SUPPLEMENTARY LECTURE NOTES : WEEK 49

FRIDAY, DECEMBER 10

Lemma 1.1. *Let ϵ, γ be positive real numbers satisfying $\gamma < 1$, $\epsilon < \gamma/3$. Let G be a tripartite graph satisfying the following conditions :*

(i) $V(G)$ is the disjoint union of sets A, B, C of equal size, n say,

(ii) each of the edge densities $d(A,B), d(A,C)$ and $d(B,C)$ is at least γ,

(iii) each of the pairs $(A,B), (A,C)$ and (B,C) is ϵ-regular.

Then G contains at least $\frac{8(1-2\epsilon)\gamma^3}{27}n^3$ triangles.

Proof. Let A_1 denote the set of those vercies $a \in A$ which have at least $2\gamma n/3$ neighbors in B. I claim that

$$|A_1| \geq (1-\epsilon)n. \tag{1.1}$$

For suppose otherwise. Then $|A\backslash A_1| > \epsilon|A|$. Since (A,B) is an ϵ-regular pair, it would follow that

$$|d(A\backslash A_1, B) - d(A,B)| \leq \epsilon. \tag{1.2}$$

Since $\epsilon < \gamma/3$, this implies that $d(A\backslash A_1, B) > 2\gamma/3$. But then there must be some vertex in $A\backslash A_1$ which has more than $2\gamma n/3$ neighbors in B, contradicting the definition of the set A_1. This proves (1.1). Similarly, if we let A_2 denote the set of those $a \in A$ which have at least $2\gamma n/3$ neighbors in C, then $|A_2| \geq (1-\epsilon)n$. Hence $|A_1 \cap A_2| \geq (1-2\epsilon)n$. Now let $a \in A_1 \cap A_2$. Let $N_B(a)$ (resp. $N_C(a)$) denote the set of neighbors of a in B (resp. in C). By assumption, each of $N_B(a)$ and $N_C(a)$ has size at least $2\gamma n/3$. Since $\epsilon < \gamma/3$ and the pair (B,C) is ϵ-regular, it follows that

$$|d(N_B(a), N_C(a)) - d(B,C)| \leq \epsilon \Rightarrow d(N_B(a), N_C(a)) \geq 2\gamma/3. \tag{1.3}$$

Hence

$$||N_B(a), N_C(a)|| \geq \frac{2\gamma}{3}|N_B(a)||N_C(a)| \geq \left(\frac{2\gamma}{3}\right)^3 n^2. \tag{1.4}$$

But for any edge $\{b,c\}$ between $N_B(a)$ and $N_C(a)$, the vertices $\{a,b,c\}$ form a triangle. Hence the vertex a is contained in at least $\left(\frac{2\gamma}{3}\right)^3 n^2$ triangles. Since this holds for any $a \in A_1 \cap A_2$, it follows that G contains at least $[(1-2\epsilon)n]\left[\left(\frac{2\gamma}{3}\right)^3 n^2\right]$ triangles, as required. □

Theorem 1.2. (Triangle Counting Lemma) *For every $\gamma > 0$, there exists $\delta = \delta(\gamma) > 0$ such that the following holds :*

If G is a graph on n vertices such that at least $\gamma \binom{n}{2}$ edges need to be removed from G in order to make it triangle-free, then G must contain at least $\delta \binom{n}{3}$ triangles.

Proof. We may assume that $\gamma \leq 1$. Let $\epsilon = \gamma/10^{100}$, say, $m = \lceil 1/\epsilon \rceil$. Let M be the integer in the statement of the Regularity Lemma, correpsonding to the pair (ϵ, m). Let G be a graph on n vertices, for some $n \geq M^M$, say. By the Regularity Lemma, there exists an ϵ-regular partition $\{V_0, V_1, ..., V_k\}$ of $V(G)$ such that $m \leq k \leq M$. Clearly, for this choice of ϵ, m, M, n, there will be no more than $\frac{\gamma}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}$ edges in G which are of the following types[1] :

(i) both vertices lie inside the same V_i,

(ii) one vertex lies in V_0,

(iii) the vertices lie in V_i and V_j, where $1 \leq i < j \leq k$ and the pair (V_i, V_j) is not ϵ-regular,

(iv) the vertices lie in V_i and V_j, where $1 \leq i < j \leq k$ and $d(V_i, V_j) < \gamma/10^{50}$, say.

