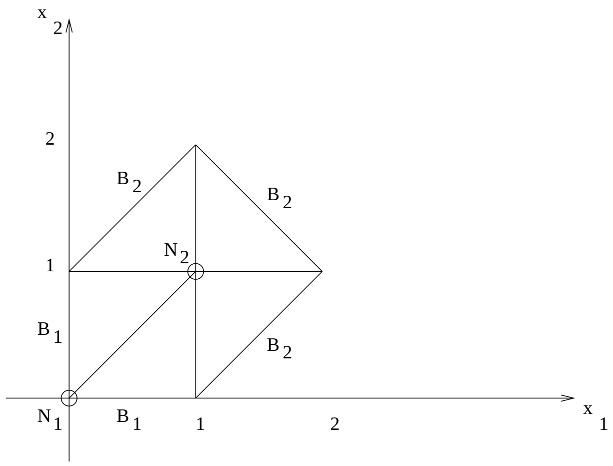
TMA371/MAN660 Partial Differential Equations TM, IMP, E3, GU
2003-04-22. Solutions

1. Let Ω be the triangulated domain below. Compute the cG(1) solution of $-\Delta u = 0$ in Ω with the Neumann data: $\partial_n u = 3$ on B_1 and Dirichlet condition: $u = 0$ on B_2.



Solution.
Variational Formulation: Using Green's formula we have that

$$
\begin{aligned}
0 &= \int_\Omega -\Delta u v\, dx = \{\text{Green's}\} = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\partial_n u) v \\
&= \{\Gamma := \partial\Omega := B_1 \cup B_2\} = \{v = 0 \text{ on } B_2, \text{ and } \partial_n u = 3 \text{ on } B_1\} \\
&= \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{B_1} 3v\, ds
\end{aligned} \tag{1}
$$

Thus we have the finite element formulation: Find piecewise linear function $U \in V_h$ such that

$$
\int_\Omega \nabla U \cdot \nabla v = \int_{B_1} 3v\, ds, \quad \forall v \in V_h. \tag{2}
$$

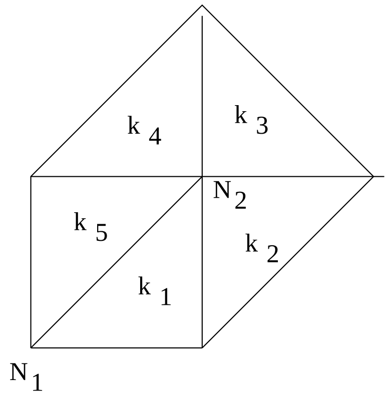
Let now

$$
U(x) = U_1\varphi_1(x) + U_2\varphi_2(x), \tag{3}
$$

where φ_i are the piecewise linears basis functions for the above discretization of Ω with $\varphi_i(N_j) = \delta_{ij},\ i,j = 1,2$. We insert (3) in (2) and let $v = \varphi_i,\ i = 1,2$ to obtain a 2×2 system viz,

$$
\begin{cases}
\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_1\, dx\, U_1 + \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_1\, dx\, U_2 = 3\int_{B_1} \varphi_1\, ds, \\
\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_2\, dx\, U_1 + \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_2\, dx\, U_2 = 3\int_{B_1} \varphi_2\, ds.
\end{cases} \tag{4}
$$

Note that using the orientation in the figur below we have



$$\begin{array}{ll}
\nabla\varphi_1\Big|_{k_1} = (-1,0) & \nabla\varphi_2\Big|_{k_1} = (0,1) \\
\nabla\varphi_1\Big|_{k_2} = (0,0) & \nabla\varphi_2\Big|_{k_2} = (-1,1) \\
\nabla\varphi_1\Big|_{k_3} = (0,0) & \nabla\varphi_2\Big|_{k_3} = (-1,-1) \\
\nabla\varphi_1\Big|_{k_4} = (0,0) & \nabla\varphi_2\Big|_{k_4} = (1,-1) \\
\nabla\varphi_1\Big|_{k_5} = (0,-1) & \nabla\varphi_2\Big|_{k_5} = (1,0)
\end{array}$$

