Partial Differential Equations

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Outline: Mathematical Tools

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Some Function Space

Regularity requirement of classical solutions on $\Omega \subset \mathbb{R}^n$

- $\mathbf{u} \in C^1(\Omega)$: Every component of $\mathbf{u}$ has a continuous 1st order derivative.
- $\mathbf{u} \in C^2(\Omega)$:All partial derivatives of $\mathbf{u}$ of order 2 are continuous.
- $\mathbf{u} \in C^1(\mathbb{R}^+; C^2(\Omega))$: $\frac{\partial \mathbf{u}}{\partial t}, \ \frac{\partial^2 \mathbf{u}}{\partial x_i \partial x_j}, \ i, j = 1, \dots, n$ are continuous.

Example

- $C[0, \ T], \quad C^k[0, \ T]$
- $\mathcal{P}^{(q)}$: Space of polynomials of degree $\leq q$
- $\mathcal{T}^{(q)}$: Space of trigonimetric polynomials of degree $\leq q$

Vector Space

Definition
A set V of functions or vectors is called a *linear space*, or a *vector space*, if for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{R}$,

$$
\begin{aligned}
&\text{(i)} && u + \alpha v \in V, \\
&\text{(ii)} && (u + v) + w = u + (v + w), \\
&\text{(iii)} && u + v = v + u, \\
&\text{(iv)} && \exists\, 0 \in V \quad \text{such that} \quad u + 0 = 0 + u = u, \\
&\text{(v)} && \forall\, u \in V,\ \exists\,(-u) \in V, \quad \text{such that} \quad u + (-u) = 0 \in V, \\
&\text{(vi)} && (\alpha + \beta)u = \alpha u + \beta u, \\
&\text{(vii)} && \alpha(u + v) = \alpha u + \alpha v, \\
&\text{(viii)} && \alpha(\beta u) = (\alpha\beta)u, \quad \text{such that} \quad 1\, u = 1(u) := 1 \times u = u,
\end{aligned}
\tag{1}
$$

Definition
A subset $U \subset V$ of a vector space V is a *subspace* of V if

$$\alpha u + \beta v \in U, \quad \forall u, v \in U, \quad \text{and} \quad \forall \alpha, \beta \in \mathbb{R}.$$

Scalar product

Definition A scalar product (inner product) is a real valued operator on $V \times V$: $\langle u, v\rangle : V \times V \to \mathbb{R}$ (or $(u, v) : V \times V \to \mathbb{R}$) such that $\forall\, u,\ v,\ w \in V$ and $\forall\, \alpha \in \mathbb{R}$,

$$
\begin{aligned}
&\text{(i)} && \langle u, v\rangle = \langle v, u\rangle && \text{(symmetry)}\\
&\text{(ii)} && \langle u + \alpha v, w\rangle = \langle u, w\rangle + \alpha\langle v, w\rangle && \text{(bi-linearity)}\\
&\text{(iii)} && \langle v, v\rangle \geq 0 \quad \forall v \in V && \text{(positivity)}\\
&\text{(iv)} && \langle v, v\rangle = 0 \iff v = 0. &&
\end{aligned}
\tag{2}
$$

Definition An inner product, or scalar product, space is a vector space W associated with a *scalar product* $\langle\cdot,\cdot\rangle$, defined on $W \times W$.

Example Spaces of continuous functions on $[a, b] : C([a, b])$; k-times continuously differentiable functions on $[a, b] : C^k((a, b))$; polynomials of degree $\leq q$ on $[a, b] : \mathcal{P}^{(q)}(a, b)$ are inner product spaces with

$$
\langle u, v\rangle := \int_a^b u(x)v(x)dx. \tag{3}
$$

Note: space of all polynomials of degree $= q$ on $[a, b]$ is not a vector space.

Orthogonality

Definition
Two real functions $u(x)$ and $v(x)$ are orthogonal $(u \perp v)$, if $\langle u, v \rangle = 0$.

