A metrized Deligne-Riemann-Roch isomorphism

Dennis Eriksson

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Contents

1 Introduction

This is the final article in a series of papers on questions that arose in Deligne's article [Del87] and is a direct sequel to [Eria]. It is in preliminary form, only the construction of the functorial Adams operations and Bott's cannibalistic class has been effectuated. However, as soon as operations of this form have been constructed, the proofs of [Rös99] apply, as they are already "functorial", but work with equivalence classes of hermitian vector bundles in arithmetic K-theory. Beyond the Adams operations and Bott's cannibalistic class, we also include a section on arithmetic Chern classes in the setting of virtual categories.

2 Review of metrized virtual categories

In this section S always denotes a regular scheme, together with a set of complex points Σ. If the point admits a complex conjugate we require that this set also contains it. If $\mathcal{X} \to S$ quasi-projective morphism and $\mathcal{X}$ a regular scheme, we say that a vector bundle E is a hermitian vector bundle if it is a hermitian vector bundle on $\mathcal{X}(\mathbb{C}) := \coprod_{s\in\Sigma} X \times_S s(\mathbb{C})$ invariant under complex conjugation. A real differential (p,p)-form ω on X will be a real differential (p,p)-form on $X(\mathbb{C})$ such that for any complex conjugation F_∞, $F_\infty^* \omega = (-1)^p \omega$. Recall that there is the Bott-Chern class associated, $\widetilde{ch}\overline{\mathcal{E}} \in \widetilde{A}(X) := \oplus_{p=0}^{\infty} A^{p,p}(X,\mathbb{R})/(\operatorname{Im}\partial, \operatorname{Im}\overline{\partial})$ to an exact sequence

$$\overline{\mathcal{E}} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0 \tag{1}$$

such that

$$\operatorname{ch}\overline{E} = \operatorname{ch}\overline{E} + \operatorname{ch}\overline{E} + dd^c \widetilde{ch}\overline{\mathcal{E}}$$

where a hermitian line bundle is denoted by a bar over it, and $\operatorname{ch}(\overline{E})$ denotes the Chern character form of $\overline{E}$. It is uniquely characterized by this property together with compatibility with pullback and by being 0 whenever E is orthogonally split by the two other vector bundles.

Definition 2.0.1. Let X be a regular scheme which is quasi-projective over a base S, which admits a set of complex points, such that it contains any complex conjugation point in case the point comes from a real point. Let $\underline{KM}(X)$ be the virtual category of metrized vector bundles on X, determined by a functor μ of commutative Picard categories $V(X)$ to the Picard category of $\widetilde{A}(X) -$ torsors. Given $x \in V(X)$ and $g \in \mu(X)$, (x,g) determines an element of $\underline{KM}(X)$. It is uniquely determined by a theory of secondary Bott-Chern classes as in [Del87].

In particular, if we are given an exact sequence of vector bundles with hermitian metrics as in (1), we have an isomorphism

$$\overline{E} \simeq \overline{E}' + \overline{E}'' + \widetilde{ch}(\overline{\mathcal{E}}).$$

3 Metrized Chow categories and Green currents

Definition 3.0.2. Let $\mathcal{X}$ be an arithmetic variety. Define for a point $x \in \mathcal{X}$, Green_x to be the set pairs $(\overline{x}, g)$ such that g is a Green current for $\overline{x}$: g is a $(p-1,p-1)$-current where p is the dimension of $\overline{x}(\mathbb{C})$ and

$$dd^c g + [\overline{x}] = \omega$$

is a smooth form, where $[Y]$ is the Lelong-integration current associated to Y (this Green current condition is understood to be void if $\overline{x}(\mathbb{C}) = \emptyset$. We define the metrized Chow categories $\widehat{\mathcal{CH}}_k(\mathcal{X})$ as the Picard category associated to the complex

$$\bigoplus_{z\in\mathcal{X}_{k+2}} K_2(k(z)) \xrightarrow{d_1} \oplus \bigoplus_{y\in\mathcal{X}_{k+1}} k(y)^* \xrightarrow{d_0} \bigoplus_{x\in\mathcal{X}_k} \mathrm{Green}_x /(\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}).$$