Thus

$$\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_2 \, dx = \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_1 \, dx = 0,$$

and

$$\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx = \sum_{i=1}^{5} |k_i| \Big(\nabla\varphi_1|_{k_i} \cdot \nabla\varphi_1|_{k_i} \Big)$$

$$\frac{1}{2} \times (-1,0)\cdot(-1,0) + \frac{1}{2} \times (0,-1)\cdot(0,-1) = \frac{1}{2} + \frac{1}{2} = 1.$$

Similarly

$$\int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_2 \, dx = \sum_{i=1}^{5} |k_i| \Big(\nabla\varphi_2|_{k_i} \cdot \nabla\varphi_2|_{k_i} \Big) = \frac{1}{2} \times \Big((0,1)\cdot(0,1)$$

$$+(-1,1)\cdot(-1,1) + (-1,-1)\cdot(-1,-1) + (1,-1)\cdot(1,-1) + (1,0)\cdot(1,0) \Big)$$

$$= \frac{1}{2} \times \Big(1+2+2+2+1 \Big) = 4.$$

As for the right hand side we have

$$3\int_{B_1} \varphi_1 = 3 \times \text{aread of the side alonge } B_1 = 3\Big(1/2 + 1/2\Big) = 3,$$

while

$$3\int_{B_1} \varphi_2 = 0.$$

Summing up we have a trivial situation as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Thus $U(x) = 3\varphi_1(x)$ and actually, with this configuration, we have a trivial one-dimensional problem.
Alternatively: We may include $N_3 = (1,0)$ and $N_4 = (0,1)$ as new nodes and extend the triangles with Neumann outer boundarys to include k_2 and k_4. This would lead to a 4×4 system which we are not considering here!

2. Consider the one-dimensional heat equation:

$$\begin{cases} \dot u - u'' = f, & 0 < x < 1, \quad t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1, \\ u(0,t) = u_x(1,t) = 0, & \qquad\qquad\quad t > 0. \end{cases}$$

a) Using appropriate variational forms show the stability estimates:
$\|u(\cdot,t)\| \le \|u_0\| + \int_0^t \|f(\cdot,s)\|\, ds$, and $\|u_x(\cdot,t)\|^2 \le \|u_0'\|^2 + \int_0^t \|f(\cdot,s)\|^2\, ds$.
b. Give physical meaning to the equation when f=9-u.

Solution: a) Multiply the equation by u and integrate over $(0,1)$ to get

$$\int_0^1 \dot u u\, dx - \int_0^1 u'' u\, dx = \int_0^1 f u\, dx.$$

Integrating by parts and using the boundary conditions we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\, dx + \int_0^1 \Big(u'\Big)^2 dx - u'(1,t)u(1,t) + u'(0,t)u(0,t) = \frac{1}{2}\frac{d}{dt}||u||^2 + ||u'||^2$$

$$= ||u||\frac{d}{dt}||u|| + ||u'||^2 = \int_0^1 f u\, dx \le ||f||||u||.$$

Consequently

$$\frac{d}{dt}||u|| \le ||f||,$$

which integrating over time:

$$\int_0^t ||f||\, ds \ge ||u(\cdot,t)|| - ||u_0||,$$

gives the first estimate in a).

Toderive the second estimate we multiply the equation by $\dot u$ and integrate over $(0,1)$ to obtain:

$$\int_0^1 \Big(\dot u\Big)^2 dx - \int_0^1 u'' \dot u\, dx = ||\dot u|| + \int_0^1 u' \dot u'\, dx - u'(1,t)\dot u(1,t) + u'(0,t)\dot u(0,t)$$

$$= ||\dot u||^2 + \frac{d}{dt}||u'||^2 = \int_0^1 f \dot u\, dx \le ||f||||\dot u|| \le \frac{1}{2}\Big(||f||^2 + ||\dot u||^2\Big).$$