Definition
The space of all square integrable functions over $\Omega \in \mathbb{R}^n$ is the $L_2(\Omega)$-space.
If $u \in L_2(\Omega)$, then the L_2-norm of u associated with the above scalar product is

$$\begin{aligned} \|u\|_{L_2(\Omega)} &:= \sqrt{\langle u, u \rangle} = \Big(\int_\Omega |u(x)|^2 dx \Big)^{1/2}, \\ L_2(\Omega) &:= \{ u : \Omega \to \mathbb{R};\ \ \|u\|_{L_2(\Omega)} < \infty \}. \end{aligned} \tag{4}$$

In the L_2 case we usually suppress the subscript and write $\|u\|$ for $\|u\|_{L_2(\Omega)}$.
General L_p-spaces and norms, $1 \le p \le \infty$, are defined below.

$$L_p(\Omega) := \{ u : \|u\|_{L_p(\Omega)} = \Big(\int_\Omega |u(x)|^p \, dx \Big)^{1/p} < \infty \}, \quad 1 \le p < \infty$$

L_2-connected inequalitirs

Cauchy Schwarz' inequality (C-S)

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

Simple proof with $\|\cdot\|^2 := \langle\cdot,\cdot\rangle$:

$$\langle u, v\rangle = \|u\|\|v\| \cos(u, v).$$

Triangle inequality (A consequence of C-S):

$$\|u + v\| \leq \|u\| + \|v\|.$$

Space of Differentiable Functions

Definition ($\Omega \subset \mathbb{R}^n$ bounded open set)
$\mathcal{C}^k(\bar\Omega)$ is the set of all functions $u \in \mathcal{C}^k(\Omega)$ such that $D^\alpha u$ can be extended from Ω to $\bar\Omega$, for all multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| \le k$. The space $\mathcal{C}^k(\bar\Omega)$ is equipped with supremum norm

$$||u||_{\mathcal{C}^k(\bar\Omega)} := \sum_{|\alpha|\le k} \sup_{x\in\Omega} |D^\alpha u(x)|.$$

Hence, e.g., for $k = 0$,

$$\mathcal{C}(\bar\Omega) := \mathcal{C}^0(\bar\Omega) = \{u : ||u||_{\mathcal{C}(\bar\Omega)} = ||u||_{\mathcal{C}^0(\bar\Omega)} < \infty\}$$

where

$$||u||_{\mathcal{C}(\bar\Omega)} := ||u||_{\mathcal{C}^0(\bar\Omega)} = \sup_{x\in\Omega} |u(x)| = \max_{x\in\Omega} |u(x)|.$$

and for $k = 1$,

$$||u||_{\mathcal{C}^1(\bar\Omega)} := \sum_{|\alpha|\le 1} \sup_{x\in\Omega} |D^\alpha u(x)| = \sup_{x\in\Omega} |u(x)| + \sum_{i=1}^{n} \sup_{x\in\Omega} |\frac{\partial u}{\partial x_i}(x)|.$$

Space of Integrable Functions

Class of Lebesgue integrable functions on an open set $\Omega \subset \mathbb{R}^n$ (or $\Omega = \mathbb{R}^n$):

$$L_p(\Omega) := \Big\{ u : \Big(\int_\Omega |u(x)|^p \, dx \Big)^{1/p} < \infty \Big\}, \quad 1 \le p < \infty.$$

$u = v \in L_p(\Omega)$, if $u = v$ except on a set of measure zero. We say $u = v$ almost everywhere (denote by $u = v$ a.e.). $L_p(\Omega)$ is associated with norm

$$\begin{aligned} ||u||_{L_p(\Omega)} &:= \Big(\int_\Omega |u(x)|^p \, dx \Big)^{1/p}, \qquad 1 \le p < \infty \\ ||u||_{L_\infty(\Omega)} &:= ess. \sup_{x \in \Omega} |u(x)|, \end{aligned}$$

The latter is the $L_\infty(\Omega)$-norm, also know as maximum norm in case Ω is bounded.