Here d_1 is the usual (generalized) tame symbol and d_0 takes a rational function f to $(\mathrm{Div} f, -\log|f|^2)$. The images of ∂ and $\overline{\partial}$ are under the differential maps on currents of the proper bidegree. It follows from Lemma 1, Ch. III, [Sou92] that this is indeed a complex. This Picard category has objects $\bigoplus_{x\in\mathcal{X}_k} \mathrm{Green}_x /(\mathrm{Im}\, d + \mathrm{Im}\, d^c)$ and a morphism from two objects s and s' is an element $f \in \bigoplus_{y\in\mathcal{X}_{k+1}} k(y)^*$ such that $d_0 f = s - s'$ modulo the image of d_1.

Notice that the category $\widehat{\mathcal{CH}}^1(\mathcal{X})$ is equivalent to the category of hermitian vector bundles on $\mathcal{X}$, via the association (D, g_D) maps to the line bundle $\mathcal{O}(D)$ with metric locally given by $\|f\|^2 = |f|^2 e^{-g_D}$. Also, there is a forgetful functor

$$\widehat{\mathcal{CH}}_k(\mathcal{X}) \to \mathcal{CH}_k(\mathcal{X})$$

sending (Z, g_Z) to Z. Suppose that Z and Z' intersect properly in X and that g_Z and $g_{Z'}$ are Green currents for Z and Z'. Then the star product is defined (see [Sou92], chapter II, for the analytical issues of multiplication of currents) as

$$g_Z \star g_{Z'} = g_Z \wedge \delta_{Z'} + \omega_Z \wedge g_{Z'}.$$

Suppose $f : X \to S$ is a flat projective morphism of integral schemes of relative dimension n. Consider the line bundle $\langle L_1, L_2, \ldots, L_{n+1}\rangle$ constructed in [Del87] and [Elk89]. If ℓ_i is (locally on S) a rational section of L_i, such that their divisors have no intersection the symbol $\langle \ell_1, \ldots, \ell_{n+1}\rangle$ defines a base (locally on S) and if S is a point it is defined this way. We equip this line bundle with the Green current-metric by setting

$$-\log|\langle \ell_1, \ldots, \ell_{n+1}\rangle|^2 = f_*(g_{D_1} * \ldots * g_{D_{n+1}})$$

where g_{D_i} is the Green current defined by $-\log|\ell_i|^2$. There is a question in [Sou92], III.5.3, whether this coincides with the metric defined in [Elk90] (where their equality in $\widehat{\mathrm{Pic}}(S)$ was asserted).

Proposition 3.1. *The Green current-metric and the Elkik metric coincide.*

Proof. First of all we can assume $S = \mathrm{Spec}\,\mathbb{C}$. Secondly, we can assume that all the line bundles are ample line bundles by a standard addition-subtraction argument. In this case, we can find by Bertini a smooth connected hyperplane section of L_{d+1}. Then both natural restriction isomorphisms

$$\langle L_1, \ldots, L_d, L_{d+1}\rangle \simeq \langle L_1|_D, \ldots L_d|_D\rangle$$

both have the norm

$$\exp\left(\int_X \log|\ell_{d+1}| \wedge_{i=1,\ldots,d} c_1(L_i)\right)$$

where ℓ_{d+1} is a section of L_{d+1} with divisor D. Since both metrics are given by $\exp\left(\int_{X/S}\log|\ell|\right)$ when $X \to S$ is finite and flat and ℓ is an invertible section of L we are done. $\square$

Definition 3.1.1. Let L be a metrized line bundle and A an object in the metrized Chow category. Given a rational section ℓ of L and and a rational section g of the sheaf associated to A over $X \setminus \mathrm{Div}\ell$. Then the object metric on the object $\widehat{\mathbf{c}}_1(L) \cap A$ is the one which associates to $g \cap \ell$ the star product of the Green currents associated to ℓ by Poincaré-Lelong and to g by definition of a metrized object.

