Partial Differential Equations

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Course Information: TMA372, MMG800, MVE455

Outline

- **Theoretical approach**
- **Approximation procedure**
- **Computational aspects**

Content

- **I. Trinities, . . .,**
- **II. Vector/Function-spaces: The working environment**
- **III. Hadamards priciples: Existance, Uniqueness, Stability**
- **IV. Polynomial Approximation/Interpolation, Quadrature rule**
- **V. Galerkin Methods, Variational formulation, Minimization problem**
- **VI. Assignments** $I\&II$**: Compulsary, Bonus generating**
- **VII. Final: (**≥ 1 **theory)+ 5 problems;**

  Breaking: 40%(3), 60%(4), 80 (5)% (Before add of bonus points)

  GU: $50\%(GK), 74\%(VG)$

Trinities

The usual three differential operators (of second order):

Laplace operator $$\Delta_n := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2},$$

Diffusion operator $$\frac{\partial}{\partial t} - \Delta_n,$$

d'Alembert operator $$\Box := \frac{\partial^2}{\partial t^2} - \Delta_n,$$

$\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the space variabl $t \in \mathbb{R}^+$ is the time,
$\partial^2/\partial x_i^2$ denotes the second partial derivative with respect to x_i, $1 \le i \le n$.
We also define the **gradient operator**:

$$\nabla_n := \Big(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\Big).$$

Classifying second order PDE in 2 dimensions

I) The constant coefficient case

$$Au_{xx}(x,y) + 2Bu_{xy}(x,y) + Cu_{yy}(x,y) + Du_x(x,y) + Eu_y(x,y) + Fu(x,y) + G = 0$$

The **Discriminant**: $d = AC - B^2$: determines the equation type:

Elliptic: if $d > 0$, *Parabolic:* if $d = 0$, and *Hyperbolic:* if $d < 0$.

Example:

Potential equation	Heat equation	Wave equation
$\Delta u = 0$	$\frac{\partial u}{\partial t} - \Delta u = 0$	$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$
$u_{xx} + u_{yy} = 0$	$u_t - u_{xx} = 0$	$u_{tt} - u_{xx} = 0$
$A = C = 1, B = 0$	$B = C = 0, A = -1$	$A = -1, B = 0, C = 1$
$d = 1$(elliptic)	$d = 0$ (parabolic)	$d = -1$ (hyperbolic).

Classifying (continued)

II) Variable coefficient case (only local classification)

Example For Tricomi equation of gas dynamics:

$$Lu(x,y) = yu_{xx} + u_{yy} \quad \Longrightarrow A = y,\ B = 0,\ C = 1,\ d = AC - B^2 = y:$$

Domain of ellipticity: $y > 0$, so on,

The usual three types of problems in differential equation

I. Initial value problems (IVP)

Example: Wave equation, on real line, augmented with the given initial data:

$$\begin{cases} u_{tt} - u_{xx} = 0, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & -\infty < x < \infty, \quad t = 0. \end{cases}$$

For the unbounded spatial domain $(-\infty, \infty)$ it is required that $u(x,t) \to 0$ (or $\to u_\infty$ =constant) as $|x| \to \infty$ (corresponds to two boundary conditions).

II. Boundary value problems (BVP)

Example: One-dimensional stationary heat equation:

$$-\Big(a(x)u'(x)\Big)' = f(x), \qquad \text{for } \ 0 < x < 1.$$

To determine $u(x)$ uniquely, the equation is complemented by boundary conditions; for example $u(0) = u_0$ and $u(1) = u_1$, (u_0 and u_1: real numbers).

Problems (continued)

III. Eigenvalue problems (EVP)

Example: vibrating string given by

$$-u''(x) = \lambda u(x), \qquad u(0) = u(\pi) = 0,$$

where λ is an eigenvalue and $u(x)$ is an eigenfunction.

The usual three types of boundary conditions

I. Dirichlet boundary condition: Solution known at the boundary of the domain,

$$u(\mathbf{x}, t) = f(\mathbf{x}), \quad \text{for} \quad \mathbf{x} = (x_1, \dots, x_n) \in \partial\Omega, \quad t > 0.$$

II. Neumann boundary condition: Derivative of solution is given in a direction:

$$\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \mathbf{grad}(u) = \mathbf{n} \cdot \nabla u = f(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega,$$

$\mathbf{n} = \mathbf{n}(\mathbf{x})$: *outward unit normal to* $\partial\Omega$ *at* $\mathbf{x} \in \partial\Omega$,

III. Robin boundary condition: (a combination of I and II),

$$\frac{\partial u}{\partial \mathbf{n}} + k \cdot u(\mathbf{x}, t) = f(\mathbf{x}), \qquad k > 0, \quad \mathbf{x} = (x_1, \dots, x_n) \in \partial\Omega.$$

The usual three studies

In Theory:

I. Existence: there exists at least one solution u.

II. Uniqueness: we have either one solution or no solutions at all.

III. Stability: the solution depends continuously on the data.

In applications:

I. Construction of the solution.

II. Regularity: how smooth is the found solution.

III. Approximation: when an exact construction is impossible.

Three general approaches for solving differential equations

I. Separation of Variables Method:

Separation of variables technique reduces the (PDEs) to simpler eigenvalue problems (ODEs). This method is known as Fourier method, or solution by eigenfunction expansion.

II. Variational Formulation Method:

Variational formulation or the multiplier method is a strategy for extracting information by multiplying a differential equation by suitable test functions and then integrating. (Referred to as The Energy Method subject of our study).

III. Green's Function Method:

Fundamental solutions, or solution of integral equations (advanced PDE).