Supplementary Notes for Friday, 4 Oct.

Here is a full solution of Exercise 15.3.6, which we didn't quite get finished in class. One wants to compute the line integral

$$\int_{\mathcal{C}} e^z \, ds, \tag{1}$$

where $\mathcal{C}$ is the parametric curve

$$\mathbf{r}(t) = (e^t \cos t, e^t \sin t, t), \quad 0 \le t \le 2\pi.$$

We computed the velocity vector

$$\frac{d\mathbf{r}}{dt} = \big(e^t(\cos t - \sin t), e^t(\sin t + \cos t), 1\big)$$

and hence, after some ugly computation, the speed

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{2e^{2t} + 1}.$$

Since $z = t$ along the curve, the integral (1) becomes

$$\int_0^{2\pi} e^t \sqrt{2e^{2t} + 1} \, dt. \tag{2}$$

After a change of variables $u = e^t$, (2) becomes

$$\int_1^{e^{2\pi}} \sqrt{2u^2 + 1} \, du. \tag{3}$$

The easiest way to evaluate this integral is to make another slight change of variables and then use a formula in the *formelblad* on the course homepage (which you will also have at the exam). Namely, set $v = \sqrt{2}\, u$, so that (3) becomes

$$\frac{1}{\sqrt{2}} \int_\alpha^\beta \sqrt{v^2 + 1} \, dv, \quad \text{where } \alpha = \tfrac{1}{\sqrt{2}}, \ \beta = \tfrac{e^{2\pi}}{\sqrt{2}}. \tag{4}$$

Now use the following formula from the formelblad:

$$\int \sqrt{x^2 + a} \, dt = \frac{1}{2}\left[x\sqrt{x^2 + a} + a \ln\left(x + \sqrt{x^2 + a}\right)\right] + C. \tag{5}$$

Here $a = 1$. So now we can evaluate (4). After some more messy computation, the final answer is

$$\int_{\mathcal{C}} e^z \, ds = \frac{1}{4\sqrt{2}} \left[\left(e^{2\pi}\sqrt{2 + e^{4\pi}} - \sqrt{3}\right) + 2 \ln\left(\frac{e^{2\pi} + \sqrt{2 + e^{4\pi}}}{\sqrt{3} + 1}\right)\right].$$