Applications:
Modelling density of particles $L_1(\Omega)$-norm corresponds to a measure for the mass.
$L_2(\Omega)$-norm can be related to measuring the energy.
$||u||_{L_2(\Omega)} = (u, u)^{1/2} \ge 0$ (with equality only if $u \equiv 0$).

Sobolev Spaces

Definition (Ω open subset of $\mathbb{R}^n$; $k \geq 0$, integer; $p \in [1, \infty]$).
Sobolev space of order k and corresponding *Sobolev norms* are defined by

$$W_p^k(\Omega) := \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega),\ |\alpha| \leq k\}, \tag{5}$$

$$||u||_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| \leq k} ||D^\alpha u||^p_{L_p(\Omega)} \Big)^{1/p}, \qquad 1 \leq p < \infty \tag{6}$$

$$||u||_{W_\infty^k(\Omega)} := \sum_{|\alpha| \leq k} ||D^\alpha u||_{L_\infty(\Omega)}. \tag{7}$$

We also define the *seminorms*

$$|u|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| = k} ||D^\alpha u||^p_{L_p(\Omega)} \Big)^{1/p}, \qquad 1 \leq p < \infty. \tag{8}$$

Thus $||u||_{W_p^k(\Omega)} = \Big(\sum_{j=0}^k |u|^p_{W_p^j(\Omega)} \Big)^{1/p}, \quad 1 \leq p < \infty$, and
$|u|_{W_\infty^k(\Omega)} := \sum_{|\alpha|=k} ||D^\alpha u||_{L_\infty(\Omega)}$ which implies $||u||_{W_\infty^k(\Omega)} = \sum_{j=0}^k |u|_{W_\infty^j(\Omega)}$.

Hilbert Spaces

For $k = 0$, $|\cdot|_{W_p^k(\Omega)}$ is the usual L_p-norm. *Seminorm* is used when $k \geq 1$.
$p = 2$ and $k = 1, 2$ called the *Hilbert spaces* and denoted by $H^k(\Omega)$:

$$H^1(\Omega) := \{u \in L_2(\Omega) : \frac{\partial u}{\partial x_j} \in L_2(\Omega),\ j = 1, \dots, n\} \tag{9}$$

$$||u||_{H^1(\Omega)} := \Big(||u||^2_{L_2(\Omega)} + \sum_{j=1}^n ||\frac{\partial u}{\partial x_j}||^2_{L_2(\Omega)}\Big)^{1/2},\ \ |u|_{H^1(\Omega)} := \Big(\sum_{j=1}^n ||\frac{\partial u}{\partial x_j}||^2_{L_2(\Omega)}\Big)^{1/2}.$$

$$H^2(\Omega) := \Big\{u : u, \frac{\partial u}{\partial x_j},\ \frac{\partial^2 u}{\partial x_i \partial x_j} \in L_2(\Omega),\ i, j = 1, \dots, n\Big\}$$

$$||u||_{H^2(\Omega)} := \Big(||u||^2_{L_2(\Omega)} + \sum_{j=1}^n ||\frac{\partial u}{\partial x_j}||^2_{L_2(\Omega)} + \sum_{i,j=1}^n ||\frac{\partial^2 u}{\partial x_i \partial x_j}||^2_{L_2(\Omega)}\Big)^{1/2},$$

$$|u|_{H^2(\Omega)} := \Big(\sum_{i,j=1}^n ||\frac{\partial u}{\partial x_i \partial x_j}||^2_{L_2(\Omega)}\Big)^{1/2}.$$

Basic Inequalities

Definition: p and q, $1 < p$, $q < \infty$ are conjugate exponents if $1/p + 1/q = 1$.
Minkowski and Hölder Inequalities:

$$||u + v||_{L_p(\Omega)} \le ||u||_{L_p(\Omega)} + ||v||_{L_p(\Omega)}, \qquad \text{(Minkowski)} \tag{10}$$

