mathrm{Set}\}$ of points of T, $x(f) : x(\mathcal{X}) \to x(\mathcal{Y})$ is a homotopy-equivalence of simplicial sets.
(b) f is called a cofibration if it is a monomorphism.
(c) f is called a fibration if it has the right lifting property with respect to trivial cofibrations, *i.e.* cofibrations which are also weak equivalences.

Theorem A.2. *[*[30]*, Theorem 2.1.4] For any (small) site with enough points T, the above equips $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ with the structure of a closed model category.*

We denote by $\mathcal{H}_s(T)$ the corresponding homotopy-category obtained by inverting the weak equivalences in $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$. To fix ideas, unless explicitly mentioned, from here on S will denote a regular scheme and T a full subsite with enough points of Sch/S_{sm} the category of S-schemes equipped with the smooth topology [2] , and denote the corresponding homotopy-category by $\mathcal{H}_s(T)$. Most often, we will be concerned with the category $\mathfrak{R}_S$ of regular S-schemes with the smooth topology. When $S = \operatorname{Spec}\mathbb{Z}$, we write $\mathfrak{R}_{\mathbb{Z}} = \mathfrak{R}$. Since any smooth morphism locally for the tale topology has a section we can identify the various topoi of sheaves of regular S-schemes with tale or smooth topology or of affine regular S-schemes with the tale or smooth topology with a "big regular tale S"-topoi. They are given a conservative set of points by regular local strict henselian rings.

Definition A.2.1. Suppose T is such that for any $X \in ob(T)$, $\mathbb{A}^1_X$ is also an object in T. We say that $\mathcal{X} \in \mathcal{H}_s(T)$ is $\mathbb{A}^1$-local with respect to T, if for any $\mathcal{Y} \in \mathbf{Shv}(T)$ the map

$$\mathrm{Hom}_{\mathcal{H}_s(T)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(\mathcal{Y}, \mathcal{X})$$

is bijective. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ is $\mathbb{A}^1$-local if for any $\mathbb{A}^1$-local object $\mathcal{Z}$, the natural map

$$\mathrm{Hom}_{\mathcal{H}_s(S)}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}_{\mathcal{H}_s(T)}(\mathcal{X}, \mathcal{Z})$$

is bijective. Now equip $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ with $\mathbb{A}^1$-local weak equivalences, cofibrations and $\mathbb{A}^1$-local fibrations. Then we have:

Theorem A.3 ([30], Theorem 2.3.2)**.** *This equips $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)$ with the structure of a closed model-category.*

[2] *i.e.* a full subcategory such that any open cover of T is an open cover of Sch/S_{Sm}.

Definition A.3.1. We denote the corresponding homotopy category by $\mathcal{H}(T)$. Whenever $T = Sm/S_{Nis}$, the corresponding homotopy-category is the $\mathbb{A}^1$-homotopy category of schemes over S defined by loc.cit., but it will not directly play a role in what we do. When the site is $T = \mathfrak{R}_{sm}$, the category of regular schemes with the smooth topology, the corresponding homotopy category is denoted by $\mathcal{H}(\mathfrak{R})$. We also have natural pointed analogues. Replacing in all previous definitions pointed versions, we obtain the $\mathbb{A}^1$-homotopy category of pointed simplicial sheaves $\mathcal{H}_\bullet(T)$ as a localization of the category of pointed simplicial sheaves; $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)_\bullet$. For two objects $(\mathcal{X}, x), (\mathcal{Y}, y) \in \mathcal{H}_\bullet(T)$ we define $X \wedge Y$ in the usual way as the coequalizer of

$$\mathcal{X} \times y, x \times \mathcal{Y} \rightrightarrows \mathcal{X} \times \mathcal{Y}.$$

For a simplicial sheaf $\mathcal{X}$, we denote by $\mathcal{X}_+$ the simplicial presheaf with a disjoint point. The functor $\mathcal{X} \to \mathcal{X}_+$ is left adjoint to the forgetful functor $\mathcal{H}_\bullet(T) \to \mathcal{H}(T)$.

The stable homotopy-category of schemes is stabilized out of the "unstable" one with the proper notion of a circle. As before, let T denote a (small) site with enough points.

Definition A.3.2. Let $\mathbf{T} \in \Delta^{\mathrm{op}}\,\mathbf{Shv}(T)_\bullet$. A $\mathbf{T}$-spectra is a set $\mathbf{E} = (d_n, \mathbf{E}_n)_{n \in \mathbb{N}}$ of objects in $\Delta^{\mathrm{op}}\,\mathbf{Shv}(T)_\bullet$ with morphisms

$$d_n : \mathbf{T} \wedge \mathbf{E}_n \to \mathbf{E}_{n+1}.$$

A morphism of $\mathbf{T}$-spectra $f : \mathbf{E} \to \mathbf{F}$ is a set of morphisms $f_n : \mathbf{E}_n \to \mathbf{F}_n$ such that the diagram commutes

$$\begin{tikzcd} \mathbf{T} \wedge E_n \arrow[r] \arrow[d] & E_{n+1} \arrow[d] \\ \mathbf{T} \wedge F_n \arrow[r] & F_{n+1} \end{tikzcd}.$$

Definition A.3.3. Let $\mathbf{E}$ be a $\mathbf{T}$-spectra, and denote by $\Omega_{\mathbf{T}}(-) = \Omega(-) = R\mathrm{Hom}(\mathbf{T}, -)$ the total derived functor (in $\mathcal{H}_\bullet(S)$) of the right adjoint to $\mathbf{T} \wedge -$. We say that $\mathbf{E}$ is a Ω-spectra if for any n the induced morphism

$$\mathbf{E}_n \to \Omega(\mathbf{E}_{n+1})$$

is in fact an isomorphism. We can naively construct "a" stable homotopy-theory by taking the category of Ω-spectras with respect to $\mathbf{T} = (\mathbb{P}^1, \infty)$, and denote it by $\mathcal{SH}_{naive}(T)$, and giving morphisms $E \to F$ by morphisms $E_n \to F_n$ in $\mathcal{H}_\bullet(S)$ for any n such that the obvious diagram commutes (cf. [34], Dfinition I.124).

