TMA371 Partial Differential Equations TM, 1999-04-06. Solutions

1. a) Multiply the equation $\dot{u} = u''$ by u and integrate over $x \in (0,1)$:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \int_0^1 \dot{u}u\,dx = \int_0^1 u''u\,dx = \{\text{part. int.}\}$$
$$= u'u|_0^1 - \int_0^1 u'u'\,dx = -\|u'\|^2 \le 0,$$

i.e., $\|u\|^2$ and hence $\|u\|$ is decreasing in t.
Now multiply the equation $\dot{u} = u''$ by $-u''$ and integrate over $x \in (0,1)$:

$$\frac{1}{2}\frac{d}{dt}\|u'\|^2 = \int_0^1 \dot{u}'u'\,dx = \dot{u}u'|_0^1 - \int_0^1 \dot{u}u''\,dx$$
$$= -\int_0^1 u''u''\,dx = -\|u''\|^2 \le 0,$$

i.e., $\|u'\|^2$ and hence $\|u'\|$ is decreasing in t.
b) According the first relation above $\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u'\|^2 = 0$. Integrating over t yields:

$$\frac{1}{2}\|u\|^2(t) + \int_0^t \|u'\|^2\,d\tau = \frac{1}{2}\|u_0\|^2.$$

Thus, it follows that $\int_0^\infty \|u'\|^2\,dt$ must converge, which is possible only if the decreasing function $\|u'\|^2$ tends to 0 as $t \to \infty$, i.e., $\|u'\| \to 0$ as $t \to \infty$.
c) In tha absence of a heat source, the temperature and heat flux are decreasing (non-increasing) in time, especially the heat flux tends to 0 as $t \to \infty$.

2. a) Let $\Gamma = \partial\Omega$ be the boundary of Ω. We have that

$$\|\Delta u\|^2 = \int_\Omega u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}.$$

Now an application of the Green's formula (partial integration first in y and then in x) gives

$$\int_\Omega u_{xx}u_{yy} = \int_\Gamma u_{xx}u_y n_y - \int_\Omega u_{xxy}u_y = \int_\Gamma u_{xx}u_y n_y - \int_\Gamma u_{xy}u_y n_x + \int_\Omega u_{xy}u_{xy},$$

where $n = (n_x, n_y)$ is the outward unit normal at the boundary. Now, on the part of the boundary Γ, where $n_y \neq 0$, we have $u_{xx} = 0$, since $u = 0$. Likewise, $u_y = 0$ on the part of the boundary, where $n_x \neq 0$. Thus $\int_\Omega u_{xx}u_{yy} = \int_\Omega u_{xy}u_{xy}$, which gives the desired identity.
b) In the case of Neumann boundary condition: $\frac{\partial u}{\partial n} = 0$ on the boundary, we have that $u_y = 0$ on the part of Γ where $n_y \neq 0$, similarly $u_{xy} = 0$ on the part of Γ where $n_x \neq 0$ (because, then $u_x = 0$ in y-direction). Thus we obtain the same identity as in a).
c) We have, using Green's formula, that $\|u\|^2 = \int_\Omega u^2\Delta\phi = -2\int_\Omega u\nabla u \cdot \nabla\phi \le 2\max_\Omega |\nabla\phi|\|u\|\|\nabla u\|$, which gives the desired (Poincare) inequality with $C_\Omega = 2\max_\Omega |\nabla\phi|$.

3. We have that

$$(1)\qquad \Big(a(x)u'(x)\Big)' = 0, \quad 0 < x < 1, \quad a(0)u'(0) = u_0,\ u(1) = 0.$$

a) Let $\mathcal{T}_h$ be a partition of $I = (0,1)$ into subintervals $I_j = (x_{j-1}, x_j)$, $j = 1, \ldots, M+1$, and let $\mathcal{V}_h$ be the space of continuous, piecewise linear functions $v(x)$ defined on $\mathcal{T}_h$ such that $v(1) = 0$. The continuous variational formulation for problem (1) is obtained by multiplying the equation in (1) by a test function v and integrating over $(0,1)$:

$$\int_0^1 \Big(a(x)u'(x)\Big)' v(x)\,dx = [PI] = \Big[a(x)u'(x)v(x)\Big]_0^1 - \int_0^1 a(x)u'(x)v'(x)\,dx$$
$$= -a(0)u'(0)v(0) - \int_0^1 a(x)u'(x)v'(x)\,dx = 0, \quad \forall v(x),\ \text{with } v(1) = 0,$$

this gives that

$$(2)\qquad \int_0^1 a(x)u'(x)v'(x)\,dx = -u_0 v(0), \quad \forall v(x),\ \text{with } v(1) = 0,$$

The cG1-method for problem (1) is the following discrete version of the variational problem (2): Find $U \in \mathcal{V}_h$ such that

$$(3)\qquad \int_0^1 a(x)U'(x)v'(x)\,dx = -u_0 v(0), \quad \forall v \in \mathcal{V}_h.$$

