SUPPLEMENTARY LECTURE NOTES ON DIRICHLET'S THEOREM

The purpose of these notes is to complete the proof of Dirichlet's theorem (Theorem 7.5). In the ordinary lecture notes, we had reduced the proof to showing that $L(1, \chi) \neq 0$, when χ is any non-trivial Dirichlet character. Actually, for this reduction to be meaningful, we must know that L-functions can be extended to the left of Re$(s) > 1$, and we glossed over the details of this a bit in the notes. So the first step is to deal rigorously with this:

Theorem 1.1. *(i) There is a function $A(s)$ which is analytic in Re$(s) > 0$ such that, when Re$(s) > 1$,*

$$\zeta(s) = \frac{s}{s-1} + A(s). \tag{1.1}$$

In other words, $\zeta(s)$ can be extended to a meromorphic function in Re$(s) > 0$, which is analytic except for a single simple pole at $s = 1$, with residue 1.
(ii) Let χ be any non-trivial Dirichlet character. Then the function $L(s, \chi)$ can be extended to an analyitc function in $Re(s) > 0$.

Proof. (i) For Re$(s) > 1$ we have of course $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$. Apply Abel summation with $a_n = 1$, $b_n = 1/n^s$. Since the series is convergent, we have

$$\zeta(s) = \sum_{n=1}^{\infty} n \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right), \quad \text{Re}(s) > 1. \tag{1.2}$$

The term in brackets can be written as $s \int_n^{n+1} x^{-s}\, dx$. Writing $x = \lfloor x \rfloor + \{x\}$, where $\{x\} \in [0, 1)$ denotes the fractional part of x, it follows that, again for Re$(s) > 1$,

$$\zeta(s) = s \int_1^{\infty} (x - \{x\}) x^{-s-1}\, dx = \frac{s}{s-1} + A(s), \tag{1.3}$$

where $A(s) = \int_1^{\infty} \{x\} x^{-s-1}\, dx$. But since $\{x\}$ is a bounded function, this integral converges in Re$(s) > 0$ and hence defines an analytic function of s in this region.

(ii) The proof is similar. For Re$(s) > 1$ we have $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. For $x \in [1, \infty)$ let $S(x) := \sum_{n \leq x} \chi(n)$. Using Abel summation as before, we obtain the expression

$$L(s, \chi) = s \int_1^{\infty} S(x) x^{-s-1}\, dx. \tag{1.4}$$

But since χ is a non-trivial character, the function $S(x)$ is bounded, by Lemma 14.2(ii). Hence the right-hand side of (1.4) defines an analytic function of s all the way down to Re$(s) > 0$. $\square$

So now we know for sure that Theorem 15.2 in the notes makes rigorous sense. In what follows, d is a fixed positive integer and $\hat{G}$ denotes the group of all extended Dirichlet characters modulo d. We continue with a lemma:

Lemma 1.2. *If $s > 1$ is real, then $\prod_{\chi \in \hat{G}} L(s, \chi)$ is a positive real number greater than one.*

Proof. From equation (15.3) in the notes we deduce that, for $\mathrm{Re}(s) > 1$,

$$\sum_{\chi\in\hat{G}} \log L(s,\chi) = \sum_{\chi}\sum_{p}\sum_{m=1}^{\infty} \frac{\chi(p^m)}{mp^{ms}}. \tag{1.5}$$

The sum is absolutely convergent so we can interchange things at will. We rewrite the above as

$$\sum_{\chi\in\hat{G}} \log L(s,\chi) = \sum_{p}\sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \sum_{\chi} \chi(p^m). \tag{1.6}$$

By Lemma 14.2(i), the inner sum is either zero or $|\hat{G}|$, i.e.: a positive integer. Hence, if $s > 1$ is real, the entire sum on the right of (1.6) is some positive real number. Thus $\sum_{\chi} \log L(s,\chi)$ is a positive real number. Exponentiating, it follows that $\prod_{\chi} L(s,\chi)$ is a real number greater than one. □

From the lemma, we can deal with complex characters.

