Matematiska institutionen
CTH&GU

TMA371 Partiella differentialekvationer TM, 1998-12-15
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Inga hjälpmedel. Kalkylator ej tillåten. Uppgifterna är värda 10 poäng vardera.

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1. Compute the piecewise linear Galerkin approximation of the Poisson equation:

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D, \\ \partial_n u = 0, & \text{on } \Gamma_N. \end{cases}$$

Hint: Use only the nodes, $N_1 = (0,0)$ and $N_2 = (1,1)$.

2. Consider the initial value problem

$$\dot{u}(t) + au(t) = 0, \quad t > 0, \qquad u(0) = u_0, \quad a > 0, \quad (a = \text{constant}).$$

Assume a constant time step k and verify the iterative formulas for $dG(0)$ and $dG(1)$ approximations U and $\tilde{U}$ respectively: i.e.

$$U_n = \Big(\frac{1}{1+ak}\Big)^n u_0, \qquad \tilde{U}_n = \Big(\frac{1-ak/2}{1+ak/2}\Big)^n u_0.$$

3. Consider the boundary value problem (BVP)

$$\begin{cases} -u''(x) + au(x) = f(x), & 0 < x < 1,\ a > 0,\ (a = \text{constant}) \\ u(0) = u(1) = 0. \end{cases}$$

Formulate the corresponding variational formulation problem (VF), the minimization problem (MP) and prove that

$$(BVP) \Longleftrightarrow (VF) \Longleftrightarrow (MP).$$

4. Consider the heat equation

$$\begin{cases} \dot{u} - u'' = f(x), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u'(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

a) Show, for the homogeneous case: i.e., $f \equiv 0$ (with $\|u\| = (\int_0^1 u^2(x)\,dx)^{1/2}$), the estimates

$$(E1) \quad \frac{d}{dt}\|u\|^2 + 2\|u'\|^2 = 0, \qquad (E2) \quad \|u(.,t)\| \le e^{-t}\|u_0\|.$$

b) Let $u_s = u_s(x)$ be the solution of the corresponding stationary problem:

$$-u_s'' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0,$$

show that $\quad \|u - u_s\| \to 0, \quad$ as $t \to \infty$.

5. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $\|u\|_{H^1}^2 = \|u'\|^2 + \|u\|^2$) for a finite element method for the problem

$$\begin{cases} -u'' + 2xu' + 2u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

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