An a posteriori error estimate

We start by defining the interpolant $\pi_h v \in \mathcal{V}_h$, of a function $v(x)$ with $v(1) = 0$ as $\pi_h v(x_j) = v(x_j)$, $j = 1, \ldots, M+1$. Let now $e = u - U$, we have using the equations (2) and (3) that

$$\begin{aligned}
\|e'\|_a^2 &= \int_I a e' e'\,dx = \int_I a u' e'\,dx - \int_I a U' e'\,dx \\
&= [(2), e(1) = 0, \text{with } v = e] = -u_0 e(0) - \int_I a U' e'\,dx \\
&= [v = \pi_h e \text{ in } (3)] = -u_0\Big(e(0) - \pi_h e(0)\Big) - \int_I a U'(e - \pi_h e)'\,dx \\
&= -\sum_{j=1}^{M+1} \int_{I_j} a U'(e - \pi_h e)'\,dx = \sum_{j=1}^{M+1} \int_{I_j} (aU')'(e - \pi_h e)\,dx \\
&= \int_I (aU')'(e - \pi_h e)\,dx \le \|h(aU')'\|_{1/a} \|h^{-1}(e - \pi_h e)\|_a \\
&\le \|h(aU')'\|_{1/a} C_i \|e'\|_a,
\end{aligned}$$

which gives the a posteriori estimate

$$(4)\qquad \|e'\|_a \le C_i \|h(aU')'\|_{1/a}.$$

b) Let now $\mathcal{T}_h$ be a partition of $I = (0,1)$ into 4 subintervals: $I_1 = (0, 1/4)$, $I_2 = (1/4, 1/2)$, $I_3 = (1/2, 3/4)$ and $I_4 = (3/4, 1)$. Then the functions $\{\phi_i\}_{i=1}^4$, where $\phi_i \in \mathcal{V}_h$, $\phi_i(x_j) = \delta_{i,j}$, $i, j = 1, 2, 3, 4$, $x_j = \frac{j-1}{4}$, form a basis for $\mathcal{V}_h$. In this way (3) is equivalent to

$$(5)\qquad \int_0^1 a(x)U'(x)\phi_i'(x)\,dx = -u_0 \phi_i(0), \quad i = 1, 2, 3, 4.$$

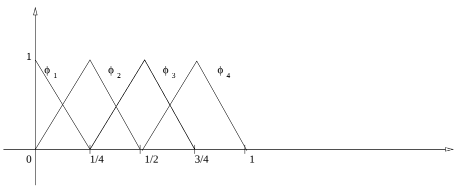

FIGURE 1. Base functions

Set now $U(x) = \sum_{j=1}^4 \xi_j \phi_j(x)$, inserting in (5) yields to the following linear system of equations:

$$\sum_{j=1}^4 \xi_j \int_0^1 a(x)\phi_j'(x)\phi_i'(x)\,dx = -u_0\phi_i(0), \quad i = 1, \dots 4 \Longleftrightarrow A\xi = b,$$

where $A = (a_{ij})$ is the massmatrix with $a_{ij} = \int_0^1 a(x)\phi_j'(x)\phi_i'(x)\,dx$ and $b = (b_i)$ is the load vector with $b_i = -u_0\phi_i(0)$. Now with $a(x) = 1/4$ for $x < 1/2$, $a(x) = 1/2$ for $x > 1/2$ and $u_0 = 3$ we have

$$b_1 = -3, \; b_2 = b_3 = b_4 = 0.$$

Further note that A is a 4×4 symmetric matrix and with the mesh size $h = 1/4$ we get

$$\begin{aligned}
a_{11} &= \int_0^{1/4} \frac{1}{4}(-\frac{1}{h})(-\frac{1}{h})\,dx = \frac{1}{4} \times 4 \times 4 \int_0^{1/4} dx = 4 \times \frac{1}{4} = 1\\
a_{22} &= \int_0^{1/4} \frac{1}{4}(\frac{1}{h})(\frac{1}{h})\,dx + \int_{1/4}^{1/2} \frac{1}{4}(-\frac{1}{h})(-\frac{1}{h})\,dx = 2\\
a_{33} &= \int_{1/4}^{1/2} \frac{1}{4}(\frac{1}{h})(\frac{1}{h})\,dx + \int_{1/2}^{3/4} \frac{1}{2}(-\frac{1}{h})(-\frac{1}{h})\,dx = 3\\
a_{44} &= \int_{1/2}^{3/4} \frac{1}{2}(\frac{1}{h})(\frac{1}{h})\,dx + \int_{3/4}^{1} \frac{1}{2}(-\frac{1}{h})(-\frac{1}{h})\,dx = 4\\
a_{12} &= \int_0^{1/4} \frac{1}{4}(-\frac{1}{h})(\frac{1}{h})\,dx = -1\\
a_{23} &= \int_{1/4}^{1/2} \frac{1}{4}(-\frac{1}{h})(\frac{1}{h})\,dx = -1\\
a_{34} &= \int_{1/2}^{3/4} \frac{1}{2}(-\frac{1}{h})(\frac{1}{h})\,dx = -2.
\end{aligned}$$

So that our matrix equation is:

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives the approximate solution $U = -3(1/2, 1, 2, 3)^t$.
c) Since a is constant and U is linear on each subinterval we have that

$$(aU')' = a'U' + aU'' = 0.$$

By the a posteriori error estimate (4) we then have $\|e'\|_a = 0$, i.e., $e' = 0$. Combining with the fact that $e(x)$ is continuous and $e(1) = 0$ we get that $e = 0$, which means that the finite element solution, in this case, would coincide with the exact solution.