Definition A.3.4. Let be a morphism $f : \mathbf{E} \to \mathbf{F}$ of $\mathbf{T}$-spectras. Then f is a projective cofibration if f_0 is a monomorphism and for any $n > 0$,

$$\mathbf{T} \wedge \mathbf{F}_n \bigvee_{\mathbf{T} \wedge \mathbf{E}_n} \mathbf{E}_{n+1} \to \mathbf{F}_{n+1}$$

is also a monomorphism. Its an $\mathbb{A}^1$-projective fibration (resp. $\mathbb{A}^1$-projective equivalence) if every map f_n is a $\mathbb{A}^1$-fibration (resp. $\mathbb{A}^1$-weak equivalence).

Theorem A.4 ([34], Premire partie). *Let $\mathbf{T} = (\mathbb{P}^1, \infty)$. The category of $\mathbf{T}$-spectras equipped with projective cofibrations as cofibrations, $\mathbb{A}^1$-projective fibrations as fibrations and $\mathbb{A}^1$-projective equivalences as weak equivalences is a closed model-category.*

Definition A.4.1. Let $\mathbf{T} = (\mathbb{P}^1, \infty)$. Then the stable homotopy-category $\mathcal{SH}(T)$ is the full subcategory, of the corresponding homotopy-category, of Ω-spectras.

Definition A.4.2. For a fixed scheme S, let $\mathrm{Gr}_{d,r}$ be the Grassmannian of locally free quotients of rank r of $\mathcal{O}_S^{d+r}$ viewed as an object of $\mathrm{Shv}(T)$. Notice that $\mathrm{Gr}_{d,r} \simeq \mathrm{Gr}_{r,d}$. Let $\mathcal{F}$ be a locally free sheaf of rank r. We have natural morphisms $\mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d+1,r}$ and $\mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d,r+1}$ by sending $\phi : \mathcal{O}^{d+r} \twoheadrightarrow \mathcal{F}$ to $\mathcal{O}^{d+r+1} \overset{(\phi,0)}{\twoheadrightarrow} \mathcal{F}$ and $\mathcal{O}^{d+r+1} \overset{\phi,\mathrm{id}}{\twoheadrightarrow} \mathcal{F} \oplus \mathcal{O}$ respectively. We denote by $\mathrm{Gr}_d = \lim_{\to} \mathrm{Gr}_{d,r}$ and $\mathrm{Gr} = \lim_{\to} \mathrm{Gr}_d$ for these maps. Here the direct limits are taken in $\mathbf{Shv}(T)$. Since all things here naturally pointed (by $\mathrm{Gr}_{\mathrm{d},0}$ for any d), we also obtain a pointed element $\mathrm{Gr} \in \mathcal{H}_\bullet(T)$. Notice that $\mathbb{P}^d = \mathrm{Gr}_{d,1} \simeq \mathrm{Gr}_{1,d}$ and denote by $\mathbb{P}^\infty = \mathrm{Gr}_1$.

By the method of [34], Dfinition III.101, it is possible to define a sheaf $(\mathbb{Z} \times \mathrm{Gr})[\frac{1}{n}]$ and $(\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}$ with a natural morphism $\mathbb{Z}\times\mathrm{Gr} \to (\mathbb{Z}\times\mathrm{Gr})[\frac{1}{n}]$ and $\mathbb{Z}\times\mathrm{Gr} \to (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}$. In a similar fashion to *loc. cit.* , to lax notation first put $\mathrm{Gr}_{d,r} = \mathrm{Gr}^{d+r,r}$ so that $\mathbb{P}^d = \mathrm{Gr}^{d+1,1}$ and define a morphism $m_{a,d} : \mathrm{Gr}^{d,1} \to \mathrm{Gr}^{d^a,1}$ by sending a surjection $p : \mathcal{O}^d \twoheadrightarrow \mathcal{L}$ to $p^{\otimes a} : (\mathcal{O}^d)^{\otimes a} \twoheadrightarrow \mathcal{L}^{\otimes a}$. One verifies the relation

$$\begin{array}{ccc} \mathrm{Gr}^{d,1} & \xrightarrow{m_{a,d}} & \mathrm{Gr}^{d^a,1} \\ \downarrow & & \downarrow \\ \mathrm{Gr}^{d+1,1} & \xrightarrow{m_{a,d+1}} & \mathrm{Gr}^{(d+1)^a,1} = \mathrm{Gr}^{(d^a+[(d+1)^a-d^a],1} \end{array}$$

and define $m_a : \mathbb{P}^\infty \to \mathbb{P}^\infty$ to be the induced morphism. The relation $m_{ab} = m_a m_b$ is easy. [3]

Definition A.4.3. One defines $\mathbb{P}^\infty[\frac{1}{n}]$ (resp. $\mathbb{P}^\infty_{\mathbb{Q}}$) as the inductive limit over m_a's ordered by division for $a = n^k, k \in \mathbb{N}$ (resp. m_a's ordered by division for all $a \in \mathbb{N}$).