4 Adams operations and Bott's cannibalistic class

Suppose that we have a functor $F : V(X) \to V(X)$. To give a lifting of F to $\underline{KM}(X)$ is equivalent to constructing a commutative diagram, for any vector bundle V,

$$\begin{tikzcd} V(X) \arrow[r] \arrow[d, "F"] & \widetilde{A}(X) - \text{torsors} \arrow[d, "\widetilde{F}"] \\ V(X) \arrow[r] & \widetilde{A}(X) - \text{torsors} \end{tikzcd}$$

which is in turn equivalent to constructing a commutative diagram

$$\begin{tikzcd} M(V) \arrow[r] \arrow[dr, "F"] & \mu(V) \arrow[d] \\ & \mu(F(V)) \end{tikzcd}$$

satisfying the compatibility conditions needed for this to descend to the virtual category, in particular compatibility with filtrations (c.f. [Del87], section 5). Notice that if there are two metrized virtual bundles (V, g) and (V', g'), and an isomorphism of the two underlying virtual bundles V and V', it follows from functoriality of the functor $\mu : V(X) \to \widetilde{A}(X) - \text{torsors}$ that the difference $(V, g) - (V', g')$ is a natural object of $\widetilde{A}(X)$. For an automorphism of an object this is quite explicit: an element in $v \in K_1(X)$ acts as $(V, g) \mapsto (V, g + \delta v)$ where $\delta : K_1(X) \to \widetilde{A}(X)$ is the natural map from the localization exact sequence. In each of the cases, we call the object constructed in $\widetilde{A}(X)$, by slight abuse of language, the Bott-Chern class of the isomorphism. Notice that if V and V' are hermitian line bundles an the isomorphism in question is given by an isomorphism of line bundles, this is nothing but the logarithm of the usual norm.

Lemma 4.1 (Splitting principle)**.** *If E is a holomorphic vector bundle on a Kähler manifold X with $p : \mathbb{P}(E) \to X$ the natural projection, then the map $p^* : \widetilde{A}(X) \to \widetilde{A}(\mathbb{P}(E))$ is injective when restricted to the subspace of* $\ker dd^c$.

Proof. Because X is Kähler the subspace in question is identified with the Dolbeault cohomology groups $\oplus_{p=0}^{\infty} H^{p,p}(X)$, which has the desired property. □

Lemma 4.2. *The category $\underline{KM}(V)$ admits a biadditive product*

$$\otimes : \underline{KM}(X) \times \underline{KM}(X) \to \underline{KM}(X)$$

which lifts the usual tensor product on $V(X)$ and such that for a pair of metrized vector bundles the product is just given by the products of the metrized vector bundles, and if η and η' are two elements of $\widetilde{A}(X)$, then $\eta \otimes \eta' = dd^c\eta \wedge \eta' = \eta \wedge dd^c\eta'$.

Proof. This is trivial, the key point being that $\widetilde{ch}(\overline{\mathcal{E}} \otimes \overline{F}) = \widetilde{ch}(\overline{\mathcal{E}}) \, \mathrm{ch}(\overline{F})$. □

Consider now the Koszul complex

$$K_k(E) : 0 \to \wedge^k E \to \wedge^{k-1} E \otimes E \to \wedge^{k-2} E \otimes S^2 E \to \ldots \to E \otimes S^{k-1} E \to S^k E \to 0,$$

which was used, following [Gra92], the functorial Adams operations in [Erib].

Definition 4.2.1. Let $\overline{E} = (E, h)$ be a metrized vector bundle, then we define,

$$\Psi^k(\overline{E}) = \sum_{p=0}^{k} (-1)^{p+1}(k-p)[S^{k-p}\overline{E} \otimes \wedge^p \overline{E}]$$

where $\wedge^k$ denotes the exterior product and S^k the symmetric product, with the following metrics: If $e_1, \ldots, e_n$ is an orthonormal frame of E, then the elements of the form $e_{i_1} \wedge \ldots \wedge e_{i_k}$ is an orthnormal frame of $\wedge^k E$. We equip $S^k E$ inductively with the virtual metric such that the isomorphism $S^k E \simeq \sum_{i=0}^{k-1} (-1)^{i+1} \wedge^i E \otimes S^{k-i} E$ deduced from the Koszul complex is an "isometry". We call it the virtual quotient metric.

