Linear differential equations and functions of operators

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The simplest example

Given initial data $x(0) \in \mathbf{C}^2$ and coefficient matrix $A := \begin{bmatrix} 7 & -9 \\ 3 & -5 \end{bmatrix}$, solve the linear system of first order ODEs

$$x'(t) + Ax(t) = 0 \quad \text{for} \quad x : \mathbf{R} \to \mathbf{C}^2.$$

Coordinates in eigenbasis $y(t) := V^{-1}x(t)$, where $V := \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$, give decoupled equations

$$y'(t) + \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} y(t) = 0 \quad \text{for} \quad y : \mathbf{R} \to \mathbf{C}^2.$$

Solution to the initial ODE is $x(t) = V \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} V^{-1}x(0)$.

Definition

$$A = V \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} V^{-1} \qquad \text{gives} \qquad e^{-tA} := V \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{bmatrix} V^{-1}.$$

Functional calculus through diagonalization of matrices

A generic matrix $A \in \mathbf{C}^{n \times n}$ is diagonalizable,

$$A = V \operatorname{diag}(\lambda_1, \ldots, \lambda_n)\, V^{-1},$$

for some invertible "change-of-basis" matrix $V \in \mathbf{C}^{n \times n}$.

Definition

For a function $\phi : \sigma(A) = \{\lambda_1, \ldots, \lambda_n\} \to \mathbf{C}$ defined on the spectrum of A, define the matrix

$$\phi(A) := V \operatorname{diag}\big(\phi(\lambda_1), \ldots, \phi(\lambda_n)\big)\, V^{-1}$$

For fixed A, the map $\phi \mapsto \phi(A)$ from symbol $\phi : \sigma(A) \to \mathbf{C}$ to matrix $\phi(A) \in \mathbf{C}^{n \times n}$ is an algebra-homomorphism:

$$(\phi\psi)(A) = \phi(A)\, \psi(A).$$

Example: $\{e^{-tA}\}_{t \in \mathbf{R}}$ is a group of matrices, $e^{-tA} e^{-sA} = e^{-(t+s)A}$.

1. Functions of matrices

2. Functions of linear Hilbert space operators

3. Spectral projections and the Kato conjecture

4. Operational calculus and maximal regularity

Functional calculus for general matrices

For general $A \in \mathbf{C}^{n\times n}$ there are natural definitions of matrices

$$A^2, A^3, A^4, \ldots, \qquad (\lambda I - A)^{-1}, \lambda \notin \sigma(A).$$

We want to define in a natural way a matrix $\phi(A)$ for any

$$\phi \in H(\sigma(A)) := \{\phi : \Omega \to \mathbf{C} \text{ holomorphic } ; \ \Omega \supset \sigma(A) \text{ open}\}.$$

Definition

For a function $\phi \in H(\sigma(A))$, define the matrix

$$\phi(A) := \frac{1}{2\pi i}\int_\gamma \phi(\lambda)(\lambda I - A)^{-1} d\lambda,$$

where γ is a closed curve in Ω counter clockwise around $\sigma(A)$.
The *holomorphic functional calculus* of A is the map
$H(\sigma(A)) \ni \phi \mapsto \phi(A) \in \mathbf{C}^{n\times n}$.

Properties of the holomorphic functional calculus

1. The holomorphic functional calculus of A is an algebra-homomorphism and its range is a commutative subalgebra of $\mathbf{C}^{n\times n}$:

$$\phi(A)\psi(A) = (\phi\psi)(A) = (\psi\phi)(A) = \psi(A)\phi(A).$$

2. For polynomials we have

$$(z^k)(A) = A^k, \qquad \text{for } k = 0, 1, 2, \ldots$$

3. If $H(\sigma(A)) \ni \phi_k \to \phi$ uniformly on compact subsets of Ω, then $\phi_k(A) \to \phi(A)$.
4. For diagonal matrices we have

$$\phi\big(\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\big) = \mathrm{diag}(\phi(\lambda_1), \ldots, \phi(\lambda_n)).$$

5. For any invertible "change-of-basis" matrix V, we have $\phi(VAV^{-1}) = V\phi(A)V^{-1}$.

Example: a Jordan block

Fix $a \in \mathbf{C}$ and consider $A = aI + N$, where N is nilpotent: $N^m = 0$ for some m.
Spectrum is $\sigma(A) = \{a\}$ and

$$\frac{1}{\lambda I - (aI + N)} = \frac{1}{\lambda - a} \frac{1}{I - N/(\lambda - a)} = \sum_{k=0}^{m-1} \frac{N^k}{(\lambda - a)^{k+1}}.$$

Cauchy's formula for derivatives gives

$$\phi(A) = \sum_{k=0}^{m-1} \left(\frac{1}{2\pi i} \int_\gamma \frac{\phi(\lambda)}{(\lambda - a)^{k+1}} d\lambda \right) N^k = \sum_{k=0}^{m-1} \frac{\phi^{(k)}(a)}{k!} N^k.$$

For example

$$\phi \left(\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \right) = \begin{bmatrix} \phi(3) & \phi'(3) & \frac{1}{2}\phi''(3) & \frac{1}{6}\phi^{(3)}(3) \\ 0 & \phi(3) & \phi'(3) & \frac{1}{2}\phi''(3) \\ 0 & 0 & \phi(3) & \phi'(3) \\ 0 & 0 & 0 & \phi(3) \end{bmatrix}.$$

Generalization to bounded operators on Hilbert spaces

Letting $n \to \infty$, we extend the above holomorphic functional calculus

from matrices $A \in \mathbf{C}^{n \times n}$, to bounded linear operators $T : \mathcal{H} \to \mathcal{H}$

on Hilbert space $\mathcal{H}$ (or more generally to Banach space operators).
The *Dunford* (or Riesz–Dunford or Dunford–Taylor) integral

$$\phi(T) := \frac{1}{2\pi i} \int_\gamma \phi(\lambda)(\lambda I - T)^{-1} d\lambda$$

defines a bounded linear operator $\phi(T) : \mathcal{H} \to \mathcal{H}$.