One of the main observations of [30] is the following theorem, which states that algebraic K-theory is represented by an infinite Grassmannian. The version presented below is proven in exactly the same way as in the article in question, with the exception of using smooth descent for rational K-theory instead of Nisnevich descent. Note that since any smooth morphism locally for the tale topology has a section we have tale descent whenever we have smooth descent, and in the former case the statement we are looking for is [42], Theorem 11.11;

Theorem A.5 ([30], Theorem 4.3.13)**.** *Let S be a regular scheme. Then we have canonical functorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}_\bullet(\mathfrak{R}_{S,sm})}(S^n \wedge X_+, (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}) = \mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_{S,sm})}(X, \Omega^n\left(\mathbb{Z}\times\mathrm{Gr}\right)_{\mathbb{Q}}) \simeq K_n(X)_{\mathbb{Q}}$$

for X a regular S-scheme, where K_n refers to Quillen's K-theory defined as above. In particular, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_{S,sm})}(X, (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}) \simeq K_0(X)_{\mathbb{Q}}.$$

Proceeding as in [34], Chapitre III, one constructs a product

$$(\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \wedge (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z}\times\mathrm{Gr})_{\mathbb{Q}}$$

in $\mathcal{H}_\bullet(\mathfrak{R}_{S,sm})$.

---

[3]To make the above a proper definition and make the diagram commute on the nose, one needs to define a natural isomorphism $\delta_{a,d} : (\mathcal{O}^d)^{\otimes a} \to \mathcal{O}^{d^a}$. It can be done as follows. We define a strict total order on $\{e_1^1, e_2^1, \ldots e_d^1\} \times \ldots \times \{e_1^a, e_2^a, \ldots e_d^a\}$, *i.e.* the structure of the category $[n^a - 1]$ inductively as follows. First $e^1_{i_1} \times e^2_{i_2} \times \ldots \times e^a_{i_a} < e^1_{j_1} \times e^2_{j_2} \times \ldots \times e^a_{j_a}$: If $\max i_k < \max j_l$. If there is equality $i_m = \max i_k = \max j_l = j_n$, then if $\max i_k \backslash i_n < \max j_l \backslash j_m$. Repeatedly removing such m, n's we obtain an order on all objects except when the i_k are a permutation of the j_l's. With these, pick the lexicographic order. We then define an isomorphism $\delta_{a,d} : (\mathcal{O}^d)^{\otimes a} \to \mathcal{O}^{d^a}$ by sending a basis-element $e^1_{i_1} \times e^2_{i_2} \times \ldots \times e^a_{i_a}$ to the basis f_i with $i \in [n^a - 1]$ via the ordering just constructed.

Proposition A.6. *Consider the natural map* $t : \mathbb{P}^1 \to \{0\} \times \mathrm{Gr} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$. *Then the data* $\mathbf{E} = (\mathbf{E}_i, d_i)$ *defined by* $\mathbf{E}_i = \mathbb{Z} \times \mathrm{Gr}$ *and the product*

$$d_i : \mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \overset{t \wedge \mathrm{id}}{\to} (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \wedge (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$$

is a naive spectrum, which we denote by K_{naive}.

Proof. We need to show that the natural map

$$(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \to R\mathrm{Hom}_{\bullet}((\mathbb{P}^1, \infty), (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}})$$

is an isomorphism. However, this follows from the fact that for any S-scheme X, the map

$$K_n(X) \to \{y \in K_n(\mathbb{P}^1_X), \infty^* y = 0 \in K_n(X)\}$$

given by $x \mapsto x \boxtimes u$, where $u = \mathcal{O}(1) - 1$, is bijective, which in turn is a consequence of the projective-bundle-formula for K-theory. □

Let S be a Noetherian, regular scheme. By the Yoneda lemma, we have a functor

$$\Phi : T \to \mathbf{Shv}(T) \to \Delta^{\mathrm{op}}\, \mathbf{Shv}(T) \to \mathcal{H}(T).$$

If G is any object of $\mathcal{H}(T)$, we denote by ϕG the presheaf on T defined by

$$T \ni U \mapsto \mathrm{Hom}_{\mathcal{H}(T)}(\Phi U, G).$$

In particular, we have an isomorphism

$$\phi(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}} \simeq K_0(-)_{\mathbb{Q}}.$$

Theorem A.7 (Thorme III.29 in [34])**.** *Let S be a regular scheme. Given two (pointed) presheaves $\mathcal{F}, \mathcal{G}$ on $\mathfrak{R}_S$ denote by* $\mathrm{Hom}_{\mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(\mathcal{F}, \mathcal{G})$ *(resp.* $\mathrm{Hom}_{\bullet, \mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(\mathcal{F}, \mathcal{G})$*) the set of (pointed) natural transformations from $\mathcal{F} \to \mathcal{G}$. Then the natural morphism*

$$\mathrm{Hom}_{\mathcal{H}(\mathfrak{R}_{S,sm})}((\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}, (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}) \to \mathrm{Hom}_{\mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}})$$

(resp.

$$\mathrm{Hom}_{\mathcal{H}_{\bullet}(\mathfrak{R}_{S,sm})}((\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}, (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}) \to \mathrm{Hom}_{\bullet, \mathfrak{R}_S^{\mathrm{op}}\,\mathrm{Set}}(K_0(-)_{\mathbb{Q}}, K_0(-)_{\mathbb{Q}}))$$

is bijective.

Theorem A.8. *[Thorme IV.72 in* [34]*] We have a natural decomposition in terms of "Adams eigenspaces",*

$$\mathbb{Z} \times \mathrm{Gr}_{\mathbb{Q}} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{H}^{(i)}$$

in $\mathcal{H}(\mathfrak{R}_S)_{\bullet}$. More precisely there is a decomposition in a certain stable homotopy category $\mathcal{SH}(\mathfrak{R}_S)$ of a stable model

$$\mathbf{BGL}_{\mathbb{Q}} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{H}^{(i)}.$$

Definition A.8.1. Let $\mathcal{C}$ be a closed simplicial model category, and suppose that $\mathcal{X}$ is an object of $\mathcal{C}$. Given a fibrant replacement $\mathcal{X} \to \mathcal{X}'$, consider the functor $V_{\mathcal{X}}$ taking an object X of $\mathcal{C}$ to the fundamental groupoid of $\mathbf{hom}(X, \mathcal{X}')$. This is independent up to unique isomorphism of the choice of fibrant replacement by abstract nonsense. We call $V_{\mathcal{X}}$ the associated category fibered in groupoids over $\mathcal{C}$. A 1- and 2-morphism of categories fibered in groupoids over $\mathcal{C}$ is the standard one and we denote by $\mathrm{Hom}_f(V_{\mathcal{X}}, V_{\mathcal{Y}})$ the set of 1-morphisms $V_{\mathcal{X}} \to V_{\mathcal{Y}}$ strictly functorial with respect to pullback.