Remark 4.2.1. In particular, for a line bundle L, $S^k L \simeq L^{\otimes k}$ is an isometry where the latter one is given the actual product metric whenever L has a metric. Unfortunately this does not induce an actual metric on $S^k E$, but rather a virtual one in general.

Proposition 4.3. *Suppose that $\overline{E} = \overline{E}' \perp \overline{E}''$. Then the natural additivity isomorphism in $V(X)$ of the Adams operations induces an isomorphism*

$$\Psi^k(\overline{E}) \simeq \Psi^k(\overline{E}') + \Psi^k(\overline{E}'').$$

Proof. Recall that the secondary Euler characteristic of a complex $E^\bullet$ is

$$\chi'(E^\bullet) = \sum (k-p)(-1)^{p+1} E^{k-p}.$$

If $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ is an exact sequence of acyclic complexes of vector bundles, it is immediate that we have the isomorphism

$$\chi'(E) = \chi'(E') + \chi'(E'')$$

of secondary Euler characteristics in the virtual category. Moreover, if we have a product of acyclic complexes, $A \otimes B$, with the total complex structure, the secondary Euler characteristic is 0, and the isomorphism is induced by the isomorphism

$$\chi'(A \otimes B) = \chi'(A)\chi(B) + \chi'(B)\chi(A) \tag{2}$$

and the acyclicity of A and B. If

$$K_k(E) : 0 \to \wedge^k E \to \wedge^{k-1} E \otimes E \to \wedge^{k-2} E \otimes S^2 E \to \ldots \to E \otimes S^{k-1} E \to S^k E \to 0$$

denotes the k-th Koszul complex of E, we have an isomorphism

$$K_n(E + F) \simeq \oplus_{i=0}^n K_i(E) \otimes K_{n-i}(F).$$

Our first observation is that with the above given metrics, this is also a isometry in the virtual category of hermitian vector bundles. Indeed, a formal identity shows that the natural isomorphism

$$S^n(E + F) \simeq \oplus_{k=0}^n S^{n-k} E \otimes S^k F$$

is an isometry when equipped with metrics and the same statement is true whenever $E + F$ is replaced by a hermitian exact sequence $\mathcal{E} : 0 \to E' \to E \to E'' \to 0$ (i.e. with trivial Bott-Chern class). In this case one sees immediately that the maps are compatible with the metrics. Note that by definition of the metric on the symmetric product the Bott-Chern class of the Koszul complex is 0. Thus, by (2), it follows that the secondary Euler characteristic of the isometry $K_n(E+F) \simeq \oplus_{i=0}^n K_i(E) \otimes K_{n-i}(F)$ induces isometries $\Psi^k E \simeq \Psi^k E' + \Psi^k E''$. $\square$

Corollary 4.4. *Consider the general case of an exact sequence of vector bundles with metrics*

$$\overline{\mathcal{E}} : 0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

and denote the norm of the isomorphism by $\Psi^k \widetilde{ch}(\overline{\mathcal{E}})$. Then this class is given by Ψ^k acting on $A^{p-1,p-1}(X, \mathbb{R})/(Im\partial, Im\overline{\partial})$ by multiplication by k^p.

Proof. By the above, the class is necessarily 0 whenever the sequence is orthogonally split, and it is clearly stable under pullback and satisfies, by definition, the equation

$$\mathrm{ch}(\Psi^k(\overline{E})) = \mathrm{ch}(\Psi^k(\overline{E}')) + \mathrm{ch}(\Psi^k(\overline{E}'')) + dd^c \Psi^k \widetilde{ch}(\overline{\mathcal{E}}).$$

It follows from the usual axioms of Bott-Chern theory that the class $\Psi^k \widetilde{ch}(\overline{\mathcal{E}})$ is necessarily given in the form

$$\int_{\mathbb{P}^1_{\mathbb{C}}} \mathrm{ch}(\Psi^k \tilde{E}) \log |z|$$

where $\tilde{E}$ is a certain auxiliary hermitian vector bundle which is orthogonally split over ∞ into $\overline{E}' + \overline{E}''$ and over 0 is the hermitian vector bundle $\overline{E}$. Since we are integrating away $(1,1)$-forms over $\mathbb{P}^1$ it suffices to verify that $\mathrm{ch}\, \Psi^k \overline{E}$ is $\mathrm{ch}\, \overline{E}$ with degree (p,p) multiplied by k^p. For this we notice that the differential form in question is a certain combination of coefficients in the curvature matrix. For a diagonal or more generally a semi-simple such matrix the equality in question is trivial, and these elements are dense in the matrix-algebra $M_{r \times r}(\mathbb{C})$. □