Thus

$$\frac{1}{2}||\dot u||^2 + \frac{1}{2}\frac{d}{dt}||u'||^2 \le \frac{1}{2}||f||^2,$$

and hence

$$\frac{d}{dt}||u'||^2 \le ||f||^2,$$

which, as in the first estimate, integrating over time: $\int_0^t \quad ds$ gives the second estimate.
b) Heat conduction with

$u(x,t) =$	temperature at x at time t.
$u(x,0) = u_0(x)$,	the initial temperature at $t = 0$.
$u(0,t) = 0$,	fixed temperatue at $x = 0$.
$u'(1,t) = 0$,	isolated at $x = 1$, (no heat flux).
$f = 9 - u$,	heat source, in this case a contril system to force $u \to 9$.

3. Let a be a positive constant. Consider the boundary value problem (BVP)

$$-u''(x) + au(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

Formulate the corresponding variational formulation (VF), and the minimization problem (MP) and prove that $(BVP) \iff (VF) \iff (MP)$.

Solution: See lecture notes, Chapter 8.

4. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $\|u\|_{H^1}^2 = \|u'\|^2 + \|u\|^2$) for a finite element method for the problem

$$-u'' + 2xu' + 2u = f, \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

Solution: We multiply the differential equation by a test function $v \in H_0^1(I),\ I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + 2xu'v + 2uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{5}$$

A *Finite Element Method* with $cG(1)$ reads as followa: Find $U \in V_h^0$ such that

$$\int_I (U'v' + 2xU'v + 2Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{6}$$

where

$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}$.

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + 2xe'v + 2ev) = 0, \quad \forall v \in V_h^0. \tag{7}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I 2xe'e = \int_I x\frac{d}{dx}(e^2) = (xe^2)|_0^1 - \int_I e^2 = -\int_I e^2, \tag{8}$$

so that

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + 2xe'e + 2ee) \\
&= \int_I ((u-U)'e' + 2x(u-U)'e + 2(u-U)e) = \{v = e \text{ in}(1)\} \\
&= \int_I fe - \int_I (U'e' + 2xU'e + 2Ue) = \{v = \pi_h e \text{ in}(2)\} \qquad (9)\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + 2xU'(e-\pi_h e) + 2U(e-\pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned}
$$

where $\mathcal{R}(U) := f + U'' - 2xU' - 2U = f - 2xU' - 2U$, (for approximation with picewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$
\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\| \|h^{-1}(e-\pi_h e)\| \\
&\le C_i \|h\mathcal{R}(U)\| \|e'\| \le C_i \|h\mathcal{R}(U)\| \|e\|_{H^1},
\end{aligned}
$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i \|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (4) and write

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + 2xe'e + 2ee) \\
&= \int_I \Big(e'(u-U)' + 2xe'(u-U) + 2e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(3)\} \\
&= \int_I \Big(e'(u-\pi_h u)' + 2xe'(u-\pi_h u) + 2e(u-\pi_h u)\Big) \\
&\le \|(u-\pi_h u)'\| \|e'\| + 2\|u-\pi_h u\| \|e'\| + 2\|u-\pi_h u\| \|e\| \\
&\le \{\|(u-\pi_h u)'\| + 4\|u-\pi_h u\|\} \|e\|_{H^1} \\
&\le C_i \{\|hu''\| + \|h^2 u''\|\} \|e\|_{H^1},
\end{aligned}
$$

this gives that

$$\|e\|_{H^1} \le C_i \{\|hu''\| + \|h^2 u''\|\},$$

which is the a priori error estimate.

5. Consider the boundary value problem

$$-div(\varepsilon \nabla u + \beta u) = f, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega,$$

where Ω is a bounded polygonal donmain in $\mathbb{R}^2$, $\varepsilon > 0$ is a constant, $\beta = (\beta_1(x), \beta_2(x))$, and $f = f(x)$. Give the conditions (based on Lax-Milgrams theorem) for existence of a unique solution for this problem. Derive stability estimates for u i terms of $||f||_{L_2(\Omega)}$, ε and $diam(\Omega)$.

