MATEMATIK
Göteborgs Universitet
Peter Hegarty

Dag : 081220 Tid : 8.30 - 13.30.
Hjälpmedel : Inga
Vakt : Peter Hegarty 076-6377873.

Tentamenskriving i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass.

1 (12p) Find all primitive roots modulo 31.

2 (12p) Prove the case $n = 4$ of Fermat's theorem.
(OBS! You may quote results on Pythagorean triples without proof).

3 (12p) State and prove Gauss' lemma on Legendre symbols.
(OBS! You may quote Euler's criterion without proof).

4 (12p) Prove that there exists a constant $c > 0$ such that

$$\pi(x) \gtrsim c\frac{x}{\log x}.$$

5 (7p+7p) Let $d(n)$ denote the number of divisors of the positive integer n, e.g.: $d(6) = 4$ since 6 is divisible by 1,2,3 and 6. This function is called the *divisor function*.
(i) Prove a relationship between $[\zeta(s)]^2$, the square of the zeta function, and the divisor function, valid when $\text{Re}(s) > 1$.
(ii) Prove that a large number N has on average about $\log N$ divisors, in the sense that

$$\frac{1}{x}\sum_{n=1}^{x} d(n) \sim \log x.$$

(HINT: Write the sum in another way).

6 (12p) Prove that the squares form a basis of order exactly 4 for $\mathbb{N}$.

7 (7p+7p) **(i)** Prove that an asymptotic basis of order 2 for $\mathbb{N}$ cannot be a Sidon set.
(ii) Let A be any asymptotic basis for $\mathbb{N}$ of order h. Prove that

$$A(n) \gtrsim (h!)^{1/h} n^{1/h},$$

where $A(n) = A \cap \{1, ..., n\}$.

8 (4p+8p) **(i)** Explain why the number of k-term arithmetic progressions

amongst $\{1, ..., n\}$ is certainly no more than $n^2/(k-1)$.
(ii) Using a probabilistic argument, or otherwise, prove that

$$W(k,l) \geq \sqrt{k-1}\, l^{\frac{k-1}{2}}.$$

Obs! Tentan beräknas vara färdigrättad den 5 januari. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.