TMA371/MAN660 Partial Differential Equations TM, E3, GU
2000-12-13. Solutions

1. Show that the linear interpolant $\pi_1 f$ of a function f on an interval $I = [a, b]$ satisfies the interpolation estimates:

$$\begin{array}{ll}(a) & \|f - \pi_1 f\|_{L^\infty(a,b)} \le (b-a)^2 \|f''\|_{L^\infty(a,b)}, \\ (b) & \|f' - (\pi_1 f)'\|_{L^\infty(a,b)} \le (b-a)\|f''\|_{L^\infty(a,b)}.\end{array}$$

Solution
(a) See lecture notes, chapter 5, Theorem 1, page 5.3.
(b) Similar. (Or see CDE, chapter 5).

2. Let U be the $cG(1)$ approximation of u satisfying the initial value problem

$$\dot{u} + au = f, \quad t > 0, \qquad u(0) = u_0.$$

Let k be the time step and show that for $a = 1$,

$$|(u-U)(T)| \le \min\Big(\|k(f - \dot{U} - U)\|_{L^\infty[0,T]}, T\|k^2 \ddot{u}\|_{L^\infty[0,T]}\Big).$$

Solution
See lecture notes, chap. 16, Lemmas 1 and 2, page 16.10. (Or see CDE, chapter 16).

3. Show that the dual of the problem 2:

$$-\dot{\varphi} + a\varphi = 0, \quad 0 \le t < t_N, \qquad \varphi(t_N) = e_N,$$

satisfies the stability estimate:

$$|\varphi(t)| \le \begin{cases} \exp(\mathcal{A} t_N)|e_N|, & \text{if } |a(t)| \le \mathcal{A} \quad \text{for all } 0 \le t \le t_N \\ |e_N|, & \text{if } a(t) \ge 0 \quad \text{for all } 0 \le t \le t_N. \end{cases}$$

Solution
See lecture notes, chapter 9, Theorem 9.2, page 9.4. (Or see CDE, chapter 9).

4. Consider the boundary value problem

$$-(au')' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0.$$

(a) Show that the solution of this problem minimizes the energy integral

$$F(v) = \frac{1}{2}\int_0^1 a(v')^2 - \int_0^1 fv,$$

i.e., $u \in V$ where V is some function space and $F(u) = \min_{v \in V} F(v)$.
(b) Show that for $a = 1$ and a corresponding discrete minimum $F(U) = \min_{v \in V_h} F(v)$, with $U \in V_h \subset V$ we have that

$$F(U) = F(u) + \frac{1}{2}\|(u-U)'\|^2.$$

(c) Let $a = 1$ and show an a posteriori error estimate for the discrete energy minimum: i.e., for $|F(U) - F(u)|$, with V_h being the space of piecewise linear functions on subintervals of length h.

Solution
(a) See lecture notes, chapter 8, page 8.3 (the only modification is that you put $g_1 = 0$). Thus from the differential equation for u it follows, after multiplication by w and using integration by parts, that

$$\int_0^1 au'w'\,dx = \int_0^1 fw\,dx. \tag{1}$$

Hence, for arbitrary $v = u + w$ we have that

$$F(v) = F(u+v) = F(u) + \int_0^1 au'w'\,dx - \int_0^1 fw\,dx + \int_0^1 a(w')^2\,dx \geq F(u), \tag{2}$$

since using (1) the first two integrals are add up to zero and the third integral is ≥ 0.
(b) Let $a = 1$ and use the following Galerkin orthogonality:

$$\int_0^1 (u-U)'v'\,dx = 0, \qquad \forall v \in V_h, \tag{3}$$

with v replaced by U to get

$$\begin{aligned} ||(u-U)'||^2 &= \int_0^1 (u-U)'(u-U)'\,dx = \int_0^1 (u-U)'(u+U)'\,dx \\ &= \int_0^1 (u')^2\,dx - \int_0^1 (U')^2\,dx = -2F(u) + 2F(U), \end{aligned} \tag{4}$$

where we have used the identities

$$2F(u) = ||u'||^2 - 2\int_0^1 fu\,dx = \{\text{with } w = u \text{ and } a = 1 \text{ in } (1)\} = -||u'||^2, \tag{5}$$