4. See lecture notes (Chapter 17: obs! a parallel version).

5. Consider

$$-div(\varepsilon\nabla u + \beta u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega. \tag{6}$$

a) Multiply the equation (6) by $v \in H_0^1(\Omega)$ and integrate over Ω to obtain the Green's formula

$$-\int_\Omega div(\varepsilon\nabla u + \beta u)v\,dx = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx = \int_\Omega fv\,dx.$$

Variational formulation for (6) is as follows: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = L(v), \qquad \forall v \in H_0^1(\Omega), \tag{7}$$

where

$$a(u,v) = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx,$$

and

$$L(v) = \int_\Omega fv\,dx.$$

According to the Lax-Milgram's theorem, for a unique solution for (7) we need to verify that the following relations are valid:

i)

$$|a(v,w)| \le \gamma\|u\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}, \qquad \forall v,\, w \in H_0^1(\Omega),$$

ii)

$$a(v,v) \ge \alpha\|v\|^2{}_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

iii)

$$|L(v)| \le \Lambda\|v\|_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

for some $\gamma,\ \alpha,\ \Lambda > 0$.
Now since

$$|L(v)| = |\int_\Omega fv\,dx| \le \|f\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} \le \|f\|_{L_2(\Omega)}\|v\|_{H^1(\Omega)},$$

thus iii) follows with $\Lambda = \|f\|_{L_2(\Omega)}$.
Further we have that

$$\begin{aligned}
|a(v,w)| &\le \int_\Omega |\varepsilon\nabla v + \beta v||\nabla w|\,dx \le \int_\Omega (\varepsilon|\nabla v| + |\beta||v|)|\nabla w|\,dx \\
&\le \Big(\int_\Omega (\varepsilon|\nabla v| + |\beta||v|)^2\,dx\Big)^{1/2}\Big(\int_\Omega |\nabla w|^2\,dx\Big)^{1/2} \\
&\le \sqrt{2}\max(\varepsilon, \|\beta\|_\infty)\Big(\int_\Omega (|\nabla v|^2 + v^2)\,dx\Big)^{1/2}\|w\|_{H^1(\Omega)} \\
&= \gamma\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)},
\end{aligned}$$

which, with $\gamma = \sqrt{2}\max(\varepsilon, \|\beta\|_\infty)$, gives i).

Finally, if $div\beta \leq 0$, then

$$\begin{aligned} a(v,v) &= \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta\cdot\nabla v)v\Big)\,dx = \int_\Omega \Big(\varepsilon|\nabla v|^2 + (\beta_1\frac{\partial v}{\partial x_1} + \beta_2\frac{\partial v}{\partial x_2})v\Big)\,dx \\ &= \int_\Omega \Big(\varepsilon|\nabla v|^2 + \frac{1}{2}(\beta_1\frac{\partial}{\partial x_1}(v)^2 + \beta_2\frac{\partial}{\partial x_2}(v)^2)\Big)\,dx = \text{Green's formula} \\ &= \int_\Omega \Big(\varepsilon|\nabla v|^2 - \frac{1}{2}(div\beta)v^2\Big)\,dx \geq \int_\Omega \varepsilon|\nabla v|^2\,dx. \end{aligned}$$

Now by the Poincare's inequality

$$\int_\Omega |\nabla v|^2\,dx \geq C\int_\Omega (|\nabla v|^2 + v^2)\,dx = C\|v\|^2{}_{H^1(\Omega)},$$

for some constant $C = C((\Omega))$, we have

$$a(v,v) \geq \alpha\|v\|^2{}_{H^1(\Omega)}, \qquad \text{with}\ \ \alpha = C\varepsilon,$$

thus ii) is valid under the *condition that* $div\beta \leq 0$.
From ii), (7) (with $v = u$) and iii) we get that

$$\alpha\|u\|^2{}_{H^1(\Omega)} \leq a(u,u) = L(u) \leq \Lambda\|u\|_{H^1(\Omega)},$$

which gives the *stability estimate*

$$\|u\|_{H^1(\Omega)} \leq \frac{\Lambda}{\alpha},$$

with $\Lambda = \|f\|_{L_2(\Omega)}$ and $\alpha = C\varepsilon$ defiened above.

MA