

MATEMATIK  
Göteborgs Universitet  
Peter Hegarty

Dag : 110825 Tid : 8.30 - 13.00 (**Obs! 4.5 hours**).  
Hjälpmedel : Inga  
Vakter : Magnus Önnheim 0703-088304,  
Peter Hegarty 0766-377873.

### Tentamenskriving i Talteori (MMA 300)

$\geq 50$  points, including bonuses from the homeworks, required to pass.

**1 (13p)** Prove that the sum of the reciprocals of the primes diverges.

**2 (12p) (i)** Determine, with proof, the  $\liminf$  and  $\limsup$ , as  $n \rightarrow \infty$ , of  $\phi(n)/n$ .

**(ii)** Determine the number of primitive roots modulo 1237.

**(iii)** Determine whether 3 is a primitive root modulo 41.

**3 (15p)** Let  $p$  be an odd prime and  $n$  a positive integer. Prove that the multiplicative group of invertible residues modulo  $p^n$  is cyclic (you may assume the result for  $n = 1$ ).

**4 (10p)** Let  $R_n$  denote the smallest positive integer  $x$  such that there are at least  $n$  primes in the interval  $(x/2, x]$ .

**(i)** Explain why the number  $R_n$  is itself prime.

**(ii)** Prove that

$$\lim_{n \rightarrow \infty} \frac{R_n}{p_{2n}} = 1,$$

where  $p_k$  denotes the  $k$ :th prime. (HINT : Use the Prime Number Theorem).

**5 (15p) (i)** State and prove Gauss' lemma on Legendre symbols.

**(ii)** Hence, or otherwise, determine with proof for which odd primes  $p$ , the number 2 is a quadratic residue modulo  $p$ .

**6 (10p) Without using generating functions**, prove that the 2-fold representation function of an asymptotic basis cannot ultimately equal one.

**7 (15p) (i)** Define the Van der Waerden number  $W(k, l)$ .

**(ii)** Determine, with proof, upper bounds for  $W(3, 2)$  and  $W(3, 3)$ .

**8 (10p)** If  $A$  is a set of integers, define

$$2 \cdot A := \{2a : a \in A\}.$$

Now, for a finite set  $A$ , prove that

$$|A + 2 \cdot A| \geq 3|A| - 2,$$

where “+” denotes a sumset.

**Obs!** Tentan beräknas vara färdigrättad den 30 augusti. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.