Supplementary Notes for Friday, 13 Sep.

These notes are meant to correct the errors I made in the solution of the final problem on the board today.

Problem: *Let* $f(x, y) = \frac{x-y}{x+y}$*. Answer the following questions with respect to the point* $(1, 1)$*.*

(a) compute the gradient of f*.*
(b) compute the directional derivative in the direction $\mathbf{i} + 2\mathbf{j}$*.*
(c) in what directions are the rates of change of f *maximal (resp. minimal) ?*
(d) compute the equation for the tangent line to the level curve of f *at* $(1, 1)$*.*

Solution:

(a) We have the general formula

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \cdots = \left(\frac{2y}{(x+y)^2}, \frac{-2x}{(x+y)^2}\right).$$

At the point $(1, 1)$ we get

$$\nabla f(1,1) = \left(\frac{2 \cdot 1}{(1+1)^2}, \frac{-2 \cdot 1}{(1+1)^2}\right) = \left(\frac{1}{2}, -\frac{1}{2}\right).$$

(b) A unit vector in the direction $\mathbf{i} + 2\mathbf{j}$ is given by

$$\hat{\mathbf{u}} = \frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{1^2 + 2^2}} = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right).$$

Hence the directional derivative in this direction is given by

$$D_{\mathbf{u}} f(1,1) = \hat{\mathbf{u}} \bullet \nabla f(1,1) = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) \bullet \left(\frac{1}{2}, -\frac{1}{2}\right) = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(-\frac{1}{2}\right) = -\frac{1}{2\sqrt{5}}.$$

(c) The scalar product formula for the directional derivative leads immediately to the fact that the rate of change is maximal (resp. minimal) in the direction of the gradient (resp. in the opposite direction). See also parts (i) and (ii) of the boxed text on the top of page 718 in the book.
Hence, in this exercise, the rate of change is maximal in the direction of $\mathbf{i} - \mathbf{j}$, and minimal in the opposite direction $-\mathbf{i} + \mathbf{j}$.

(d) Along a level curve, $f(x, y)$ is constant, so the directinal derivative in the direction of the tangent to the curve must be zero. From the scalar product formula, we see that the directional derivative is zero in the direction perpendicular to the gradient (since the scalar product of two vectors is zero if and only if they are perpendicular). See also part (iii) of the boxed text on page 718.
In this exercise we have already computed $\nabla f(1, 1) = (1/2, -1/2)$. A vector perpendicular to this is $(1, 1)$. Hence the tangent to the level curve is in the direction $\mathbf{i} + \mathbf{j}$, and has slope $k = 1$. The base point is $(1, 1)$, so the equation of the tangent line becomes $y - 1 = 1 \cdot (x - 1) \Rightarrow y = x$.