Supplementary Problem Solutions, Oct. 13

Exercise 14.6.5. We use cylindrical coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{1}$$

Integrating over z first, we'll then be left with a double integral

$$\int\int_R (2a - r)\, r\, dr\, d\theta. \tag{2}$$

The region R in the xy-plane is bounded by the curve

$$x^2 + y^2 = 2ay. \tag{3}$$

After completing squares, we can write this equation as $x^2 + (y-a)^2 = a^2$, from which we see that R is a disc of radius a centred at $(0, a)$. The important thing to get right now is the integration limits, i.e.: the description of R in terms of polar coordinates. To do this, we write (3) in terms of the coordinates in (1), and it becomes

$$r^2 = 2ar\sin\theta \quad \Rightarrow \quad r = 2a\sin\theta. \tag{4}$$

Since r must be positive, (4) implies that θ must lie in $[0, \pi]$. Hence, (2) becomes

$$\int_0^{\pi} d\theta \int_0^{2a\sin\theta} (2a - r) r\, dr. \tag{5}$$

After integrating over r, we'll be left with the following integral over θ:

$$\frac{4a^3}{3}\int_0^{\pi} (3\sin^2\theta - 2\sin^3\theta)\, d\theta. \tag{6}$$

To evaluate this integral, one can make use of the trigonometric identities

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta), \tag{7}$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta). \tag{8}$$

Then it's just a matter of plugging these in and integrating, which I will leave you to do. ANSWER: $a^3\left(2\pi - \frac{32}{9}\right)$.

Exercise 15.5.17. We saw in the redovisning how to solve this problem using the formula on the top half of page 874. An alternative solution involves observing that

$$e^{2z} = e^{2u} = [(e^u\cos v)^2 + (e^u\sin v)^2] = x^2 + y^2,$$

hence that

$$z = \frac{1}{2}\ln(x^2 + y^2). \tag{9}$$

Since the surface is thus given explicitly as the graph $z = g(x, y)$ of a function, we can instead use the formula at the bottom of page 874. Note that

$$\delta = \sqrt{1 + e^{2u}} = \sqrt{1 + (x^2 + y^2)}. \tag{10}$$

Furthermore, since $0 \leq u \leq 1$ and $u = z$, plugging this into (9) we find that $1 \leq x^2 + y^2 \leq e^2$. This describes an annulus of inner radius 1 and outer radius e. However, we also have $0 \leq v \leq \pi$, so we are only integrating over the upper half of this annulus. In terms of polar coordinates, the region is thus

$$1 \leq r \leq e, \quad 0 \leq \theta \leq \pi. \tag{11}$$

From (9), (10) and the formula on page 874, we obtain the surface integral

$$\int\int \sqrt{1 + (x^2 + y^2)} \sqrt{1 + \left(\frac{x}{x^2 + y^2}\right)^2 + \left(\frac{y}{x^2 + y^2}\right)^2} \, dx \, dy = \int\int \frac{1 + (x^2 + y^2)}{\sqrt{x^2 + y^2}} \, dx \, dy.$$

We switch to polar coordinates and obtain, by (11), the integral

$$\int_0^\pi d\theta \int_1^e (1 + r^2) \, dr = \cdots = \frac{\pi}{3}(3e + e^3 - 4).$$