TABLE 1. FOURIER SERIES

The functions $f$ in this table are all understood to be $2\pi$-periodic. The formula for $f(\theta)$ on either $(-\pi,\pi)$ or $(0,2\pi)$ (except perhaps at its points of disconinuity) is given in the left column; the Fourier series of $f$ is given in the right column. Verify these Fourier series.

| | | |
|---|---|---|
| 1. | $f(\theta)=\theta \quad (-\pi<\theta<\pi)$ | $2\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin n\theta$ |
| 2. | $f(\theta)=\lvert\theta\rvert \quad (-\pi<\theta<\pi)$ | $\frac{\pi}{2}-\frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2}$ |
| 3. | $f(\theta)=\pi-\theta \quad (0<\theta<2\pi)$ | $2\sum_1^\infty \frac{\sin n\theta}{n}$ |
| 4. | $f(\theta)=\begin{cases} 0 & (-\pi<\theta<0) \\ \theta & (0<\theta<\pi)\end{cases}$ | $\frac{\pi}{4}-\frac{2}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} + \sum_1^\infty \frac{(-1)^{(n+1)}}{n}\sin n\theta$ |
| 5. | $f(\theta)=\sin^2\theta$ | $\frac{1}{2}-\frac{1}{2}\cos 2\theta$ |
| 6. | $f(\theta)=\begin{cases} -1 & (-\pi<\theta<0) \\ 1 & (0<\theta<\pi)\end{cases}$ | $\frac{4}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$ |
| 7. | $f(\theta)=\begin{cases} 0 & (-\pi<\theta<0) \\ 1 & (0<\theta<\pi)\end{cases}$ | $\frac{1}{2}+\frac{2}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$ |
| 8. | $f(\theta)=\lvert\sin\theta\rvert$ | $\frac{2}{\pi}-\frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1}$ |
| 9. | $f(\theta)=\lvert\cos\theta\rvert$ | $\frac{2}{\pi}-\frac{4}{\pi}\sum_1^\infty \frac{(-1)^n\cos 2n\theta}{4n^2-1}$ |
| 10. | $f(\theta)=\begin{cases} 0 & (-\pi<\theta<0) \\ \sin\theta & (0<\theta<\pi)\end{cases}$ | $\frac{1}{\pi}-\frac{2}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1}+\frac{1}{2}\sin\theta$ |

TABLE 1 (continued)

| | | |
|---|---|---|
| 11. | $f(\theta) = \begin{cases} \theta & (-a < \theta < a) \\ a\frac{\pi-\theta}{\pi-a} & (a < \theta < \pi) \\ a\frac{\pi+\theta}{a-\pi} & (-\pi < \theta < -a) \end{cases}$ | $\frac{2}{\pi - a}\sum_1^\infty \frac{\sin na}{n^2}\sin n\theta$ |
| 12. | $f(\theta) = \begin{cases} (2a)^{-1} & (\|\theta\| < a) \\ 0 & (a < \|\theta\| < \pi) \end{cases}$ | $\frac{1}{2\pi} + \frac{1}{\pi}\sum_1^\infty \frac{\sin na}{na}\cos n\theta$ |
| 13. | $f(\theta) = \begin{cases} (2a)^{-1} & (\|\theta - \theta_0\| < a) \\ 0 & (a < \|\theta - \theta_0\| < \pi) \end{cases}$ | $\frac{1}{2\pi} + \frac{1}{\pi}\sum_1^\infty \frac{\sin na}{na}(\cos n\theta_0 \cos n\theta)$ $+ \sin n\theta_0 \sin n\theta)$ |
| 14. | $f(\theta) = \begin{cases} 1 & (-a < \theta < a) \\ -1 & (2a < \theta < 4a) \\ 0 & \text{elsewhere in } (-\pi, \pi) \end{cases}$ | $\sum_1^\infty \frac{\sin na}{n}[(1 - \cos 3na)\cos n\theta$ $- \sin 3na \sin n\theta]$ |
| 15. | $f(\theta) = \begin{cases} a^{-2}(a - \|\theta\|) & (\|\theta\| < a) \\ 0 & (a < \|\theta\| < \pi) \end{cases}$ | $\frac{1}{2\pi} + \frac{2}{\pi}\sum_1^\infty \frac{1 - \cos na}{n^2 a^2}\cos n\theta$ |
| 16. | $f(\theta) = \theta^2 \quad (-\pi < \theta < \pi)$ | $\frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$ |
| 17. | $f(\theta) = \theta(\pi - \|\theta\|) \quad (\pi < \theta < \pi)$ | $\frac{8}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{(2n-1)^3}$ |
| 18. | $f(\theta) = e^{b\theta} \quad (-\pi < \theta < \pi)$ | $\frac{\sinh b\pi}{\pi}\sum_{-\infty}^\infty \frac{(-1)^n}{b - in}e^{in\theta}$ |
| 19. | $f(\theta) = e^{b\theta} \quad (0 < \theta < 2\pi)$ | $\frac{e^{2\pi b} - 1}{2\pi}\sum_{-\infty}^\infty \frac{e^{in\theta}}{b - in}$ |
| 20. | $f(\theta) = \sinh\theta \quad (-\pi < \theta < \pi)$ | $\frac{2\sinh\pi}{\pi}\sum_1^\infty \frac{(-1)^{n+1}n}{n^2+1}\sin n\theta$ |

## EXERCISES

1. Which of the following functions are continuous, piecewise continuous, or piecewise smoth on $[-\pi, \pi]$?

   a. $f(\theta) = \csc\,\theta$. b. $f(\theta) = (\sin\theta)^{1/3}$ c. $f(\theta) = (\sin\theta)^{4/3}$.
   d. $f(\theta) = \cos\theta$ if $\theta > 0, f(\theta) = -\cos\theta$ if $\theta \le 0$.
   e. $f(\theta) = \sin\theta$ if $\theta > 0, f(\theta) = \sin 2\theta$ if $\theta \le 0$.
   f. $f(\theta) = (\sin\theta)^{1/3}$ if $\theta < \frac{1}{2}\pi, f(\theta) = \cos\theta$ if $\theta \ge \frac{1}{2}\pi$.

