Complexity for Probabilists

Comments and partial solutions to homework 4

1. There exists an algorithm (`‘‘repeat every digit three times’’`) which translates $(X_0, \ldots, X_n)$ into $(Y_0, \ldots, Y_{3n})$. Conversely, there also exists an algorithm (`‘‘remove all but every 3rd digit’’`) which translates $(Y_0, \ldots, Y_{3n})$ into $(X_0, \ldots, X_n)$. Hence there exist constants c, c' such that

$$K(X_0, \ldots, X_n) - c \leq K(Y_0, \ldots, Y_{3n}) \leq K(X_0, \ldots, X_n) + c' \tag{1}$$

for all n. By Theorem 7.3.5 in Cover & Thomas (or a combination of theorems I proved in the lectures), we have that

$$\frac{K(X_0, \ldots, X_n)}{n} \to 1 \text{ as } n \to \infty \tag{2}$$

with probability 1. Combining (1) and (2) gives the desired limiting behavior of $K(Y_0, \ldots, Y_n)$.

2. (a) Write $\mathcal{A}$ for the class of all computable a.s. properties. If $A \in \mathcal{A}$, then conditions (i)–(iii) together imply that $\mathbf{Pr}(X \notin A) \leq 2^{-k}$ for any k. Hence

$$\mathbf{Pr}(X \notin A) = 0\,.$$

Now

$$\begin{aligned} \mathbf{Pr}(X \text{ is not Martin-Löf random}) &= \mathbf{Pr}(\exists A \in \mathcal{A} \text{ such that } X \notin A) \\ &= \mathbf{Pr}\left(\bigcup_{A \in \mathcal{A}} \{X \notin A\} \right) \\ &\leq \sum_{A \in \mathcal{A}} \mathbf{Pr}(X \notin A) \\ &= 0 \end{aligned}$$

using the fact that $\mathcal{A}$ is a countable set (and countable additivity of probability measures).

(b) Let $b_1, b_2, \ldots$ be a (computable) sequence of numbers growing fast enough so that

$$\mathbf{Pr}(10010 \text{ does not occur among the first } b_k \text{ bits of a Bernoulli}(\tfrac{1}{2}) \text{ sequence}) \leq 2^{-k}\,.$$

For instance $b_k = 108k$ is more than enough (check this!). Then let

$$M((a_0, \ldots, a_n), k) = \begin{cases} 1 & \text{if } n < b_k \\ 1 & \text{if } n \geq b_k \text{ and } 10010 \text{ appears in } (a_0, \ldots, a_{b_k}) \\ 0 & \text{otherwise.} \end{cases}$$

It is now easy to check that (i)–(iii) hold with

$$A = \{10010 \text{ appears somewhere}\}\,.$$

Hence A holds for every Martin-Löf random sequence.

(c) and (d) Sorry, the amount of time I had alloted to this course just ran out!!! `;-)`