Proposition 1.3. *If χ is a complex character, then $L(1,\chi) \neq 0$.*

Proof. If χ is complex, then $\overline{\chi} \neq \chi$. From the definition of the L-series we see immediately that, if $s > 1$ is real, then $L(s,\overline{\chi}) = \overline{L(s,\chi)}$. Taking limits as $s \to 1^+$, it follows that if $L(1,\chi) = 0$ then also $L(1,\overline{\chi}) = 0$. Hence, in this case, at least two terms of the product $\prod_{\chi\in\hat{G}} L(s,\chi)$ tend to zero as $s \to 1^+$. But only one term in the product tends to infinity, and it has a simple pole at $s = 1$ - this follows from Theorem 1.1 above and equation (15.8) in the notes. Hence, the entire product would tend to zero as $s \to 1^+$. This contradicts Lemma 1.2. □

To complete the proof of Dirichlet's theorem, it remains to show that $L(1,\chi) \neq 0$ when χ is a non-trivial, real extended Dirichlet character. This is actually by far the hardest part of Dirichlet's original proof. We shall present a much shorter (though still highly non-trivial) proof due to de la Vallée Poussin (1896).

Let χ be a non-trivial real character and assume $L(1,\chi) = 0$. We shall obtain a contradiction by considering the function

$$\psi(s) = \frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}. \tag{1.7}$$

CLAIM 1: $\psi(s)$ is analytic in $\mathrm{Re}(s) > 1/2$.

Since we're assuming $L(1,\chi) = 0$, the simple pole of $L(s,\chi_0)$ at $s = 1$ will be cancelled by this zero, so we don't have a problem at $s = 1$. For all other values of s in the range $\mathrm{Re}(s) > 1/2$, it follows from Theorem 1.1 and (15.8) that $\psi(s)$ is analytic.

CLAIM 2: $\psi(s) \to 0$ as $s \to \frac{1}{2}^+$.

This also follows from Theorem 1.1 and (15.8), since the denominator $L(2s,\chi_0)$ will head to infinity as $s \to \frac{1}{2}^+$, whereas both terms in the numerator will converge to finite values.

Next, let us confine attention to Re$(s) > 1$, in which case all three L-functions making up ψ have Euler product representations, as in Lemma 15.1. Using the facts that (i) all Euler products converge absolutely, so we can regroup and cancel terms willy-nilly (ii) χ is real, so $\chi(p) \in \{\pm 1\}$ for all p not dividing d, (iii) the identity $(1 - p^{-2s}) = (1 + p^{-s})(1 - p^{-s})$, one finds eventually that, for Re$(s) > 1$,

$$\psi(s) = \prod_{p\colon \chi(p)=+1} \frac{1 + p^{-s}}{1 - p^{-s}}. \tag{1.8}$$

Using the binomial theorem, it follows that

$$\psi(s) = \prod_{p\colon \chi(p)=+1} (1 + p^{-s}) \sum_{m=0}^{\infty} p^{-ms}. \tag{1.9}$$

This can be expanded to a Dirichlet series

$$\psi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.10}$$

where, and this is the only relevant point, every coefficient a_n is a non-negative real number[1] and $a_1 = 1$. The Dirichlet series is absolutely convergent, so can be differentiated term by term ad nauseum. Hence, for any $m \in \mathbb{N}_0$ and Re$(s) > 1$,

$$\psi^{(m)}(s) = (-1)^m \sum_{n=1}^{\infty} \frac{a_n (\log n)^m}{n^s}. \tag{1.11}$$

By Claim 1 and standard complex analysis tjafs, $\psi(s)$ can be expanded as a Taylor series about $s = 2$ with radius of convergence at least $3/2$. Hence, for Re$(s) > 1/2$,

$$\psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!} \psi^{(m)}(2)(s-2)^m. \tag{1.12}$$

Substituting from (1.11) we have

$$\psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!} b_m (2-s)^m, \tag{1.13}$$

where

$$b_m = \sum_{n=1}^{\infty} \frac{a_n (\log n)^m}{n^2}. \tag{1.14}$$

All that's relevant is that, since the a_i are all non-negative reals, the same is true of the b_i. Hence, for s real and $s \in \left(\frac{1}{2}, 2\right]$, the right-hand side of (1.13) is a real-valued, decreasing function of s. In particular, this means that $\psi(s)$ has a limit as $s \to \frac{1}{2}^+$ and

$$\lim_{s \to \frac{1}{2}^+} \psi(s) \geq \psi(2). \tag{1.15}$$

But $\psi(2) \geq 1$, by (1.10), since $a_1 \geq 1$ and all $a_n \geq 0$. This contradicts Claim 2 and completes the proof.

[1] Some coefficients will be zero, since only those primes for which $\chi(p) = +1$ appear in (1.9).