MAN 640 : Taltaori

Tentamen 170107

Lösningar

F.1 By Fermat, the 72nd power of any integer is congruent to 0 or 1 (mod 73). Hence an expression of the form $a \cdot b^{72}$ can only be congruent to 0 or a (mod 73), i.e.: can attain 2 possible values (mod 73). Hence a sum/difference of 6 such expressions can at best attain $2^6 = 64$ possible values (mod 73). Since there are 73 congruence classes (mod 73), we conclude that, at the very best, at most 64/73 of all integers (in the obvious sense) can be represented in the given form.

F.2 Theorem 8 in my lecture notes from 2004.

F.3 Theorem 12 in my lecture notes from 2004.

F.4 (i) Let $\{a, b, c\}$ be a reduced positive-definite form of discriminant -83. Since $b^2 - 4ac = -83$ is odd, we must have b odd. Since the form is reduced we have

$$0 < a \leq \sqrt{\frac{-d}{3}} \Rightarrow a \in \{1, 2, 3, 4, 5\}.$$

If $a = 1$ then, since $b \in (-a, a]$, the only possibility is $b = 1$. This gives $c = 21$, so we have the form $\{1, 1, 21\}$.

If $a = 2$ then $b \in \{\pm 1\}$, in which case $c = (b^2 + 83)/4a = 21/2 \notin \mathbf{Z}$, so we get nothing there.

If $a = 3$ then $b \in \{\pm 1, 3\}$. We'll get an integer-valued c if $b = \pm 1$, thus giving us two further reduced forms, namely $\{3, \pm 1, 7\}$.

If $a = 4$ then $b \in \{\pm 1, \pm 3\}$ and no choice of b leads to an integer-valued c.

Finally, if $a = 5$, then $b \in \{\pm 1, \pm 3, 5\}$, but still no choice of b yields an integer-valued c.

We conclude that there are three reduced forms of discriminant -83, namely

$$x^2 + xy + 21y^2 \quad \text{and} \quad 3x^2 \pm xy + 7y^2.$$

(ii) Denote the given form as $f(x, y) = \{127, 67, 9\}$. We apply the following sequence of transformations to reduce the form :

$$\begin{aligned} S &: \{127, 67, 9\} \mapsto \{9, -67, 127\}, \\ T^4 &: \{9, -67, 127\} \mapsto \{9, 5, 3\}, \\ S &: \{9, 5, 3\} \mapsto \{3, -5, 9\}, \\ T &: \{3, -5, 9\} \mapsto \{3, 1, 7\}. \end{aligned}$$

Hence f is equivalent to the reduced form $3x^2 + xy + 7y^2$. To work out the variable substitution which accomplishes this transformation, we compute

$$\begin{aligned} ST^4ST = (ST^4)(ST) &= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}. \end{aligned}$$

Hence the desired variable substitution is

$$f(-x - y, 4x + 3y) = 3x^2 + xy + 7y^2.$$

F.5 Call the number x. Let $t := x - 3$. The definition of the continued fraction expansion implies that

$$t = \frac{1}{1 + \frac{1}{2+t}}.$$

From this we get that t satisfies the quadratic equation

$$t^2 + 2t - 2 = 0,$$

hence $t = -1 \pm \sqrt{3}$. Since $t \in (0, 1)$ we choose the positive root. Finally, then, $x = t + 3 = 2 + \sqrt{3}$.

F.6 (i) For $\mathrm{Re}(s) > 1$, the following representation is valid :

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

(ii) Theorem 8.24 in NZM.

F.7 The case $W(3,3)$ of Van der Waerden's theorem. See the handout from the book *Ramsey Theory.*

F.8 The result holds for $n = 1$. Now use induction on n. If $A \cap (n/4, n/2]$ is empty there is nothing more to prove. Similarly, we may assume $n \in A$. Now let $x \in A \cap (n/4, n/2]$. Note that then $1 \leq 4x - n \leq n$. For any $y \in A \cap [4x - n, n]$ we must have that $4y - n \in [4x - n, n] \backslash A$, as otherwise $n + (4y - n) = 4y$ would be a solution to our equation in A.

Thus A contains at most half of the elements in the interval $[4x - n, n]$. By the induction hypothesis, A contains at most $\lceil \frac{3}{4}(4x - n - 1) \rceil$ numbers in the interval $[1, 4x - n)$. Clearly, then, $|A| \leq \lceil 3n/4 \rceil$ and we're done, cerberusskallar jag i tritonal vansinnesdissonans !