Corollary 4.5. *It follows that Ψ^k has a natural additive lifting from $V(X)$ to $\underline{KM}(X)$, by defining Ψ^k to act by multiplication by k^p on the graded piece $A^{p-1,p-1}(X, \mathbb{R})/(Im\partial, Im\overline{\partial})$.*

Proposition 4.6. *The Adams operations in question are ring homomorphisms in the sense that there is a natural biadditive isomorphism*

$$\Psi^k(V) \otimes \Psi^k(V') \simeq \Psi^k(V \otimes V').$$

Proof. The difference of the natural isomorphism on the level of $V(X)$ shows that $\Psi^k(V) \otimes \Psi^k(V') - \Psi^k(V \otimes V')$ is in $\widetilde{A}(X)$, and one verifies easily that if V' is a sum of two virtual metrized bundles, both sides transform in the same way leaving the excess term constant and thus we can assume that V and V' are actual hermitian vector bundles. It is also stable under pullback, and as such we can pass to a flag variety and filter both V and V' by flags with quotients being line bundles. Thus we can finally assume that both V and V' are hermitian line bundles, in which case the statement is trivial. Finally to deduce that the class is already zero before passing to the flag we use the above computations for the Chern character of the Adams operations to deduce that the class is in $\ker dd^c$. The statement then follows by the splitting principle. □

By the same method of proof, we have:

Proposition 4.7. *The Adams operations also satisfy $\Psi^k \Psi^l \simeq \Psi^{kl}$.*

Remark 4.7.1. It should be noted that the Adams operations constructed by [Fel10] differ by some factors. In our case we have equipped the symmetric product with a virtual metric, whereas in the loc.cit. it is equipped with the symmetrization metric. In this case unfortunately we do not obtain the above Propositions giving the ring-type structure. Indeed, for the symmetrization metric, for a hermitian line bundle $\overline{L}$, $\Psi^k_E L = A_k \overline{L}^{\otimes k}$ where A_k denotes multiplying the natural product metric with the constant $1/\sqrt{(k-1)!}$. In particular, considering the operations in $\widehat{K}_0(\mathbb{Z}) \simeq \mathbb{R} \oplus \mathbb{Z}$, we can compute formally and get that in this group that: $\Psi^k_E \overline{L} \otimes \Psi^k_E \overline{M} = A_k^2 \overline{L}^{\otimes k} \otimes \overline{M}^{\otimes k} \neq A_k \overline{L}^{\otimes k} \otimes \overline{M}^{\otimes k} = \Psi^k_E (L \otimes M)$ and we would have equality in $\mathbb{R}$ if and only if $A_k^2 = A_k$.

Finally, let Ω_f be the cotangent bundle of the map f. We can define a Bott cannibalistic class element $\theta_k(\Omega_f)$ with metrics defined as $1 + L + \ldots + L^{k-1}$ in case L is a line bundle and compatible with metrics. More precisely, we can express $\theta_k(E)$ in terms of γ operations, and the multiplicativity of the Bott class then follows from multiplicativity properties of the γ operations. This in turn follows from multiplicativity of λ-operations with the usual metrics.

Proposition 4.8. *There is a natural arithmetic Bott cannibalistic class $\theta(\overline{E}) \in \underline{KM}(X)$, associated to a hermitian vector bundle $\overline{E}$. Moreover, for a projective generically smooth arithmetic variety X/S with a fixed Kähler metric, there is a class $\theta_k(\Omega_{X/S})^{-1} \in \underline{KM}(X)$. Moreover, this is the inverse of a natural element $\theta_k(\Omega_{X/S}) \in \underline{KM}(X)$.*