Very often, these groupoids have the structure of Picard categories and they form Picard categories fibered in $\mathcal{C}$. Recall from [11], 3.6 that a Picard category fibered over $\mathcal{C}$, P, is, for every object X of $\mathcal{C}$, a Picard category P_X and for every morphism $X \to Y$ an additive functor $P_Y \to P_X$ compatible with composition in the obvious sense.
The following proposition is formal, and is surely known in more generality:

Proposition A.9. *[Pre-rigidity, proof of [34], Chapitre III, section 10] Let T be as above and consider the category of pointed or unpointed simplicial (pre-)sheaves on T. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are objects thereof, with $\mathcal{X}$ cofibrant and $\mathcal{Y}$ fibrant, with associated fibered categories in groupoids $V_{\mathcal{X}}, V_{\mathcal{Y}}$, and suppose that*

- $\mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(\mathcal{X}, \Omega\mathcal{Y}) = 0.$
- $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$ *is an H-group.*

Then we have a canonical map

$$\mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_f(V_{\mathcal{X}}, V_{\mathcal{Y}})$$

which associates to an element of the left a functor of fibered categories $\phi : V_{\mathcal{X}} \to V_{\mathcal{Y}}$, canonical up to unique isomorphism.

Proof. If Φ is in $\mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(\mathcal{X}, \mathcal{Y}) = \pi_0(\mathbf{hom}(\mathcal{X}, \mathcal{Y}))$, it induces for any $X \in \mathcal{C}$ a map $\phi_X : \mathcal{V}_{\mathcal{X}}(X) \to \mathcal{V}_{\mathcal{Y}}(X)$, functorial in X, by choice of a representative ϕ of Φ in $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$. If ϕ and ϕ' induce the same homotopy-class, there is a homotopy $h : \Delta^1 \times \mathcal{X} \to \mathcal{Y}$ from ϕ to ϕ' which gives an isomorphism $\mathrm{iso}_{h,X} : \phi_X \to \phi'_X$. Moreover, it is easy to see that if there are two homotopies h and h' which are homotopic, they induce the same isomorphism of functors. The obstruction for $\mathrm{iso}_{h,X}$ to be canonical lies in the fundamental group of $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$ which can be identified with $\mathrm{Hom}_{\mathcal{H}(\mathcal{C})}(\mathcal{X}, \Omega\mathcal{Y})$ which is 0 by assumption. □

Appendix B. Algebraic stacks

In this section we recall the necessary facts about algebraic stacks that will be needed. It is neither self-contained nor complete, and we refer the reader to for example [26] or [1] for more exhaustive treatments. We refer to Dfinition 3.1 and Dfinition 4.1, [26] for the necessary definitions of algebraic stacks. In particular an S-stack is a sheaf in groupoids on $(Aff/S)_{et}$. Furthermore, let T be a full subsite of $(Aff/S)_{et}$ (*i.e.* a full subcategory with a Grothendieck topology such that a cover in the former is one in the latter). By abuse of language, we say that that a category fibered in groupoids over T is resp. a stack, an algebraic stack or Deligne-Mumford stack if it is the restriction of a stack, algebraic stack or Deligne-Mumford stack.

Again to fix notation we recall [26], Application 14.3.4:

Definition B.0.1. We say that a representable 1-morphism $F : X \to Y$ of algebraic stacks is projective (or quasi-projective) if there is a coherent locally free sheaf $\mathcal{E}$ on Y and 2-commutative diagram

$$\begin{tikzcd} X \arrow[r, "I"] \arrow[dr] & \mathbb{P}(\mathcal{E}) \arrow[d, "P"] \\ & Y \end{tikzcd}$$

with I a (representable) closed immersion (resp. quasi-compact immersion) and P is the canonical projection.[4]

By [41] there are many examples of when an equivariant scheme which is projective as a scheme is also equivariantly projective. Given an algebraic stack $\mathcal{X}$, an algebraic space X and a fppf-morphism $X \to \mathcal{X}$, the associated groupoid $[X_1 \rightrightarrows X_0]$ is an fppf-presentation of $\mathcal{X}$ (cf. [26], Corollaire 10.6). We now recast the above in a setting which will make it more natural to apply various auxiliary results, which is that of a simplicial setting.

Definition B.0.2. Let T be a site. The category of presheaves and sheaves on this site is denoted by $pShv(T)$ and $Shv(T)$ respectively.

The category of simplicial objects of a category $\mathcal{C}$, *i.e.* functors $\Delta^{op} \to \mathcal{C}$, is denoted by $\Delta^{op}\mathcal{C}$ or $s\mathcal{C}$.

Recall that whenever T has enough points a morphism of simplicial presheaves in T is said to be a local equivalence if it induces weak equivalences of simplicial sets on all stalks.
Let $U \to X$ be a morphism of an object X in T. The nerve of this morphism is the simplicial object $\mathcal{N}(U/X)$ whose n-simplices are given by the product $U \times_X U \times_X U \ldots \times_X U$ (n times). Given a presheaf of simplicial sets on T we have an associated cosimplicial functor $\Delta \to \mathrm{Set}$, $[n] \mapsto \mathcal{F}(\mathcal{N}(U/X)_n)$. The Cech cohomology with respect to the covering $U \to X$ is the simplicial set

$$\mathbf{H}(U/X, \mathcal{F}) := \operatorname{holim}_{\Delta} \mathcal{F}(\mathcal{N}(U/X)_n).$$

We say that $\mathcal{F}$ satisfies descent if for any X and any covering $U \to X$ in T, the map

$$\mathcal{F}(X) \to \mathbf{H}(U/X, \mathcal{F})$$

is a weak equivalence.