If $u \in L_p(\Omega)$, $v \in L_q(\Omega)$ and $1/p + 1/q = 1$, then

$$\int_\Omega u(x)v(x)\,dx \le ||u||_{L_p(\Omega)}||v||_{L_q(\Omega)} \qquad \text{(Hölder)} \tag{11}$$

Poincaré inequality: (u, solution of a homogeneous Dirichlet problem)
u, $|\nabla u| \in L_2(\Omega)$, $\Omega \subset \mathbb{R}^d$ (bdd). Then, $\exists C_\Omega$, independent of u such that

$$\|u\| \le C_\Omega \|\nabla u\|. \tag{12}$$

Trace inequality: Let $u \in W_p^1(\Omega)$, then $\exists C$ constant such that, for $1 \le p \le \infty$,

$$||u||_{L_p(\partial\Omega)} \le C||u||^{1-1/p}_{L_p(\Omega)}||u||^{1/p}_{W_p^1(\Omega)}. \tag{13}$$

In particular for $p = 2$ we have that

$$||u||^2_{L_2(\partial\Omega)} \le C||u||_{L_2(\Omega)}||u||_{H^1(\Omega)}. \tag{14}$$

Gronwall's lemma

Suppose that u is a non-negative continuous function such that

$$\varphi(t) \le \alpha(t) - \int_0^t \beta(s)\varphi(s)\, ds, \qquad t > 0,$$

(a) If β is nonnegative then

$$\varphi(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\Big(\int_s^t \beta(r)\, dr\Big)\, ds, \qquad t > 0.$$

(b) If, in addition, α is non-decreasing, then

$$\varphi(t) \le \alpha(t) \exp\Big(\int_0^t \beta(s)\, ds\Big).$$

Boundary value problem

$u(x)$: displacement of the bar at a point $x \in I = (0, 1)$
$a(x)$: *modulus of elasticity*
$f(x)$: *load function*.
Then u satisfies the boundary value problem:

$$(BVP)_1 \qquad \begin{cases} -\Big(a(x)u'(x)\Big)' = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{15}$$

Equation (15) is modelling also the stationary heat flux derived in Chapter 1.
Assume $a(x)$ is continuous in $(0, 1)$ and bounded for $0 \le x \le 1$.
Let v, $v' \in L_2(0, 1)$, and recall the L_2-based Sobolev space: *Hilbert space*

$$H_0^1(0,1) = \Big\{ v : \int_0^1 (v(x)^2 + v'(x)^2) dx < \infty, \quad v(0) = v(1) = 0 \Big\}. \tag{16}$$

As a consequence of Poincaré inequality $H_0^1(0, 1)$ is identically defined as

$$H_0^1(0,1) = \Big\{ v : \int_0^1 v'(x)^2 \, dx < \infty, \quad v(0) = v(1) = 0 \Big\}. \tag{17}$$

Variational Formulation (VF)

Multiply $(\mathrm{BVP})_1$ by a test function $v \in H_0^1(0,1)$ and integrate over $(0,1)$:

$$-\int_0^1 (a(x)u'(x))'v(x)dx = \int_0^1 f(x)v(x)dx. \qquad (18)$$

Integration by parts yields

$$-\Big[a(x)u'(x)v(x)\Big]_0^1 + \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx. \qquad (19)$$

Since $v(0) = v(1) = 0$ we obtain the *variational formulation* for the problem (15):
Find $u \in H_0^1(0,1)$ such that

$$(\mathrm{VF})_1 \qquad \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \quad \forall v \in H_0^1. \qquad (20)$$

Thus, we have shown that if u satisfies $(\mathrm{BVP})_1$, then u satisfies $(\mathrm{VF})_1$:

$$(\mathrm{BVP})_1 \Longrightarrow (\mathrm{VF})_1.$$

We shall show the reverse implication is also true, i.e., $(\mathrm{VF})_1 \Longrightarrow (\mathrm{BVP})_1$.