Proof. First of all, the Bott cannibalistic classes are in [Eria] constructed as classes in the virtual category lifted by rigidity from the standard Bott-class, which are characterized by functoriality with respect to pullback, $\theta_k(L) = 1 + L + \ldots + L^{k-1}$ for a line bundle, and $\theta_k(E + F) = \theta_k(E)\theta_k(F)$ for general vector bundles E and F. By rigidity, these classes also have unique expressions in terms of the γ-operations, which come with natural metrics from the λ-operations. Since the λ-operations satisfy $\lambda^n(E + F) \simeq \oplus\lambda^k E\lambda^{n-k}F$, even with metrics, the same properties are inherited by the γ-operations. We equip $\theta_k(E)$ with this natural metric on the γ-operations. By rigidity again, the multiplicitativity relation on θ_k is reflected on the multiplicativity relation on the γ-operations, and it follows directly that $\theta_k(\overline{L}) = 1 + \overline{L} + \ldots + \overline{L}^{k-1}$ and $\theta_k(\overline{E} + \overline{F}) = \theta_k(\overline{E})\theta_k(\overline{F})$ (with metrics) as well.
The equivalence class of $\theta_k(\overline{E})$ in $\widehat{K}_0(X)_{\mathbb{Q}}$ is invertible by [Rös99], Proposition 4.2. It follows by general nonsense (see [Eria], "Explicit construction of characteristic classes"), that there is an element $\theta_k(\overline{E})^{-1}$, characterized up to unique isomorphism by an isomorphism $\theta_k(\overline{E}) \otimes \theta_k(\overline{E})^{-1} \simeq 1$ in $\underline{KM}(X)$.
Consider a factorization of X/S as $X \to \mathbb{P}(\mathcal{E}) \to S$, where the first map is a closed immersion with normal bundle N, and the second map is a projection with tangent bundle T. We are given a short exact sequence

$$\mathcal{N} : 0 \to N^\vee \to T^\vee \to \Omega_{X/S} \to 0.$$

Endow $\mathbb{P}(E)$ with a Kähler metric and the normal bundle with a hermitian metric and let $\overline{\mathcal{N}}$ represent the exact sequence with metrics. We define

$$\theta_k(\Omega_{X/S})^{-1} = \theta_k(N^\vee)\widetilde{\theta}_k(\overline{\mathcal{N}}) + \theta_k(N^\vee)\theta_k(i^*T^\vee)^{-1}$$

where $\widetilde{\theta}_k(\overline{\mathcal{N}})$ denotes the Bott-Chern secondary class associated to θ_k. By the argument of [Rös99], after Proposition 7.3, this is independent of the factorization and the auxiliary metrics and has, by the same token, an inverse determined up to unique isomorphism. $\square$

5 Higher analytic torsion

To define the higher analytic torsion, we will need some yet not introduced notions. For simplicity, we will already assume that $f : X \to Y$ is a proper smooth Kähler fibration of complex smooth varieties, and that all the higher direct images of a vector bundle E on X vanishes. In this case $Rf_*E = f_*E$ is necessarily a vector bundle. For any holomorphic bundle E, denote by $E_{\mathbb{R}}$ the underlying real bundle, so that $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = E$. There is a C^∞-splitting (normally by considering the orthogonal complement of T_f with respect to some metric) $T_X = f^*T_Y \oplus T_f$, and for a tangent-vector $v \in T_S$, we denote by v^H its "horizontal lift" to T_X. If we have a metric on T_X and T_S, we get induced metrics on T_S and T_f. On $T_{S,\mathbb{R}}$, take the Levi-Civita-connection, on $T_{f,\mathbb{R}}$ the real connection induced by the Chern-Weil-connection on $T_{f,\mathbb{C}}$. The corresponding connection on T_X has a torsion-element T, which is a real 2-form

with values in $T_{M,\mathbb{R}}$. Now, a super vector-bundle is a vector bundle E with an involution ϵ, which in turn defines an involution on $\mathrm{End}(E)$ by $\alpha \mapsto \epsilon\alpha\epsilon$. Henceforth, for super-vector-bundles, $\widehat{\otimes}$ denotes the obvious tensor-product of super-vector-bundles, and we will freely use the natural concepts such as "End" of super-objects.
The metric on E gives rise to a Clifford-action of $A^1(X)$ on $\oplus_q A^{0,q}(X,E)$ as follows. Let $W \in A^1(X)$ and $W = W' + W''$ with $W' \in A^{1,0}(X)$ and $W'' \in A^{0,1}(X)$. We define for $s \in \oplus_q A^{0,q}(X,E)$,

