A general reference for the material covered in this talk is Chapter 24 of the Handbook of Combinatorics.

3. 'Lattice' problems

PROOF OF SPERNER'S THEOREM : Let $\mathcal{A}$ be an antichain in $2^{[n]}$. Consider the set S of all pairs (A, C) where $A \in \mathcal{A}$ and C is a maximal chain in $2^{[n]}$ which passes through A.

On the one hand, since $\mathcal{A}$ is an antichain, each maximal chain in $2^{[n]}$ passes through at most one point of $\mathcal{A}$. The number of maximal chains in $2^{[n]}$ is $n!$. Hence, $|S| \leq n!$.

On the other hand, if $A \in \mathcal{A}$ is a set of size k, then there are exactly $k!(n-k)!$ maximal chains passing through A. Hence if $\mathcal{A}$ contains α_k sets of size k, we find that $|S| = \sum_{k=0}^{n} \alpha_k k!(n-k)!$.

Thus we may conclude that

$$\sum_{k=0}^{n} \alpha_k k!(n-k)! \leq n!,$$

from which it follows that $|\mathcal{A}| = \sum \alpha_k \leq \binom{n}{\lfloor n/2 \rfloor}$, with equality if and only if either (i) $\alpha_{\lfloor n/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor}$ and all other $\alpha_k = 0$ or (ii) $\alpha_{\lceil n/2 \rceil} = \binom{n}{\lfloor n/2 \rfloor}$ and all other $\alpha_k = 0$.

REFERENCES FOR FRANKL'S CONJECTURE

[1] G.L. Faro, Union-closed sets conjecture : improved bounds, *J. Comb. Math. and Comb. Computing* **16** (1994), 97-102.

[2] B. Poonen, Union-closed families, *J. Comb. Theory* (Series A) **59** (1992), 253-68.

[3] D.G. Sarvate and J-C. Renaud, On the union-closed sets conjecture, *Ars Comb.* **27** (1989), 149-54.

[4] T.P. Vaughan and R.J. Johnson, On union-closed families I, *J. Comb. Theory* (Series A) **xx** (1999), xxx-xxx.

PROOF THAT FC IMPLIES WC : Suppose FC holds and let $\mathcal{F}$ be a UC family of subsets of $[n]$. We proceed by induction on n. Trivially, WC holds if $n = 1$, so suppose it holds for $n < m$ and now consider $\mathcal{F}$ as a collection of subsets of $[m]$. For any $1 \le i \le m$ set $\mathcal{G}_i = \{A - i : A \in \mathcal{F}_i\}$, $\mathcal{H}_i = \mathcal{F} - \mathcal{F}_i$. Then each of $\mathcal{G}_i$ and $\mathcal{H}_i$ is a UC family of subsets of an $(m-1)$-set. If $|\mathcal{F}_i| = \alpha_i |\mathcal{F}|$, then the induction hypothesis implies that

$$a(\mathcal{F}_i) \ge 1 + \frac{1}{2}\log_2\left(\alpha_i|\mathcal{F}|\right), \tag{1}$$

$$a(\mathcal{H}_i) \ge \frac{1}{2}\log_2\left((1-\alpha_i)|\mathcal{F}|\right). \tag{2}$$

From (1) and (2) it follows, by a simple computation, that $a(\mathcal{F}) \ge \frac{1}{2}\log_2|\mathcal{F}|$ provided that

$$\alpha_i \log_2 \alpha_i + (1-\alpha_i)\log_2(1-\alpha_i) \ge -1. \tag{3}$$

Simple calculus shows that (3) holds if and only if $\alpha_i \ge \frac{1}{2}$ and FC guarantees that this is so for at least one index i.

4. Intersecting families

PROOF OF ERDŐS-KO-RADO THEOREM : Arrange the integers $\{1, ..., n\}$ on a circle. Let $\mathcal{G} = \{G_1, ..., G_n\}$ be the family of k-sets consisting of all possible choices of k consecutive integers on this circle. Since $\mathcal{F}$ is intersecting we have $|\mathcal{F} \cap \mathcal{G}| \le k$. Since $\mathcal{F}^\pi$ is also intersecting for any permutation $\pi \in S_n$, it follows that

$$\sum_{\pi \in S_n} |\mathcal{F}^\pi \cap \mathcal{G}| \le k \cdot n! \tag{4}$$

On the other hand, for each $A \in \mathcal{F}$ and $G_j \in \mathcal{G}$, there are exactly $k!(n-k)!$ permutations π such that $A^\pi = G_j$. Hence,

$$\sum_{\pi \in S_n} |\mathcal{F}^\pi \cap \mathcal{G}| = |\mathcal{F}| \cdot n \cdot k!(n-k)! \tag{5}$$

It follows immediately from (4) and (5) that $|\mathcal{F}| \le \binom{n-1}{k-1}$.