The minimization Problem

For problem (15), we formulate yet another equivalent problem:

Find $u \in H_0^1$: $F(u) \leq F(w)$, $\forall w \in H_0^1$,

$$(\mathrm{MP})_1 \qquad F(w) = \underbrace{\frac{1}{2}\int_0^1 a(w')^2 dx}_{\text{Internal (elastic) energy}} - \underbrace{\int_0^1 fw dx}_{\text{Load potential}}. \tag{21}$$

This means that the solution u minimizes the energy functional $F(w)$.

Theorem

$$(BVP)_1 \text{ "} \Longleftrightarrow \text{" } (VF)_1 \Longleftrightarrow (MP)_1.$$

Recall that " $\Longleftrightarrow$ " is a conditional equivalence, requiring u to be twice differentiable, for the reverse implication.

An Abstract Framework

Consider the simple one-dimensional boundary value problem:

$$(BVP): \qquad -u''(x) = f(x), \quad 0 < x < 1 \qquad u(0) = u(1) = 0, \tag{22}$$

Let $V = \mathcal{H}_0^1$ and define

$$a(u, v) := (u, v) := \int_0^1 u'(x)v'(x)dx, \tag{23}$$

then $(\cdot, \cdot)$ is *symmetric*, i.e. $(u, v) = (v, u)$, *bilinear* , and *positive definite*:

$$(u, u) \geq 0, \quad \text{and } (u, u) = 0 \Longleftrightarrow u \equiv 0.$$

Further, for $f \in L_2(0, 1)$, let

$$\ell(v) = \int_0^1 fv\, dx, \qquad \forall v \in \mathcal{H}_0^1, \tag{24}$$

Then our (VF) can be restated as : Find $u \in \mathcal{H}_0^1$ such that

$$a(u, v) = \ell(v), \qquad \forall v \in \mathcal{H}_0^1. \tag{25}$$

General form, Hilbers spce, coercivity

Generalizing the above *(e.g. to a Hilbert space defined below)*, to a bilinear form $a(\cdot,\cdot)$, and a linear form $L(\cdot)$, we get the *abstract problem*: Find $u \in V$, such that

$$a(u,v) = L(v) \qquad \forall v \in V. \tag{26}$$

Definition. A linear space V (vector space) with the norm $\|\cdot\|$ is called *complete* if every Cauchy sequence in V is convergent.

Definition. A *Hilbert space* is a *complete* linear space with a scalar product.

Definition. Let $\|\cdot\|_V$ be a norm corresponding to a scalar product $(\cdot,\cdot)_V$ defined on $V \times V$. Then the bilinear form $a(\cdot,\cdot)$ is called *coercive* (*V-elliptic*), and $a(\cdot,\cdot)$ and $L(\cdot)$ are continuous, if there are constants c_i, $i = 1, 2, 3$ such that:

$$a(v,v) \geq c_1\|v\|_V^2, \quad \forall v \in V \qquad \text{(coercivity)} \tag{27}$$

$$|a(u,v)| \leq c_2\|u\|_V\|v\|_V, \quad \forall u,v \in V \qquad (a \text{ is bounded}) \tag{28}$$

$$|L(v)| \leq c_3\|v\|_V, \quad \forall v \in V \qquad (L \text{ is bounded}). \tag{29}$$

Existence, Uniqueness; Riesz and Lax-Milgram Theorem

Recalling

$$(u, v) = \int_0^1 u'(x)v'(x)dx \qquad \text{and} \quad \ell(v) = \int_0^1 f(x)v(x)dx,$$

we may redefine variational formulation (VF) and minimization problem (MP) in an abstract form as (V) and (M):

(V) Find $u \in \mathcal{H}_0^1$, such that $(u, v) = \ell(v)$ for all $v \in \mathcal{H}_0^1$

(M) Find $u \in \mathcal{H}_0^1$, such that $F(u) = \min_{v \in \mathcal{H}_0^1} F(v) \qquad F(v) = \frac{1}{2}\|v\|^2 - \ell(v).$

Riesz and Lax-Milgram Theorem:

There exists a unique solution for the, equivalent, problems (V) and (M).