Hence, even if we remove all edges of types (i)-(iv) from G, we will still need to remove at least $\frac{\gamma}{2}\begin{pmatrix} n \\ 2 \end{pmatrix}$ further edges in order to make G triangle-free. Pick any remaining triangle $\{a, b, c\}$ in G. Then $a \in V_i$, $b \in V_j$, $c \in V_l$, where i, j, l are distinct elements of $\{1, ..., k\}$ and each of the pairs (V_i, V_j), (V_i, V_l) and (V_j, V_l) is both ϵ-regular and has density at least $\gamma/10^{50}$. Note furthermore that each of the sets V_i, V_j, V_l has size at least $\left(\frac{1-\epsilon}{M}\right) n$. Hence, it follows from Lemma 1.1 that the number $N_{i,j,l}$ of triangles in G with one vertex in each of V_i, V_j and V_l satisfies

$$N_{i,j,l} \geq \frac{8(1-2\epsilon)}{27}\left(\frac{\gamma}{10^{50}}\right)^3 \left[\frac{(1-\epsilon)n}{M}\right]^3. \tag{1.5}$$

Hence we can take

$$\delta = \delta(\gamma) = \frac{8(1-2\epsilon)(1-\epsilon)^3\gamma^3}{27 \cdot 10^{50} \cdot M^3}. \tag{1.6}$$

□

We can now prove Roth's theorem.

Theorem 1.3. (Roth 1956) *Let $\epsilon > 0$. Then there exists a positive integer n_ϵ such that, if $n \geq n_\epsilon$ and S is a subset of $\{1, ..., n\}$ of size at least ϵn, then S contains a non-trivial 3-term arithmetic progression.*

Proof. Fix $\epsilon > 0$, let $n \in \mathbb{N}$ and let $S \subseteq \{1, ..., n\}$ be given satisfying $|S| \geq \epsilon n$. Let G be the following tripartite graph : $V(G) = A \sqcup B \sqcup C$, where

$$A = \{(i, 1) : 1 \leq i \leq n\},$$
$$B = \{(j, 2) : 1 \leq j \leq 2n\},$$
$$C = \{(k, 3) : 1 \leq k \leq 3n\},$$

and the edges of G are defined as follows :

$$(i, 1) \text{ is joined to } (j, 2) \text{ if and only if } j - i \in S,$$
$$(j, 2) \text{ is joined to } (k, 3) \text{ if and only if } k - j \in S,$$
$$(i, 1) \text{ is joined to } (k, 3) \text{ if and only if } k - i \in 2 * S = \{2s : s \in S\}.$$

By construction, $(i, 1), (j, 2)$ and $(k, 3)$ form a triangle in G if and only if $(j - i, \frac{k-i}{2}, k - j)$ is a 3-term arithmetic progression in S. Here, the progression may be trivial. Now $|S| \geq \epsilon n$. Given $s \in S$ and $i \in \{1, ..., n\}$, there is a unique choice of $j \in \{1, ..., 2n\}$, $k \in \{1, ..., 3n\}$ such that $j - i = k - j = s$. Hence G contains at least

[1] The details of the verification are left to the reader. Note that our choice of ϵ is extreme - one could get away with a far bigger $\epsilon = \epsilon(\gamma)$.

ϵn^2 triangles corresponding to trivial AP:s. The crucial point is that these triangles are pairwise edge-disjoint. Hence, in order to make G triangle-free, we need to remove at least one edge from each of these 'trivial' triangles, hence at least ϵn^2 edges in all. Since $|V(G)| = 6n$, we thus need to remove at least $\frac{\epsilon}{18} \binom{|V(G)|}{2}$ edges in order to make G triangle-free. By Theorem 1.2, there exists $\delta = \delta(\epsilon) > 0$ such that G has at least δn^3 triangles. But each triangle comes from some 3-term AP in S, and the number of trivial APs cannot, by the same argument as above, exceed $|S|n \leq n^2$. Hence, if $n > 1/\delta$, then S must contain a non-trivial 3-term AP. $\square$