$$c(W)s := \sqrt{2}(W'' \wedge s - i_{W'}(s))$$

where $i_{W'}$ is the contraction; left adjoint to $\overline{W'} \wedge -$ with respect to the pairing defined by the metric on E. Let $f_1, \dots, f_{2n}$ be a local frame for $T_{\mathbb{R}}Y$ and $f^1, \dots, f^{2n}$ be its dual frame in $T^*_{\mathbb{R}}Y$. Define the element

$$c(T) \in f^*(\wedge T^*_{\mathbb{C}}Y)\widehat{\otimes}\mathrm{End}(\wedge T_f^{*(0,1)} \otimes E)^{odd}$$

by the formula

$$c(T) = \frac{1}{2}\left(\sum f^\alpha \wedge f^\beta \widehat{\otimes} c(T(f^H_\alpha, f^H_\beta))\right)$$

We will need some normalization-operators. Let ϕ be the operator that operates as $(2\pi i)^{-q/2}$ on $T^q(X)$, N_V the operator acting as multiplication by p on $\wedge^p T^{*(0,1)} f \otimes E$. Also, let $N_u := N_V + \frac{i}{u}\omega^H$, where ω^H is a section of $f^* \wedge^2 T_{S,\mathbb{R}}$ defined by the formula $\omega^H(U,V) = \omega(U^H, V^H)$.

For $u > 0$, let B_u be the Bismut super-connection on E, defined by:

$$B_u = \nabla^E + \sqrt{u}(\overline{\partial}^Z + \overline{\partial}^{Z*}) - \frac{1}{2\sqrt{2u}} c(T).$$

Here ∇^E is a certain superconnection built out of the connections already in play, and $\overline{\partial}^Z$ is the fiber-wise Dolbeault-operator and $\overline{\partial}^{Z*}$ its formal adjoint. We say that an operator A is trace-class if the sum

$$\mathrm{Tr}(A) = \sum_i < Ae_i, e_i >$$

is absolutely convergent, for some, hence any, orthonormal basis $\{e_i, i \in I\}$ of the space in question. In this case, the sum is its "trace". The super-trace of a super-operator A is by definition $\mathrm{Tr}_s(A) = \mathrm{Tr}(\epsilon A)$. It is possible to define a certain element $\exp(-B_u^2)$, which is trace-class; it is possible to take its super-trace.

Definition 5.0.1. The higher analytic torsion of E is defined as

$$T(E, h_f) = -\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \phi(\mathrm{Tr}_s(N_u \cdot \exp(-B_u^2)) - \mathrm{Tr}_s(N_V \exp(-(\nabla^{f_* E})^2))) du.$$

Remark 5.0.1. The degree zero part of this form is the usual analytic torsion.

Proposition 5.1. *Suppose $f : X \to Y$ is a flat and generically smooth projective Kähler fibration of schemes. Then there is a direct image functor*

$$Rf_* : \underline{KM}(X) \to \underline{KM}(Y).$$

Proof. If η is a form on X, we define $Rf_*\eta = \int_{X/Y} \eta \,\mathrm{Td}\, T_f$. If $\overline{E}$ is a hermitian vector bundle, resolve it by f_*-acyclic vector bundles $\overline{E}^*$ with total complex $\mathcal{E}$, then there is an equivalence in $\underline{KM}(X)$

$$\overline{E} \simeq \overline{E}^* + \widetilde{ch}\overline{\mathcal{E}}.$$

We define $Rf_*\overline{E} = f_*E^*_{L^2} - T(\overline{E})$ where $f_*E_{L^2}$ denotes the L^2-metric given fiberwise by Hodge theory. We have to verify that if

$$\mathcal{E} : 0 \to E' \to E \to E'' \to 0$$

is an exact sequence, there is a canonical isomorphism

$$Rf_*E \simeq Rf_*E' + Rf_*E'' + f_*\widetilde{\mathrm{ch}}\mathcal{E}.$$

But this is one of the standard properties of the analytic torsion. □

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