and similarly $2F(U) = -||U'||^2$.
(c) Recall that in the one dimensional case, we have the interpolation estimate, (see problem 1),

$$||u' - U'|| \leq C_i||hf||,$$

where C_i is an interpolation constant. This gives using (b) that

$$|F(U) - F(u)| \leq C_i^2||hf||^2.$$

5. Apply the finite element method $cG(1)$ and prove an a posteriori error estimate for the following Neumann problem:

$$-\nabla \cdot (a\nabla u) + u = f, \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad a\partial_n u = g, \quad \text{on } \Gamma = \partial\Omega.$$

Solution
We start to give the continuous and discrete variational formulations: Multiply by a test function $v \in V = \{v : \int_\Omega (a|\nabla v|^2 + v^2)\,dx < \infty\}$ and integrate over Ω.

$$-\int_\Omega \nabla \cdot (a\nabla u)v\,dx + \int_\Omega uv\,dx = \int_\Omega fv\,dx, \quad \forall v \in V.$$

Using Green's formula we have that

$$
\begin{aligned}
-\int_\Omega \nabla\cdot(a\nabla u)v\,dx &= -\int_\Gamma (a\nabla u)\cdot nv\,ds + \int_\Omega (a\nabla u)\cdot\nabla v\,dx\\
&= \Big\{a\nabla u\cdot n = a\partial_n u = g \text{ on } \Gamma\Big\} = \int_\Omega (a\nabla u)\cdot\nabla v\,dx - \int_\Gamma gv\,ds.
\end{aligned}
$$

Thus we have the <u>continuous variational</u> formulation:
Find $u\in V$ such that

$$
\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega uv\,dx = \int_\Omega fv\,dx + \int_\Gamma gv\,ds, \quad \forall v\in V. \tag{6}
$$

The corresponding <u>$cG(1)$ method</u> for this problem reads
Find $U\in V_h$ such that

$$
\int_\Omega (a\nabla U)\cdot\nabla v\,dx + \int_\Omega Uv\,dx = \int_\Omega fv\,dx + \int_\Gamma gv\,ds, \quad \forall v\in V_h \tag{7}
$$

Thus $(6)-(7)$ yields the <u>Galerkin orthogonality</u>

$$
\int_\Omega a\nabla(u-U)\cdot\nabla v\,dx + \int_\Omega (u-U)v\,dx = 0, \quad \forall v\in V_h. \tag{8}
$$

<u>A Posteriori Error Estimates:</u> Let $e=u-U, \quad U\in V_h$,

$$
\begin{aligned}
||e||_E^2 &= (e,u-U)_E = (e,u)_E - (e,U)_E = \{(6),\ e\in V\}\\
&= \int_\Omega fe\,dx + \int_\Gamma ge\,ds - (e,U)_E = \{(7),\ \pi_h e\in V_h\}\\
&= \int_\Omega f(e-\pi_h e)\,dx + \int_\Gamma g(e-\pi_h e)\,ds - (e-\pi_h e,U)_E.
\end{aligned} \tag{9}
$$

Recall that

$$
\begin{aligned}
(e-\pi_h e,U)_E &= \int_\Omega a\nabla U\cdot\nabla(e-\pi_h e)\,dx + \int_\Omega U(e-\pi_h e)\,dx\\
&= \sum_K \int_K a\nabla U\cdot\nabla(e-\pi_h e)\,dx + \int_\Omega U(e-\pi_h e)\,dx.
\end{aligned} \tag{10}
$$

Hence using the following version of Green's formula

$$
\int_\Omega \nabla U\cdot\nabla w\,dx = \int_\Gamma a\partial_n U w\,ds - \int_\Omega w\nabla\cdot(a\nabla U)\,dx,
$$

we may write

$$
\begin{aligned}
(e-\pi_h e,U)_E &= \sum_K \Big(\int_{\partial K} a\partial_n U(e-\pi_h e)\,ds - \int_K \nabla\cdot(a\nabla U)(e-\pi_h e)\,dx\Big)\\
&\quad + \int_\Omega U(e-\pi_h e)\,dx\\
&= \int_\Omega (U-\nabla\cdot(a\nabla U))(e-\pi_h e)\,dx + \sum_K \int_{\partial K} a\partial_n U(e-\pi_h e)\,ds.
\end{aligned}
$$