Definition B.0.3. A presheaf $\mathcal{F}$ of simplicial sets on a site T is said to be flabby, if for any (and thus each) simplicially fibrant replacement $\mathcal{F} \to \mathcal{F}'$, and any $X \in T$, the map

$$\mathcal{F}(X) \to \mathcal{F}'(X)$$

is a weak equivalence of simplicial sets.

Theorem B.1 ([43], Thorme 1.2)**.** *$\mathcal{F}$ is flabby if and only if it satisfies descent.*

Thus any simplicially fibrant simplicial presheaf satisfies descent. It follows from the definition that a groupoid is flabby if and only if it is a stack.
If $\mathcal{X}$ is an S-stack, there is sheaf of simplicial sets defined as follows: Let U be an object in (Aff/S), and let $\overline{\mathcal{X}}$ be the associated fibered category over (Aff/S). The category $F_{\mathcal{X}}(U) := Hom_{Cat/S}(\overline{U}, \mathcal{X})$ is a groupoid, and its nerve is a simplicial set $BF_{\mathcal{X}}$.

Definition B.1.1. Let T be a site, and consider the category of simplicial presheaves on T, $\Delta^{op}p\,\mathbf{Shv}(T)$. The full subcategory of simplicial sheaves is denoted by $\Delta^{op}\,\mathbf{Shv}(T)$. If $\mathfrak{Ch}$ is the category of stacks on T, we call the functor $B : \mathfrak{Ch}(T) \to \Delta^{op}\,\mathbf{Shv}(T)$ constructed above the extended Yoneda functor.

[4]This is what many authors call a "strongly projective" (or "strongly quasi-projective") 1-morphism.

Furthermore, a (cartesian) quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module on an algebraic stack $\mathcal{X}$ viewed as a simplicial set is an assignment of a quasi-coherent (resp. coherent, locally free, etc) $\mathcal{F}_n$ on each $\mathcal{X}_n$ such that for any $\phi : [n] \to [m]$ we have an isomorphism $\phi^* : \phi^* \mathcal{F}_n \to \mathcal{F}_m$ compatible with compositions $[n] \to [n'] \to [n'']$. Coherent and locally free sheaves are defined analogously.
As an example (cf. [2], 6.1.2), let G be a group scheme, finitely presented, separated and faithfully flat over a scheme S. Let X be an algebraic space over S. We say that G acts on X if there is a morphism $\mu : G \times_S X \to X$ satisfying the usual associativity and unit-constraints. If $\mathcal{F}$ is a $\mathcal{O}_X$-module, we say that G acts on $\mathcal{F}$, or that $\mathcal{F}$ is G-equivariant, if there is an isomorphism of $\mathcal{O}_{G \times_S X}$-modules

$$\phi : \mu^* \mathcal{F} = p_2^* \mathcal{F}$$

satisfying the associativity constraint, on $G \times_S G \times_S X$:

$$p_{23}^* (1 \times \mu)^* \phi = (\mu \times 1)^* \phi.$$

We employ the analogous definition for complexes of quasi-coherent $\mathcal{O}_X$-modules. To an algebraic space X with a group action G, we can form the following simplicial algebraic space:

$$[X/G/S] := X \; \substack{\longrightarrow \\ \longleftarrow \\ \longleftarrow} \; G \times_S X \; \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow} \; G \times_S G \times_S X \ldots$$

Here the maps are either projection or multiplication-maps, and the non-written arrows in the other directions are given by repeated applications of the unit-map e. The above condition that $\mathcal{F}$ is G-equivariant can equivalently be rephrased as that $\mathcal{F}$ is the degree 0-part of a cartesian $\mathcal{O}_{[X/G/S]}$-module on $[X/G/S]$ with descent-data.

Yet another way of defining a quasi-coherent $\mathcal{O}_{\mathcal{X}}$ on an algebraic stack $\mathcal{X}$, is in the following way: Given an algebraic space U and a 1-morphism with U an algebraic space, $s : U \to \mathcal{X}$, we have a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_s$ on U. Given two 1-morphisms of algebraic spaces $s : U \to \mathcal{X}, t : V \to \mathcal{X}$, a morphism $f : U \to V$, and a 2-isomorphism $h : t \circ f \Rrightarrow s$, an isomorphism

$$\phi_{f,t,h} : f^* \mathcal{F}_t \simeq \mathcal{F}_s.$$

Given morphisms of algebraic spaces $U \xrightarrow{f} V \xrightarrow{g} W$, and 1-morphisms $s : U \to \mathcal{X}, t : V \to \mathcal{X}, w : W \to \mathcal{X}$, and 2-isomorphisms $h : t \circ f \Rrightarrow s$ and $j : w \circ g \Rrightarrow t$ an equality

$$\phi_{f,t,h} \circ f^* \phi_{g,w,j} = \phi_{f \circ g, w, h \circ j}.$$

Given two quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ and $\mathcal{E}$, a morphism between them is morphism $\mathcal{F}_s \to \mathcal{E}_s$ for every morphism $s : U \to \mathcal{X}$ with U an algebraic space compatible with the isomorphism ϕ in the obvious way.

Definition B.1.2. The Quillen K-theory space of an algebraic stack $\mathcal{X}$, $K(\mathcal{X})$ is defined to be the space ΩBQC, with C being the exact category of (coherent) vector bundles on X. The K-theory groups $K_i(\mathcal{X})$ are defined to be π_i of the corresponding loops-space ΩBQC. Similarly, one defines the G-theory space and G-theory of an algebraic stack $\mathcal{X}$, $G_i(\mathcal{X})$, as the corresponding object considering the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules instead.