5. Ideals

PROOF OF KLEITMAN'S THEOREM : Induction on n, the case $n=0$ being trivial. Set $\mathcal{F}_0 := \mathcal{F} - \mathcal{F}_1$, $\mathcal{G}_0 := \mathcal{G} - \mathcal{G}_1$, $f_i := |\mathcal{F}_i|$ and $g_i := |\mathcal{G}_i|$ for $i = 0, 1$. Observe that

$$|\mathcal{F} \cap \mathcal{G}| = |\mathcal{F}_0 \cap \mathcal{G}_0| + |\mathcal{F}_1 \cap \mathcal{G}_1|. \tag{6}$$

By the induction hypothesis, the rhs of (6) is

$$\begin{aligned} &\geq \frac{f_0 g_0 + f_1 g_1}{2^{n-1}} \\ &= \frac{(f_0+f_1)(g_0+g_1)}{2^n} + \frac{(f_0-f_1)(g_0-g_1)}{2^n} \\ &= \frac{|\mathcal{F}||\mathcal{G}|}{2^n} + \frac{(f_0-f_1)(g_0-g_1)}{2^n}. \end{aligned}$$

But since both $\mathcal{F}$ and $\mathcal{G}$ are ideals, we have $f_0 \geq f_1$ and $g_0 \geq g_1$, which completes the proof.

THE SEYMOUR-HAJEDA THEOREM

A proof of this theorem can be found in

[5] D. Hajela and P. Seymour, Counting points in hypercubes and involution measure algebras, *Combinatorica* **5** (No. 3) (1985), 205-14.

This journal is not in the library, but I have a copy of the paper if anyone wants to look at it.

The proof that the SH theorem implies the theorem on $\nabla(\mathcal{F}, \mathcal{G})$ goes as follows : since $\mathcal{F}$ and $\mathcal{G}$ are ideals, there is a natural bijection $\mathcal{F} + \mathcal{G}^c \to \nabla(\mathcal{F}, \mathcal{G})$ given by

$$\mathcal{F} + \mathcal{G}^c \ni X \mapsto (\text{set of 2's in } X, \text{ set of 0's in } X).$$

REFERENCES FOR CHVÁTAL'S CONJECTURE

[6] C. Berge, A theorem related to the Chvátal conjecture, in : C.St.J.A. Nash-Williams and J. Sheehan eds., Proc. of 5th British Comb. Conf., Aberdeen 1975, pp. 35-40. (also : Utilitas Math., Winnipeg 1976).
[7] D. Miklós, Great intersecting families of edges in hereditary hypergraphs, *Discrete Math.* **48** (1984), 95-99.

6. Isoperimetric problems

References for KKS theorem

The first three references below are the original (independent) proofs of the three authors whose names are attached to the theorem. The fourth reference is to a simple proof which uses the technique of shifting.

[8] M.P. Schützenberger, A characteristic property of certain polynomials of E.F. Moore and C.E. Shannon, in : *RLE Quarterly Progress Report No. 55*, Research Laboratory of Electronics, M.I.T., 1959, 117-118.
[9] J.B. Kruskal, The number of simplices in a complex, in : Mathematical Optimization Techniques, pp. 251-78, University of California Press, Berkeley, CA, 1963.
[10] G.O.H. Katona, A theorem of finite sets, in : *Theory of Graphs*, Proc. Colloq. Tihany, pp. 187-207, Akadémiai Kiadó, Budapest, 1966.
[11] P. Frankl, A new short proof of the Kruskal-Katona theorem, *Discrete Math.* **48** (1984), 327-9.

References for the Isoperimetric theorem

Once again, the first reference contains the original proof and the second contains a much simpler proof employing the technique of shifting.

[12] K.H. Harper, Optimal numberings and isoperimetric problems, *J. Comb. Theory* **1** (1966), 385-93.
[13] P. Frankl and Z.Füredi, A short proof of a theorem of Harper about Hamming spheres, *Discrete Math.* **34** (1981), 311-13.

Finally, the theorem about $\mathcal{P}(\mathcal{F})$ appears to be a new result of mine. Hopefully, a preprint of a paper containing a proof of this result will shortly appear in the departmental preprint series.