   **answers**

   a. Not piecewise continuous. b. Continuous. c. Continuous and piecewise smooth. d. Piecewise smooth. e. Continuous and piecewise smooth. f Continuos and piecewise smooth.

2. To what values do the series in entries 6, 7, 12, and 18 of Table 1, §2.1, coverge at the points where their sums are disconinuous?

   **answers**

   6.0 7.1/2 12.$(4a)^{-1}$. 18. $\cosh(b\pi)$.

The Fourier series for a number of piecewise smooth functions are listed in Table 1, and Theorem 2.1 tells what the sums of these series are. By using this information and choosing suitable values of $\theta$ (usually $0, \frac{1}{2}\pi$, or $\pi$), derive the following formulas for the sums of numerical series. (The relevant entries from Table 1 are indicated in parentheses.)

3. $\displaystyle\sum_1^\infty \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-1}{4}$ (8).

4. $\displaystyle\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (16)

5. $\displaystyle\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$ (17)

6. $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2+b^2} = \frac{\pi}{2b}\text{csch }b\pi - \frac{1}{2b^2}$ (18 or 19)

7. $\displaystyle\sum_1^\infty \frac{1}{n^2+b^2} = \frac{\pi}{2b}\text{coth }b\pi - \frac{1}{2b^2}$ (18 or 19; be careful!)

8. Derive the result of entry 16 of Table 1 by using entry 1 and Theorem 2.4 (the integration theorem for Fourier series).

9. Starting from entry 16 of Table 1 and using Theorem 2.4, show that

   a. $\theta^3 - \pi^2\theta = 12\sum_1^\infty \frac{(-1)^n \sin n\theta}{n^3} \quad (-\pi \le \theta \le \pi);$

   b. $\theta^4 - 2\pi^2\theta^2 = 48\sum_1^\infty \frac{(-1)^{n+1}\cos\theta}{n^4} - \frac{7\pi^4}{15} \quad (-\pi \le \theta \le \pi)$

   c. $\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$

10. Use the entry 17 of Table 1, and verify that

$$\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^4} = \frac{\pi|\theta|^3}{24} - \frac{\pi^2\theta^2}{16} + \frac{\pi^4}{96}, \quad \text{on } (-\pi, \pi).$$

In Exercise 11-16, find both the Fourier cosine series and the Fourier sine series of the given function on the interval $[0, \pi]$. Try to use the results of Table 1, rather than working from scratch.To what values do these series converge when $\theta = 0$ and $\theta = \pi$?

11. $f(\theta) = 1$

    **answer**

$$\frac{4}{\pi}\sum_{n=1}^\infty \frac{\sin(2n-1)\theta}{2n-1}.$$

12. $f(\theta) = \pi - \theta$

    **answer**

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} \quad \text{and} \quad 2\sum_{n=1}^\infty \frac{\sin n\theta}{n}.$$

13. $f(\theta) = \sin\theta$

    **answer**

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^\infty \frac{\cos 2n\theta}{4n^2-1}, \quad \text{and} \quad \sin\theta.$$

14. $f(\theta) = \cos\theta$

    **answer**

$$\cos\theta, \quad \text{and} \quad \frac{8}{\pi}\sum_{n=1}^\infty \frac{n\sin 2n\theta}{4n^2-1}.$$

15. $f(\theta) = \theta^2$. (For the sine series, use entries 1 and 17 of Table 1.)

**answer**

$$\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\theta; \quad \text{and} \quad 2\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\theta-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\theta}{(2n-1)^3}.$$

16. $f(\theta) = \theta$ for $0 \le \theta \le \frac{1}{2}\pi$, $f(\theta) = \pi - \theta$ for $\frac{1}{2}\pi \le \theta \le \pi$. (For the sine series, use entry 11 of Table 1, and for the cosine series, entry 2.)

**answer**

$$\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(4n-2)\theta}{(2n-1)^2}; \quad \text{and} \quad \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(2n-1)\theta}{(2n-1)^2}.$$

In Exercises 17-21, expand the function in a series of the indicated type. For example, "sine series on $[0, \ell]$" means a series of the form $\sum b_n \sin(n\pi x/\ell)$. Again, use previously derived results as much as possible.

17. $f(x) = 1$; sine series on $[0, 6\pi]$.

**answer**

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)x}{6}.$$

18. $f(x) = 1 - x$; cosine series on $[0, 1]$.

**answer**

$$\frac{1}{2}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}.$$

19. $f(x) = 1$ for $0 < x < 2$, $f(x) = -1$ for $2 < x < 4$; cosine series on $[0, 4]$.

**answer**

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\frac{(2n-1)\pi x}{4}.$$

20. $f(x) = \ell x - x^2$; sine series on $[0, \ell]$.

**answer**

$$\frac{8\ell^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{\ell}.$$

21. $f(x) = e^x$; series of the form $\sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$ on $[0, 1]$.

**answer**

$$(e-1)\sum_{n=-\infty}^{\infty}\frac{e^{2\pi i n x}}{1-2\pi i n}.$$