The main standard properties of $K-$ and G-theory are summarized in the following theorem (compare with [43], Proposition 2.2, note however that it does not seem to be true that most of the results in this proposition automatically generalize from the case of schemes. Indeed, this is the main point of the article [40] where the equivariant versions of non-cohomological K and G-theory are studied):

Theorem B.2. *Fix a separated algebraic stack $\mathcal{X}$. Then we have*

- $K(-)$ *is contravariantly functorial with respect to 1-morphisms of algebraic stacks, and is covariantly functorial with respect to representable projective morphisms between algebraic stacks with the resolution property.*
- $G(-)$ *is covariantly functorial with respect to proper representable 1-morphisms.*
- *Let $\mathcal{E}$ be a vector bundle of rank n on X, and consider the canonical bundle $\mathcal{O}(1)$ on $\pi : Proj_X(Sym^\bullet \mathcal{E}) = \mathbb{P}(\mathcal{E}) \to \mathcal{X}$. Then we have a homotopy equivalence*
$$\bigvee_{j=0}^{n-1} K(X) \to K(\mathbb{P}(\mathcal{E}))$$
induced by $(f_j)_{j=0}^{n-1} \mapsto \sum_{j=0}^{n-1} \pi^* f_j \otimes \mathcal{O}(-j)$. Same formula holds for G.
- *Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$, and T a torsor of $\mathcal{E}$ over $\mathcal{X}$. Then $G(\mathcal{X}) \to G(T)$ is a homotopy equivalence.*

Proof. The first result is proven as in [40], Theorem 3.1. and most of the results are proven using the classical techniques or modifying the same using loc.cit. As we shall only need the above theorems in the special cases of their associated virtual categories we will contend ourselves with the above statements without proofs. □

An additional object will enter onto our stage, K-cohomology, which in this form is borrowed from [43].

Definition B.2.1. Let $T = Aff/S_{sm}$, the category of affine S-schemes with the smooth topology. Denote by $K_{\mathbb{Q}}^{TT}$ a T-simplicially fibrant model of the simplicial presheaf on T that represents rational Thomason algebraic K-theory and let X be a simplicial T-sheaf. The K-cohomology K^{sm} is the simplicial presheaf (automatically flabby) $X \mapsto K^{sm}(X) := \mathbf{hom}(X, K_{\mathbb{Q}}^{TT})$. We define the K-cohomology groups $K_i^{sm}(X)$ to be $\pi_i(\mathbf{hom}(X, K_{\mathbb{Q}}^{TT}))$. Also define $G_{\mathbb{Q}}^{sm}$ to be the G-cohomology of [43].

The definition of $K^{sm}(X)$ of [43] is different, and exhibits $K^{sm}(\mathcal{X})$ more properly as a S^1-spectrum. But by *ibid* Proposition 2.2, the given spectrum is flabby when restricted to the small smooth site on the algebraic stack (*i.e.* a smooth presentation is a cover) and equal to ordinary (rational) Thomason K-theory for a regular Noetherian finite dimensional algebraic space or scheme. Because holim preserves weak equivalences, for a regular stack with smooth presentation $X \to \mathcal{X}$, we have weak equivalences $K^{sm}(\mathcal{X}) = \mathbf{H}(X/\mathcal{X}), K^{sm}) = \mathbf{H}(X/\mathcal{X}, K_{\mathbb{Q}}^{TT}) = \mathbf{hom}(\mathcal{X}, K_{\mathbb{Q}}^{TT})$ so Toen's K-cohomology necessarily coincides with our K-cohomology in this case. Also recall that for a scheme in addition to being finite dimensional Noetherian admit an ample family of line bundles $K_{\mathbb{Q}}^{TT}(X)$ represents rational Quillen K-theory. By [38], Theorem 2.15 rational G-theory has tale descent for separated Noetherian schemes of finite Krull-dimension and thus rational G-theory has descent for algebraic spaces. It should be noted that Toen's corresponding G-cohomology theory does not have smooth descent in general so cannot be defined as values of an algebraic stack in some simplicial sheaf representing G-theory in $\mathbb{A}^1$-homotopical theory.
By [43], Proposition 1.6, there is a natural transformation $K \to K_{\mathbb{Q}}^{TT}$ that can be realized as,

for a smooth presentation $X \to \mathcal{X}$, the augmentations $K(\mathcal{X}) \to \mathbf{H}(X/\mathcal{X}, K_{\mathbb{Q}})$.
With these remarks it follows from Theorem A.5 that we have the following proposition:

Proposition B.3. *Let $T = \mathfrak{R}_{S,sm}$ be the category of regular S-schemes with the smooth topology. Then for any regular algebraic S-stack $\mathcal{X}$ there is an $\mathbb{A}^1$-weak equivalence $K_{\mathbb{Q}}^{TT} \to (\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}$ so that*

$$K_i^{sm}(\mathcal{X}) = \mathrm{Hom}_{\mathcal{H}(T)}(\mathcal{X}, R\Omega^i(\mathbb{Z} \times \mathrm{Gr})_{\mathbb{Q}}).$$

Proposition B.4 ([43], Proposition 2.2)**.** *The conclusions of Theorem B.2 hold with K (resp. G) replaced by K^{sm} (resp. G^{sm}), at least whenever restricted to the category of regular stacks. Moreover, for a regular algebraic stack there is Poincar duality; the natural map $K^{sm}(\mathcal{X}) \to G^{sm}(\mathcal{X})$ is a weak equivalence.*

Appendix C. Various virtual categories and some fundamental properties

We will freely use the language of Appendix B in this chapter where we expand slightly on the concept of a virtual category of an algebraic stack. We will always consider the a stack as a simplicial sheaf via the extended Yoneda functor B.1.1. Also, for the purposes of this section, all algebraic stacks are separated locally of finite type over some (non-fixed) Noetherian scheme S.

