Supplementary Notes for Friday, 6 Sep.

In class, I rather quickly mentioned the trajectory in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = (\cos t, \sin t, t^2),$$

which represents a helix around the z-axis, winding upwards at increasing speed (the vertical speed increasing like the square of time). We observed that the length of the curve sketched out between $t = 0$ and $t = T$ was given by the integral

$$\int_0^T \sqrt{1 + 4t^2}\, dt.$$

This is a messy but doable integral. In fact it can be shown that

$$\int_0^T \sqrt{1 + t^2}\, dt = \frac{1}{2}\left[T\sqrt{1 + T^2} + \ln\left(T + \sqrt{1 + T^2}\right)\right] \tag{1}$$

and hence, by a change of variables, that

$$\int_0^T \sqrt{1 + 4t^2}\, dt = \frac{1}{4}\left[2T\sqrt{1 + 4T^2} + \ln\left(2T + \sqrt{1 + 4T^2}\right)\right]. \tag{2}$$

To prove (1), make the substitution $t = \sinh\theta$. I will leave the details of the calculation as an (optional) exercise. The material in Section 3.6 of the book should be helpful if you get stuck. Note that the above calculations are also related to Kryssuppgift 11.3.17.