MATEMATIK
Göteborgs Universitet
Peter Hegarty

Dag : 121219 Tid : 8.30 - 13.00 (**Obs! 4.5 hours**).
Hjälpmedel : Inga
Vakter : Dawan Mustafa 0703-088304,
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Tentamenskriving i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass. In Problems 1,3,5,7, any results that you use from the lecture notes may be just stated without proof.

1 (9p+3p) (i) Determine for which primes p the congruence

$$3x^2 + 9x + 5 \equiv 0 \pmod{p}$$

has a solution.
(ii) Let S be the set of primes determined in part **(i)** and, for $x \in \mathbb{R}_+$, let $\pi_S(x)$ denote the number of primes in S up to x. Determine

$$\lim_{x \to \infty} \frac{\pi_S(x)}{\pi(x)}.$$

2 (13p) Classify, with proof, those non-negative integers n which are sums of two squares.

3 (10p) A natural number n is said to be *perfect* if it equals the sum of its proper divisors, e.g.: $6 = 1 + 2 + 3$. Prove that an even number is perfect if and only if it is of the form $2^{p-1}(2^p - 1)$, where p is a prime such that $2^p - 1$ is also prime.

4 (15p) Prove that

$$(\ln 2)\frac{x}{\ln x} \lesssim \pi(x).$$

5 (10p) A set A of elements in an abelian group is said to be *sum-free* if $(A + A) \cap A = \{\}$.

Let p be a prime. Determine, with proof, the maximum size of a sum-free subset of $\mathbb{Z}_p$, as a function of p.

6 (2p+10p) (i) Let $h \in \mathbb{N}$. Define what is meant by the *h-fold representation function* of a subset $A \subseteq \mathbb{N}_0$.
(ii) Prove that the 2-fold representation function of a subset of $\mathbb{N}_0$ cannot be ultimately constant and non-zero.

7 (2p+9p) (i) Define the Van der Waerden number $W(k,l)$.
(ii) Using a probabilistic method, or otherwise, show that

$$W(k,l) > \sqrt{2(k-1)}l^{(k-1)/2}.$$

8 (2p+15p) (i) State the Regularity Lemma.
(ii) Using the Regularity Lemma, give a complete proof of Roth's theorem, i.e.: of the fact that, if $f(n)$ is the maximum size of a 3-AP-free subset of $\{1,...,n\}$, then $f(n) = o(n)$.

Obs! Tentan beräknas vara färdigrättad den 28 december. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.