Definition C.0.1. Given an algebraic stack $\mathcal{X}$, there are for our purposes four main candidates for virtual categories one might consider, namely any one of the following Picard categories

(a) the virtual category of locally free sheaves on $\mathcal{X}$, $V(\mathcal{X}) = V_{naive}(\mathcal{X})$.
(b) the virtual category of coherent $\mathcal{O}_{\mathcal{X}}$-modules on $\mathcal{X}$, $C(\mathcal{X})$.
(c) if $\mathcal{Z}$ is a closed substack of $\mathcal{X}$, the fundamental groupoid of the K-theory of the category of finite complexes of vector bundles on $\mathcal{X}$ with support on $\mathcal{Z}$, $V^{\mathcal{Z}}(\mathcal{X})$.
(d) if $\mathcal{Z}$ is a closed substack of $\mathcal{X}$, the fundamental groupoid of the K-theory of the category of complexes of vector bundles on $\mathcal{X}$ with support on $\mathcal{Z}$, $C_{\mathcal{Z}}(\mathcal{X})$.
(e) the fundamental groupoid of $K^{sm}(\mathcal{X})$, the cohomological virtual category, $W(\mathcal{X})$.
(f) the fundamental groupoid of $G^{sm}(\mathcal{X})$, the coherent cohomological virtual category, $WC(\mathcal{X})$.

By the remarks concluding the Appendix B we have additive functors of fibered Picard categories, $V(-) \to W(-)$ and $C(-) \to WC(-)$. Notice that since the automorphism-group of any object of $W(\mathcal{X})$ or $WC(\mathcal{X})$ is a $\mathbb{Q}$-vector space they are automatically strictly commutative.

Definition C.0.2. Since K^{sm} is flabby, to give operations involving $W(\mathcal{X})$ it is sufficient to construct functorial homotopies on the K-theory spaces of the vertices of simplicial algebraic space $\mathcal{N}(X/\mathcal{X})$ for some presentation of $\mathcal{X}$. The same remark applies to $WC(\mathcal{X})$. We will say that any such constructed operations are given by *cohomological descent*.

Given a morphism $F : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks locally of finite type over a Noetherian scheme S, recall that for a coherent sheaf $\mathcal{F}$ we can define $R^iF_*\mathcal{F}$ by a Cech-cohomology argument (compare [2], Dfinition 5.2.2.). We know by [31], Theorem 1.2, that whenever F is moreover proper, $R^iF_*\mathcal{F}$ is coherent whenever $\mathcal{F}$ is coherent. Suppose in addition that F is of finite cohomological dimension so that $R^iF_*(\mathcal{F}) = 0$ for large enough i. Then the usual formula

$$RF_*(\mathcal{F}) = \sum(-1)^i R^iF_*\mathcal{F}$$

defines a pushforward on $RF_* : C(\mathcal{X}) \to C(\mathcal{Y})$. It is more subtle to define the corresponding functor $WC(\mathcal{X}) \to WC(\mathcal{Y})$. If $F : \mathcal{X} \to \mathcal{Y}$ is a proper morphism, and given a proper surjective morphism $X \to \mathcal{X}$ with X a scheme, we obtain a diagram of

$$\begin{tikzcd} \mathcal{N}(X/\mathcal{X}) \arrow[d, "p"] \arrow[dr, "q"] & \\ \mathcal{X} \arrow[r, "F"] & \mathcal{Y} \end{tikzcd}$$

with proper morphisms and applying the functor $G^{sm}()$ we obtain a diagram

$$\begin{tikzcd} G^{sm}(\mathcal{N}(X/\mathcal{X})) = G(\mathcal{N}(X/\mathcal{X})) \arrow[d, "p_*"] \arrow[dr, "q_*"] & \\ G^{sm}(\mathcal{X}) & G^{sm}(\mathcal{Y}) \end{tikzcd} .$$

By [43], Théorème 2.9, given a proper surjective morphism $X \to \mathcal{X}$ with X a scheme and $\mathcal{X}$ is Deligne-Mumford, there is a weak equivalence $G(\mathcal{N}(X/\mathcal{X})) \to G^{sm}(\mathcal{X})$. Applying the fundamental groupoid-construction thus gives an equivalence of categories $\Pi_f(G^{sm}(\mathcal{N}(X/\mathcal{X}))) \to WC(\mathcal{X})$ and we define $RF_* = q_*(p_*)^{-1} : WC(\mathcal{X}) \to WC(\mathcal{Y})$ (compare [44], Section 3.2.2). We have essentially proved:

Proposition C.1. *Suppose $F : \mathcal{X} \to \mathcal{Y}$ is a proper of finite cohomological dimension morphism of separated Deligne-Mumford stacks of finite type over a Noetherian base-scheme S. It is possible to define a functor $RF_* : WC(\mathcal{X}) \to WC(\mathcal{Y})$ such that the diagram*

$$\begin{tikzcd} C(\mathcal{X}) \arrow[r] \arrow[d, "RF_*"] & WC(\mathcal{X}) \arrow[d, "RF_*"] \\ C(\mathcal{Y}) \arrow[r] & WC(\mathcal{Y}) \end{tikzcd}$$

is commutative up to canonical equivalence of functors.

Proof. The statement is clear as soon as we can show that there is always a choice of a proper surjective $X \to \mathcal{X}$ with X a scheme. It is clearly independent of such a choice. But this is [31], Theorem 1.1, which moreover shows we can pick X to be quasi-projective over S. □

The following uses a standard argument factorizing a projective morphism as a closed immersion and a projective bundle projection, we refer to [13], chapter V for the definition.

Proposition C.2. *Suppose $F : \mathcal{X} \to \mathcal{Y}$ is a (representable) projective local complete intersection morphism of algebraic stacks with $\mathcal{Y}$ quasi-compact and $\mathcal{Y}$ has the resolution property, i.e. any coherent sheaf is the quotient of a locally free sheaf. Then there is a natural functor*

$$RF_* : V(\mathcal{X}) \to V(\mathcal{Y})$$

compatible with the functor defined on C under the natural functor $V(-) \to C(-)$.

