TMA372/MAN660 PARTIAL DIFFERENTIAL EQUATIONS
COMPUTER ASSIGNMENT 2

Introduction

This assignment treats adaptive Finite Element Methods for:

- Poisson's equation,
- Convection-diffusion problems,
- Selected applications.

To solve the problems in this assignment you are proposed to use the MATLAB program FEMLAB. As an alternative, for most of the applications below it is also possible to use the (somewhat simpler) PDE Toolbox in MATLAB.

The main objective is twofold. First we study the qualitative behaviour of the partial differential equations and develop a basic understanding and intuition for the dependence of the solutions on the coefficients in the equation. We also study the qualitative properties of the adaptive finite element method. Next we study mathematical modelling of real world problems using the finite element methodology to solve the equations in our models.

1. Qualitative study: Some two-dimensional model problems

In all of your computations below try different visualisation options. Study in particular the mesh refinement procedure (see e. g. page 1-19 in "Minicourse on FEMLAB" or pages 3-25 to 3-26 in "PDE Toolbox User's Guide" to learn how to turn on adaptivity). Does the automatic mesh refinement agree with your intuition?

1.1. **The Poisson equation.** Elliptic equations arise in many different physics and engineering problems, they often represent the asymptotic solution to a time dependent problem when $t \to \infty$. The simplest and most important example of an elliptic equation is the following, which is called Poisson's equation when $c = 1$,

$$
\begin{aligned}
-\nabla \cdot (c\nabla u) &= f && \text{in } \Omega, \\
u &= g && \text{on } \Gamma_{Dir}, \\
\frac{\partial u}{\partial n} &= h && \text{on } \Gamma_{Neu}, \\
-c\frac{\partial u}{\partial n} &= \alpha(u-g) && \text{on } \Gamma_{Rob}.
\end{aligned}
\tag{1}
$$

- Give a simple physical interpretation of the equation. What do the different boundary conditions mean?
- Solve (1) on a polygonal domain Ω in the plane with the simplest choice of data, $c = 1$, $f = 1$ and the Dirichlet boundary condition $g = 0$ on $\Gamma = \Gamma_{Dir}$. Modify the domain Ω, and investigate the behaviour of the solution in the neighbourhood of convex and non convex corners. Compare with theory.
- Choose a domain and compute the solution for different right hand sides f, change to Neumann/Robin conditions on part of the boundary. Compare with your physical interpretation.

1.2. **The convection-diffusion equation.** The convection-diffusion equation is a simple example of hyperbolic type problems. Hyperbolic problems typically describe transport processes and they appear in many different applications, e.g. fluid mechanics. The convection-diffusion equation also serves as a very simple but useful model problem for the Navier-Stokes equations as it inherits typical properties such as boundary layers and rapidly changing solutions. The equation takes the form

$$\begin{aligned} \alpha u + \beta \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_{inflow}, \\ u \text{ or } \frac{\partial u}{\partial n} &= h \quad \text{on } \Gamma_{outflow}. \end{aligned} \tag{2}$$

Note that Dirichlet conditions must be given on the inflow part of the boundary but on the outflow part both Dirichlet and Neumann/Robin conditions are possible. In order to get a rough idea about the solution when the viscosity/diffusivity, ϵ, is small it is often useful to consider the purely hyperbolic case where ϵ is zero and the equation can be solved by integration along the characteristics.

- Compute[1] the solution on some polygonal domain Ω in a situation with small diffusion and large convection, say $\epsilon = 1$ and $|\beta| = 100$. Change the direction of the vector β, describe the corresponding changes in the solution.
- Consider the solution to (2) with homogeneous Dirichlet boundary data on some polygonal domain Ω vary the diffusion coefficient, study the change towards a more elliptic equation when the diffusion is increasing and the change towards a more hyperbolic behaviour when the diffusion is decreasing.
- Investigate the different possibilities for boundary conditions. Consider in detail the different behaviour at the outflow boundary for Dirichlet and Neumann/Robin conditions. What is the relationship between the width of the outflow boundary layer and the size of ϵ? Compare with theory.
- Solve the problem with discontinuous Dirichlet data on the inflow boundary, investigate how far this discontinuity in data propagates into the domain for different choices of ϵ. What can you say about the smoothing property of the equation?

2. Selected applications

Select one of the following applied problems, or solve a problem of your own. The objective is to solve an applied problem of interest using FEMLAB or the PDE Toolbox, to evaluate the results obtained and draw some conclusions concerning the nature of the exact solution and the numerical approximation. Use your fantasy and focus on features of interest. Note that the problems are not precisely formulated. You thus have to think of:

- An interesting real world problem.
- Mathematical modelling including for instance the choice of boundary conditions and truncation of the computational domain in case of unbounded domains.
- Computational aspects.

[1] For this type of convection dominated problems you can not use the PDE Toolbox. If you want, at http://www.math.chalmers.se/~mohammad/pde1_lab/lab2/example.html you can find an example of how to solve such a problem using FEMLAB.

- Analytical aspects, seek to simplify the model so that it is possible to obtain an analytical solution. Solve the simplified problem and think about the extra assumptions you have made, are these realistic?

