MATEMATIK
Göteborgs Universitet
Peter Hegarty

Dag : 130402 Tid : 8.30 - 13.00 (**Obs! 4.5 hours**).
Hjälpmedel : Inga
Vakter : Jakob Hultgren 0703-088304,
Peter Hegarty 0766-377873.

Tentamenskrivning i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass. In Problems 1,3,5,7, any results that you use from the lecture notes may be just stated without proof.

1 (5p+4p+3p) (i) Determine all primitive roots modulo 43.
(ii) Without finding them, determine the number of solutions in $\mathbb{Z}_{105}$ to the congruence $x^2 \equiv 1$. Explain your reasoning.
(iii) Now find all the solutions in **(ii)** above.

2 (14p) Prove that every non-negative integer is a sum of four non-negative integer squares.

3 (1p+6p+5p) (i) Define the Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$.
(ii) Determine, with proof, an infinite series representation for the function $1/\zeta(s)$, valid in the range $\text{Re}(s) > 1$.
(iii) Let $A \subseteq \mathbb{N}$ be the set of squarefree numbers, i.e.: the set of those numbers not divisible by any perfect square. Prove that the set A has a non-zero asymptotic density, and determine its value.

4 (3p+11p) (i) State Gauss' Lemma.
(ii) With the help of this lemma, or otherwise, state and prove the Law of Quadratic Reciprocity.

5 (10p) Let $n \in \mathbb{N}$. We showed in class that, if A is a set of n integers, then

$$2n - 1 \leq |A + A| \leq \frac{n(n+1)}{2}.$$

Now prove that, for every $n \in \mathbb{N}$ and every integer $t \in \left[2n-1, \frac{n(n+1)}{2}\right]$, there is a set of integers A such that $|A| = n$ and $|A + A| = t$.

6 (2p+13p) (i) Let $h \in \mathbb{N}$. Define what is meant by the *h-fold representation function* $r_h(A, n)$ of a subset $A \subseteq \mathbb{N}_0$.
(ii) Prove that there is a set $A \subseteq \mathbb{N}$ such that $r_2(A, n) = \Theta(\log n)$.

7 (2p+11p) (i) Define the Van der Waerden number $W(k, l)$.
(ii) Assuming that the numbers $W(3, l)$ exist for all l, prove that the number $W(4, 2)$ exists.

8 (10p) Let A be the subset of $\mathbb{N}$ which is free of 3-term APs and is obtained by the following greedy algorithm: First choose $1 \in A$. Given $n > 1$ and $A_n = A \cap \{1, ..., n-1\}$, choose $n \in A$ if and only if $A_{n-1} \cup \{n\}$ is free of 3-term APs.

Let $a(n) = |A_{n+1}|$. Prove that $a(n) \sim n^{2/3}$.

Obs! Tentan beräknas vara färdigrättad den 8 april. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.