Solution: Consider

$$-div(\varepsilon \nabla u + \beta u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega. \qquad (10)$$

a) Multiply the equation (6) by $v \in H_0^1(\Omega)$ and integrate over Ω to obtain the Green's formula

$$-\int_\Omega div(\varepsilon\nabla u + \beta u)v\,dx = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx = \int_\Omega fv\,dx.$$

Variational formulation for (6) is as follows: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v), \qquad \forall v \in H_0^1(\Omega), \tag{11}$$

where

$$a(u,v) = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx,$$

and

$$L(v) = \int_\Omega fv\,dx.$$

According to the Lax-Milgram's theorem, for a unique solution for (7) we need to verify that the following relations are valid:
i)

$$|a(v,w)| \le \gamma\|u\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}, \qquad \forall v,\, w \in H_0^1(\Omega),$$

ii)

$$a(v,v) \ge \alpha\|v\|^2_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

iii)

$$|L(v)| \le \Lambda\|v\|_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

for some $\gamma,\ \alpha,\ \Lambda > 0$.
Now since

$$|L(v)| = |\int_\Omega fv\,dx| \le \|f\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)}\|v\|_{H^1(\Omega)},$$

thus iii) follows with $\Lambda = \|f\|_{L_2(\Omega)}$.
Further we have that

$$\begin{aligned}
|a(v,w)| &\le \int_\Omega |\varepsilon\nabla v + \beta v||\nabla w|\,dx \le \int_\Omega (\varepsilon|\nabla v| + |\beta||v|)|\nabla w|\,dx \\
&\le \Big(\int_\Omega (\varepsilon|\nabla v| + |\beta||v|)^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla w|^2\,dx\Big)^{1/2} \\
&\le \sqrt{2}\max(\varepsilon, \|\beta\|_\infty)\Big(\int_\Omega (|\nabla v|^2 + v^2)\,dx\Big)^{1/2}\|w\|_{H^1(\Omega)} \\
&= \gamma\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)},
\end{aligned}$$

which, with $\gamma = \sqrt{2}\max(\varepsilon, \|\beta\|_\infty)$, gives i).
Finally, if $div\beta \le 0$, then

$$\begin{aligned}
a(v,v) &= \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta\cdot\nabla v)v\Big)\,dx = \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta_1\frac{\partial v}{\partial x_1} + \beta_2\frac{\partial v}{\partial x_2})v\Big)\,dx \\
&= \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{2}(\beta_1\frac{\partial}{\partial x_1}(v)^2 + \beta_2\frac{\partial}{\partial x_2}(v)^2)\Big)\,dx = \text{Green's formula} \\
&= \int_\Omega \Big(\varepsilon|\nabla v|^2 - \frac{1}{2}(div\beta)v^2\Big)\,dx \ge \int_\Omega \varepsilon|\nabla v|^2\,dx.
\end{aligned}$$

Now by the Poincare's inequality

$$\int_\Omega |\nabla v|^2\,dx \ge C\int_\Omega (|\nabla v|^2 + v^2)\,dx = C\|v\|^2_{H^1(\Omega)},$$

for some constant $C = C(diam(\Omega))$, we have

$$a(v,v) \geq \alpha \|v\|^2_{H^1(\Omega)}, \qquad \text{with } \alpha = C\varepsilon,$$

thus ii) is valid under the *condition that* $div\beta \leq 0$.
From ii), (7) (with $v = u$) and iii) we get that

$$\alpha \|u\|^2_{H^1(\Omega)} \leq a(u,u) = L(u) \leq \Lambda \|u\|_{H^1(\Omega)},$$

which gives the *stability estimate*

$$\|u\|_{H^1(\Omega)} \leq \frac{\Lambda}{\alpha},$$

with $\Lambda = \|f\|_{L_2(\Omega)}$ and $\alpha = C\varepsilon$ defiened above.

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