Remark C.2.1. Whenever we are working in a category of stacks where perfect complexes can be used to define algebraic K-theory the above is just a consequence of preservation of perfectness of a complex under proper local complete intersection morphisms. The compatibility under composition is given by Grothendieck's spectral sequence.

Similarly, if E is a vector bundle on $\mathcal{Y}$, and $F : \mathcal{X} \to \mathcal{Y}$ is any morphism, we define a functor $LF^* : V(\mathcal{Y}) \to V(\mathcal{X})$ via $LF^*[E] = [F^*E]$.

Let us just recall the usual definition of the basechange morphism, which always exists. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a Cartesian diagram of schemes. By adjointness, we have an equality of morphisms in the derived category of quasi-coherent complexes schemes;

$$Hom(Lf^*Rg_*E, Rg'_*Lf'^*E) = Hom(Rg_*E, Rf_*Rg'_*Lf'^*E)$$

and since $Rf_*Rg'_* \simeq Rg_*Rf'_*$ this is equal to

$$Hom(Rg_*E, Rg_*Rf'_*Lf'^*E).$$

By the adjunction morphism $E \to Rf'_*Lf'^*E$ we thus obtain a map

$$Hom(Rg_*E, Rg_*E) \to Hom(Lf^*Rg_*E, Rg'_*Lf'^*E).$$

The basechange morphism is the morphism which is the image under the identity-map on the left-hand-side.

Definition C.2.1. Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a commutative diagram of schemes. We say the diagram is transversal or Tor-independent or that X and Y' are transversal or Tor-independent over Y ([19], III, Dfinition 1.5) if the diagram is a Cartesian diagram of schemes, with Y quasi-compact, f quasi-compact and quasi-separated and if for any $x \in X, y' \in Y'$ mapping to the same point $y \in Y$, we have

$$Tor_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0, \text{ for } i > 0,$$

and f is of finite Tor-dimension.

Lemma C.3. *[SGA6, IV 3.1] Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Y \end{array}$$

be a transversal diagram, and let $E \in D^b(X)$ be a complex with quasi-coherent cohomology. In this case the basechange morphism is an isomorphism

$$Lf^*Rg_*E \simeq Rg'_*Lf'^*E.$$

Since it is natural it also satisfies descent with respect to any smooth equivalence relationship and thus we have

Corollary C.4. *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

be a transversal Cartesian diagram of quasi-compact algebraic stacks with the resolution property and representable morphisms, f and f' local complete intersection projective morphisms. Then there is a natural transformation

$$Lg^* Rf_* = Rf'_* Lg'^*$$

of functors $V(\mathcal{Y}') \to V(\mathcal{X})$.

Proof. From the above one readily obtains that if a vector bundle E is f_*-acyclic, f_*E is also g^*-acyclic and that g'^*E is f'_*-acyclic, inducing an isomorphism $g^*f_*E \to f'_*g'^*E$. If f is a projective bundle-projection we can, by Theorem B.2, assume that E is of the form $\sum f^*E_i \otimes \mathcal{O}(-i)$ which is a sum of f_*-acyclic objects. In the case f is a closed immersion f_* is automatically exact. The general case is obtained via the composition of the two which by standard techniques is seen to be independent of the choice of the factorization. $\square$

We record the following:

Lemma C.5. *The following diagrams are commutative whenever all of the morphisms are defined:*

(a) *Let*

$$\begin{tikzcd} X'' \arrow[r, "g''"] \arrow[d, "e'"] & Y'' \arrow[d, "e"] \\ X' \arrow[r, "g'"] \arrow[d, "f'"] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

be the composition of two transversal cartesian diagrams. Then the third diagram is also transversal and the diagram

$$\begin{tikzcd} Lg^*R(fe)_* \arrow[rr] \arrow[d, equal] & & R(f'e')_*Lg''^* \arrow[d, equal] \\ Lg^*Rf_*Re_* \arrow[r] & Rf'_*Lg'^*Re_* \arrow[r] & Rf'_*Re'_*Lg''^* \end{tikzcd}$$

is commutative.

(b) *Let*

$$\begin{tikzcd} X'' \arrow[r, "h'"] \arrow[d, "f''"] & X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, "h"] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be composition of two transversal cartesian diagrams. Then the third diagram is also transversal and the diagram

$$\begin{tikzcd} L(gh)^*Rf_* \arrow[rr] \arrow[d] & & Rf'_*L(g'h')^* \arrow[d] \\ Lh^*Lg^*Rf_* \arrow[r] & Lh^*Rf'_*Lg'^* \arrow[r] & Rf'_*Lh'^*Lg'^* \end{tikzcd}$$

Proof. Left to the reader (compare the unproved result of [3], XII, Proposition 4.4). □

The following is trivial:

Lemma C.6 (Projection formula). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a local complete intersection projective morphism of algebraic stacks with the resolution property. Suppose F is a virtual bundle on $\mathcal{Y}$ and E is a virtual bundle on $\mathcal{X}$. Then there is a functorial isomorphism $Rf_*(E \otimes Lf^*F) \to Rf_*(E) \otimes F$ compatible with transversal basechange, i.e. for a diagram as in Corollary C.4, there is a commutative diagram*

$$\begin{tikzcd} Lg^*Rf_*(E \otimes Lf^*F) \arrow[r] \arrow[d] & Lg^*(Rf_*(E) \otimes Lf^*F) \arrow[d] \\ Rf'_*(Lg'^*E \otimes Lgf'^*F) \arrow[r] & Rf'_*(Lg'^*E) \otimes Lg^*Lf^*F) \end{tikzcd}$$

where the horizontal lines are given by the projection-formula and the vertical lines are given by basechange. Moreover it is stable under composition in the naive way.

Remark C.6.1. We also have a projection formula isomorphism in the case instead of the virtual category of vector bundles we consider the virtual category of coherent sheaves as input for E.

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