Substituting into (9) we get

$$
\begin{aligned}
||e||_E^2 = &\int_\Omega f(e-\pi_h e)\,dx + \int_\Gamma g(e-\pi_h e)\,ds + \int_\Omega (\nabla\cdot(a\nabla U)-U)(e-\pi_h e)\,dx \\
&-\sum_K \int_{\partial K} a\partial_n U(e-\pi_h e)\,ds \\
= &\int_\Omega (f+\nabla\cdot(a\nabla U)-U)(e-\pi_h e)\,dx + \int_\Gamma g(e-\pi_h e)\,ds \\
&-\sum_K \int_{\partial K} a\partial_n U(e-\pi_h e)\,ds.
\end{aligned}
$$

Let now

$$
R_1(U) = |f+\nabla\cdot(a\nabla U)-U|. \tag{11}
$$

Then

$$
\begin{aligned}
||e||_E^2 \le &\int_\Omega R_1(U)|e-\pi_h e|\,dx + \Big|\sum_K \int_{\partial K} a\partial_n U(e-\pi_h e)\,ds\Big| \\
&+ \Big|\int_\Gamma g(e-\pi_h e)\,ds\Big| := I + II + III.
\end{aligned} \tag{12}
$$

Below we shall estimate each term separately:

$$
\begin{aligned}
I = &\Big|\int_\Omega R_1(U)|e-\pi_h e|\,dx\Big| = \Big|\int_\Omega \frac{h}{\sqrt{a}} R_1(U)\frac{\sqrt{a}}{h}|e-\pi_h e|\,dx\Big| \\
&\le \Big(\int_\Omega a^{-1}(hR_1(U))^2\Big)^{1/2}\Big(\int_\Omega a(h^{-1}(e-\pi_h e))^2\Big)^{1/2} \\
&\le ||hR_1(U)||_{a^{-1}}\cdot ||h^{-1}(e-\pi_h e)||_{L_2(\Omega)}\cdot \max_\Omega \sqrt{a} \\
&\le ||hR_1(U)||_{a^{-1}}\cdot C_i||\nabla e||_{L_2(\Omega)}\cdot \max_\Omega \sqrt{a} \\
&\le ||hR_1(U)||_{a^{-1}}\cdot C_i||\nabla e||_a\cdot \max_\Omega \sqrt{a}/\min_\Omega \sqrt{a} \\
&\le C_i\Big(\max_\Omega \sqrt{a}/\min_\Omega \sqrt{a}\Big)||hR_1(U)||_{a^{-1}}\cdot ||e||_E
\end{aligned}
$$

Similarly

$$
\begin{aligned}
II = &\Big|\sum_K \int_{\partial K} a\partial_n U(e-\pi_h e)\,ds\Big| \le \sum_K \Big|\int_{\partial K} a\partial_n U(e-\pi_h e)\,ds\Big| \\
&\le \max_\Omega(a)\sum_K \Big|\int_{\partial K} h_K^{1/2}\partial_n U h_K^{-1/2}(e-\pi_h e)\,ds\Big| \\
&\le \max_\Omega(a)\sum_K \Big(\int_{\partial K} h_K|\partial_n U|^2\,ds\Big)^{1/2}\cdot\Big(\int_{\partial K} h_K^{-1}(e-\pi_h e)^2\,ds\Big)^{1/2} \\
&\le \max_\Omega(a)\sum_K \Big(\int_{\partial K} h_K|\partial_n U|^2\,ds\Big)^{1/2}\cdot\Big(\sum_K h_K\int_{\partial K}(e-\pi_h e)^2\,ds\Big)^{1/2} \\
= &\max_\Omega(a)\sum_K \Big(\int_{\partial K} h_K|\partial_n U|^2\,ds\Big)^{1/2}\cdot\Big(\sum_K h_K||e-\pi_h e||_{L_2(\Omega)}^2\Big)^{1/2}.
\end{aligned}
$$