2.1. **Convection-diffusion-absorption/reaction.** Consider a 2d convection-diffusion-absorption/reaction problem of the form

$$\alpha u + \beta \cdot \nabla u - \nabla \cdot (\epsilon \nabla u) = f,$$

together with suitable boundary conditions on the boundary Γ of Ω, where u is an unknown concentration, $\epsilon = \epsilon(x)$ is a given (small) diffusion coefficient, $\beta = \beta(x)$ is a given velocity field, $\alpha = \alpha(x)$ is a given absorption/reaction coefficient and $f = f(x)$ is a given production term. Solve a convection-dominated problem of this form for instance related to pollution control, where f is a delta-function at some point $P \in \Omega$. Determine for instance the width of the "smoke plume" and compare with theory.

2.2. **Electrostatics.** Consider the basic problem of 2d electrostatics

$$\begin{aligned} \nabla \cdot (\epsilon E) &= \rho, \\ E &= -\nabla \phi, \end{aligned}$$

together with suitable boundary conditions corresponding to a part of the boundary of Ω being a perfect conductor and the remaining part being insulated. Here E is the electric field, ϕ the electric potential, $\epsilon = \epsilon(x)$ the dielectricity coefficient, and ρ the charge density. Solve a problem of this form in a configuration of interest for instance with the boundary containing a sharp non-convex corner. Study the behaviour of the electric field in the vicinity of the corner and compare with theory.

2.3. **2d fluid flow.** The velocity $u = (u_1, u_2)$ of an incompressible irrotational 2d fluid may be expressed through a potential ϕ by $u = \nabla \phi$. Coupled with the incompressibility equation $\nabla \cdot u = 0$ this gives the Laplace equation for ϕ:

$$\nabla \cdot (\nabla \phi) = \Delta \phi = 0,$$

together with suitable boundary conditions expressing for instance that $u \cdot n = 0$ on solid boundaries. Note that it is not possible to use Neumann conditions on the entire boundary. Solve a problem of the following type, using a potential:

(a) flow through a 2d nozzle
(b) flow around a disc or wing profile

Use the gradient plot to visualize the flow.

In FEMLAB, also compute the flow by solving the incompressible Navier-Stokes equations. Compare the results.

2.4. **Membrane problem.** Derive the basic equation for a membrane supported by an elastic half space subject to a transversal load f:

$$-\nabla \cdot (\nabla u) + Eu = f,$$

together with suitable boundary conditions, where u is the vertical displacement of the membrane, E is the modulus of elasticity of the supporting half space. Solve a problem of this type in a configuration of interest. Derive an eigenvalue problem describing the eigenmodes of the corresponding vibrating membrane and solve this problem.

2.5. **Heat conduction.** Consider the 2d stationary heat equation

$$\nabla \cdot q = f, \quad q = -\kappa \nabla u,$$

together with suitable boundary conditions, where u is the temperature, q the heat flow, κ the heat conduction coefficient and f a given production term. Solve for instance a problem of this form modelling a hot water pipe buried in a half space and determine the temperature on the boundary of the half space above the pipe using a Robin boundary condition on the surface.

2.6. **Quantum physics.** Consider the 2d stationary Schrödinger eigenvalue problem

$$-\frac{\hbar^2}{2m}\Delta u + V(x)u = \lambda u,$$

where V is a given potential, $\hbar$ is Planck's constant divided by 2π and m is the particle mass. Give a quantum physical interpretation of the eigenvalues and corresponding eigenfunctions determined by this equation. Normalize the constants and solve the problem for some suitable domain and potential. Discuss your computational results from a quantum physical viewpoint.

2.7. **Flow in porous media.** In the simplest model of flow in porous media we assume that the flow q is proportional to the pressure gradient

$$q = k\nabla p,$$

where k is a constant depending on the media. The pressure will then satisfy the equation.

$$\nabla \cdot k\nabla p = 0,$$

together with suitable boundary conditions. Consider for example the flow under a dam. What happens with the flow for various geometries of the dam. We may expect that the media right under the dam will be more "dense" due to the weight of the dam. How can you take this into account in your model.

2.8. **Scattering.** Consider the wave equation in two space dimensions

$$\frac{\partial^2 u}{\partial^2 t} - \Delta u = f(t, x),$$

make the ansatz $u(x, t) = u(x)\exp(i\omega t)$ to find the Helmholtz equation for 2d acoustic scattering

$$\Delta u + \omega^2 u = -e^{-i\omega t} f(t, x),$$

Solve this problem with $f = \exp(i\omega t)$ corresponding to periodic forcing with a given frequency ω and suitable boundary conditions on a bounded domain Ω. Let $f = 0$ and consider a scatterer with boundary Γ and let Ω be the unbounded region outside the scatterer. u is the reflected field, and ω is a given frequency. Solve a problem of this form with Dirichlet boundary conditions on Γ and suitable boundary conditions (in the simplest case homogeneous Dirichlet conditions) approximating the Orr-Sommerfeld conditions on a truncated bounded domain approximating Ω.