Homework 3 (due Friday, Dec. 14)

A total of 8 points or more gives 5 bonus points on the exam. Exercise 6 is worth 2 points. Each * problem is worth 3 points. Note that, in Exercise 6(iv), only the upper bound is a * problem. All your work must be properly motivated !

In Exercises 1-3 you will require the following terminology : Let $\mathcal{L} : \sum_{i=1}^{n} a_i x_i = a_0$ be a linear Diophantine equation. Let $c \in \mathbb{N}$. We say that the equation $\mathcal{L}$ is c*-irregular* if there exists a c-coloring of $\mathbb{N}$ for which there are no non-trivial monochromatic solutions to $\mathcal{L}$. We say that $\mathcal{L}$ is *irregular* if it is c-irregular for some $c \in \mathbb{N}$. Otherwise, $\mathcal{L}$ is said to be *(partition) regular*.

Q.1. Let $\mathcal{L}$ be a linear Diophantine equation. Suppose that there exists a $c \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, there exists a c-coloring of $\{1, ..., n\}$ which induces no non-trivial monochromatic solutions to $\mathcal{L}$. Prove that $\mathcal{L}$ is then irregular.

Q.2 Prove that the equation $4x = 2y + z$ is 3-irregular.
(REMARK : The question of whether one can 3-color the reals such that there are no monochromatic solutions to this equation is known to be undecidable in ZFC-set theory).

Q.3 Prove that the equation $x = y + z$ is regular.
(HINT : Apply Ramsey's theorem to a suitable graph).

Q.4 Let $h \geq 2$. Recall that an asymptotic basis A for $\mathbb{N}_0$ of order h is said to be *thin* if the counting function $A(n)/n^{1/h}$ is bounded. Let's call A *skinny* if the representation function $r_{A,h}(n)$ is bounded.

Prove that a skinny asymptotic basis of order h is also thin of order h. On the other hand, give an example for each $h \geq 2$ of a thin asymptotic basis of order h which is not skinny.

Q.5 Prove that the Erdős-Turán Conjecture fails in $\mathbb{Z}$ by exhibiting, for each $h \geq 1$, a basis A for $\mathbb{Z}$ of order h such that $r_{A,h}(n) = 1 \ \forall \ n \in \mathbb{Z}$.

Q.6. Let A be an asymptotic basis for $\mathbb{N}_0$. An element $a \in A$ is said to be *essential* if the set $A \backslash \{a\}$ is no longer an asymptotic basis, of any order.

***(i)** Prove that if A is an asymptotic basis for $\mathbb{N}_0$ and $a \in A$, then a is

essential if and only if $A\backslash\{a\}$ is contained inside some non-trivial, homogeneous arithmetic progression, i.e.: inside $n\mathbb{Z}$ for some $n > 1$.
(ii) Deduce from part (i) that an asymptotic basis contains only finitely many essential elements.
(iii) For each $k \geq 1$, give an example of an asymptotic basis with exactly k essential elements.
***(iv)** Prove that there exists a function $X : \mathbb{N} \to \mathbb{N}$ with the following property :

For every $h \in \mathbb{N}$, every asymptotic basis A for $\mathbb{N}_0$ of order h and every $a \in A$, either a is essential or the order of $A\backslash\{a\}$ as an asymptotic basis is at most $X(h)$. In fact, prove that, as $h \to \infty$,

$$\frac{h^2}{4} \lesssim X(h) \lesssim \frac{5h^2}{4}.$$

Q.7 If $A \subseteq \mathbb{Z}$, then the *difference set* $A - A$ is defined as

$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}.$$

(i) For a finite set A, give upper and lower bounds for $|A - A|$ in terms of $|A|$, analogous to those given in the lectures for the sumset $A + A$.
(ii) Give any example whatsoever of a finite set A for which $|A + A| > |A - A|$.
***(iii)** Prove that there exists a real number $C > 0$ such that, for all $n \in \mathbb{N}$, there are at least $C \cdot 2^n$ subsets A of $\{1, ..., n\}$ which satisfy $|A + A| \geq |A - A|$.

Q.8 For each $n \in \mathbb{N}$, let

$$A_n := \{k^2 : 1 \leq k \leq n\}.$$

Prove that, for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{|A_n - A_n|}{n^{2-\epsilon}} = +\infty.$$

Q.9 Let $A \subseteq \mathbb{Z}$ be a finite set, $|A| = k > 3$. Prove that if $|A+A| \leq 2k$ then there exists an arithmetic progression B such that $|B| \leq k+1$ and $A \subseteq B$.

***Q.10** Let A be a subset of $\mathbb{Z}$. The *product set* $A \cdot A$ is defined in the same way as the sumset $A + A$, but with multiplication replacing addition. In other words,

$$A \cdot A \stackrel{\text{def}}{=} \{x \in \mathbb{Z} : x = a_1 a_2 \text{ for some } a_1, a_2 \in A\}.$$

Now let $n \in \mathbb{N}$ and define

$$f(n) := \min_{|A|=n} \left(\max\{|A \cdot A|, |A + A|\}\right).$$

Prove that $f(n)/n \to \infty$ as $n \to \infty$.