Thus usinfg Theorem 14.2, (14.9) yields

$$
\begin{aligned}
II &\le \max_\Omega(a)\sum_K \Big(\int_{\partial K} h_K|\partial_n U|^2\,ds\Big)^{1/2}\cdot C_i||\nabla e||_{L_2(\Omega)}\\
&\le \Big(\max_\Omega(a)/\min_\Omega(\sqrt{a})\Big)\sum_K \Big(\int_{\partial K} h_K|\partial_n U|^2\,ds\Big)^{1/2}\cdot C_i||\nabla e||_a\\
&\le C_i\Big(\max_\Omega(a)/\min_\Omega(\sqrt{a})\Big)\sum_K \Big(\int_{\partial K} h_K|\partial_n U|^2\,ds\Big)^{1/2}||\nabla e||_E \le \dots\\
&\le C_i\Big(\max_\Omega(a)/\min_\Omega(a)\Big)\Big(\max\sqrt{h}\Big)\frac{\max_\Omega\Big(\int_{\partial K}\,ds\Big)^{1/2}}{\min_\Omega\Big(\int_K\,dx\Big)^{1/2}}||hR_2(U)||_{L_2(\Omega)}||\nabla e||_E\\
&\le C_i\Big(\max_\Omega(a^{3/2})/\min_\Omega(a)\Big)\Big(\max\sqrt{h}\Big)\frac{\max_\Omega\Big(\int_{\partial K}\,ds\Big)^{1/2}}{\min_\Omega\Big(\int_K\,dx\Big)^{1/2}}||hR_2(U)||_{a^{-1}}||\nabla e||_E,
\end{aligned}
$$

where

$$R_2(U) = h_K^{-1}\max_K|\partial_n U|. \tag{13}$$

Now it remains to estimate the last term:

$$
\begin{aligned}
III &= \Big|\int_\Gamma g(e-\pi_h e)\Big|\,ds \le \sum_K \Big|\int_{\partial K} h_K^{1/2} g h_K^{-1/2}(e-\pi_h e)\Big|\,ds\\
&\le \sum_K\Big[\Big(h_K\int_{\partial K} g^2\,ds\Big)^{1/2}\Big(h_K^{-1}\int_{\partial K}(e-\pi_h e)^2\,ds\Big)^{1/2}\Big]\\
&\le \Big(\sum_K h_K\int_{\partial K} g^2\,ds\Big)^{1/2}\Big(\sum_K h_K^{-1}||e-\pi_h e||^2_{L_2(\partial K)}\Big)^{1/2}\\
&\le \Big\{\text{Theorem 14.2, (14.9)}\Big\} \le \Big(\sum_K h_K\int_{\partial K} g^2\,ds\Big)^{1/2} C_i||\nabla e||_{L_2(\Omega)}\\
&\le \frac{C_i}{\min_\Omega\sqrt{a}}\cdot\frac{1}{\min_\Omega\sqrt{h}}\Big(\sum_K h_K\int_{\partial K} g^2\,ds\Big)^{1/2}||\nabla e||_a\\
&\le \frac{C_i}{\min_\Omega\sqrt{a}}\cdot\frac{1}{\min_\Omega\sqrt{h}}\frac{\max_\Omega\int_{\partial K}\,ds}{\min_\Omega\int_K\,dx}\Big(\sum_K\int_K\Big(h_K\max_K|g|\Big)^2\,dx\Big)^{1/2}||\nabla e||_E\\
&\le \frac{C_i\max_\Omega\sqrt{a}}{\min_\Omega\sqrt{a}\,\min_\Omega\sqrt{h}}\frac{\max_\Omega\int_{\partial K}\,ds}{\min_\Omega\int_K\,dx}||hR_3(U)||_{a^{-1}}||\nabla e||_E,
\end{aligned}
$$

where

$$R_3(U) = \max_K|g|. \tag{14}$$

Inserting I, II and III in (12) we get the following a posteriori error estimate:

$$||e||_E \le C_1||hR_1(U)||_{a^{-1}} + C_2||hR_2(U)||_{a^{-1}} + C_3||hR_3(U)||_{a^{-1}},$$

with $R_i(U)$'s $i = 1,2,3$ given by (11), (13) and (14) above.

MA