MATEMATIK
Chalmers tekniska högskola

Hjälpmedel: Inga hjälpmedel är tillåtna
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2001–12–18 kl. 14.15-19.15

TMA372/MAN660 Partiella differentialekvationer TM, E3, GU

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

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1. a) Derive the fundamental solution for the initial value problem

$$\dot{u}(t) + a(t)u(t) = f(t), \quad 0 < t \leq T, \qquad u(0) = u_0.$$

   b) Prove the stability estimates

$$\begin{aligned} &i) \quad a(t) \geq \alpha > 0 \quad \Longrightarrow \quad |u(t)| \leq e^{-\alpha t}|u_0| + \tfrac{1}{\alpha}\Big(1 - e^{-\alpha t}\Big) \max_{0\leq s\leq t} |f(s)| \\ &ii) \quad a(t) \geq 0 \qquad \Longrightarrow \quad |u(t)| \leq |u_0| + \textstyle\int_0^t |f(s)|\, ds. \end{aligned}$$

2. Show that for $a(t) > 0$, and for $N = 1, 2, \ldots$, the piecewise linear approximate solution U for the problem 1 satisfies the a posteriori error estimate

$$|u(t_N) - U_N| \leq \max_{[0,t_N]} |k(\dot{U} + aU - f)|, \quad k = k_n, \text{ for } t_{n-1} < t \leq t_n.$$

3. Prove *a priori* and *a posteriori* error estimates, in the energy norm $||v||_E^2 = ||v'||^2 + a||v||^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x) + u'(x) + au(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad a \geq 0.$$

4. a) Formulate a $cG(1)$ finite element method for the following system

$$\begin{cases} u(x) + v''(x) = f(x), & v(0) = v(1) = 0, \quad < x < 1, \\ u''(x) - v(x) = 0, & u(0) = u(1) = 0, \end{cases}$$

   and show how the approximate solution (U, V) can be computed from the load vector F, using mass- and stiffness matrises.

   b) Derive stability estimates for u and v, in terms of f, (e.g., through multiplying the first equation by u and the second by $-v$).

5. Consider the following *Schrödinger* equation

$$\dot{u} + i\Delta u = 0, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

   where $i = \sqrt{-1}$ and $u = u_1 + iu_2$. a) Show that the the L_2 norm of the solution, i.e., $\int_\Omega |u|^2$ is time independent.

   Hint: Multiply the equation by $\bar{u} = u_1 - iu_2$, integrate over Ω and consider the real part.

   b) Consider the corresponding *eigenvalue problem*, of finding $(\lambda, u \neq 0)$, such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

   Show that $\lambda > 0$, and give the relation between $||u||$ and $||\nabla u||$ for the corresponding eigenfunction u.

   c) What is the optimal constant C (expressed in terms of smallest eigenvalue λ_1), for which the inequality $||u|| \leq C||\nabla u||$ can fullfil for all functions u, such that $u = 0$ on $\partial\Omega$?

MA