- Spectrum $\sigma(T) \subset \mathbf{C}$ is compact (but typically not finite).
- Symbol $\phi : \Omega \to \mathbf{C}$ is holomorphic on open neighbourhood $\Omega \supset \sigma(T)$.
- Closed curve $\gamma \subset \Omega$ encircles $\sigma(T)$ counter clockwise.

Example: self-adjoint operators

Let $T : \mathcal{H} \to \mathcal{H}$ be self-adjoint: $\langle Tf, g\rangle = \langle f, Tg\rangle$, $f, g \in \mathcal{H}$. The spectral theorem shows

$$T = VM_\lambda V^{-1},$$

where

- $V : L_2(X, d\mu) \to \mathcal{H}$ is a bijective isometry, with a Borel measure $d\mu$ on a σ-compact space X, and
- $M_\lambda : L_2(X, d\mu) \to L_2(X, d\mu)$ is multiplication by a real-valued function $\lambda \in L_\infty(X, d\mu)$: $(M_\lambda f)(x) := \lambda(x)f(x)$ for $f \in L_2(X, d\mu)$ and a.e. $x \in X$.

Then

1. the spectrum $\sigma(T) \subset \mathbf{R}$ is the essential range of $\lambda : X \to \mathbf{R}$, and
2. $\phi(T)$ is similar to multiplication by $\phi(\lambda(x))$, or more precisely

$$\phi(T) = VM_{\phi\circ\lambda}V^{-1}.$$

A boundedness conjecture

Even though ϕ is assumed to be holomorphic on a neighbourhood of $\sigma(T)$ in the definition of $\phi(T)$, it is natural to ask whether $\phi(T)$ only depends on $\phi|_{\sigma(T)}$? Do we have the estimate

$$\|\phi(T)\|_{\mathcal{H}\to\mathcal{H}} \leq \sup_{\lambda\in\sigma(T)} |\phi(\lambda)| \quad ?$$

For self-adjoint operators we have

$$\|T\|_{\mathcal{H}\to\mathcal{H}} = \|M_{\phi\circ\lambda}\|_{L_2\to L_2} = \sup_{\lambda\in\sigma(T)} |\phi(\lambda)|.$$

The conjecture is false!

The estimate

$$\|\phi(T)\|_{\mathcal{H}\to\mathcal{H}} \leq \sup_{\lambda\in\sigma(T)} |\phi(\lambda)|$$

cannot be true for general non-selfadjoint operators, since changing to an equivalent norm on $\mathcal{H}$ may change the RHS, but not the LHS (not $\sigma(T)$!).

Example

A simple counter example is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $A^2 = 0$, so $\sigma(A) = \{0\}$. However,

$$A = \begin{bmatrix} \phi(0) & \phi'(0) \\ 0 & \phi(0) \end{bmatrix}.$$

We cannot bound A ($\phi(0)$ and $\phi'(0)$) by $\phi(0)$, uniformly in ϕ.

A classical positive result

The estimate

$$\|\phi(T)\|_{\mathcal{H}\to\mathcal{H}} \leq \sup_{\lambda\in\sigma(T)} |\phi(\lambda)|$$

holds if we replace the spectrum $\sigma(T)$ in the RHS by the disk

$$\{|\lambda| \leq \|T\|\} \supset \sigma(T).$$

The following is a classical result by J. von Neumann (*Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes.* Math. Nachr., 1951).

Theorem (von Neumann)

Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then

$$\|\phi(T)\|_{\mathcal{H}\to\mathcal{H}} \leq \sup_{|\lambda|\leq\|T\|} |\phi(\lambda)|$$

holds for all ϕ holomorphic on some neighbourhood of $|\lambda| \leq \|T\|$.

A sharpening of von Neumann's theorem

M. Crouzeix (*Numerical range and functional calculus in Hilbert space.* J. Funct. Anal., 2007) has proved the following.

Theorem (Crouzeix)

Let $W(T) := \{\langle Tf, f\rangle \ ; \ \|f\| = 1\}$ *be the numerical range of a linear Hilbert space operator* $T : \mathcal{H} \to \mathcal{H}$. *Then*

$$\|\phi(T)\|_{\mathcal{H}\to\mathcal{H}} \leq 11.08 \sup_{\lambda\in\overline{W(T)}} |\phi(\lambda)|$$

holds uniformly for all ϕ *holomorphic on some neighbourhood of* $\overline{W(T)}$.

- Hausdorff–Toeplitz theorem: $W(T)$ is a convex set.
- $\sigma(T) \subset \overline{W(T)} \subset \{|\lambda| \leq \|T\|\}$
- The second inclusion quite sharp: $\|T\| \leq 2\sup\{|\lambda| \ ; \ \lambda \in W(T)\}$.
- For a normal ($A^*A = AA^*$) matrix A, $W(A)$ is the convex hull of $\sigma(A)$.

Spectral projections: a simple example

Often one needs to apply ϕ which are not holomorphic on a convex neigbourhood of $\sigma(T)$ (like $\overline{W(T)}$).

Example

Let $A := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix}$, so that $\sigma(A) = \{3, 7\}$ (algebraic multiplicity 2 for $\lambda = 7$). For $\phi(\lambda) = \begin{cases} 0, & |\lambda - 3| < 1, \\ 1, & |\lambda - 7| < 1, \end{cases}$ we have $\phi(A) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ being the spectral projection onto the two-dimensional generalized eigenspace at $\lambda = 7$.

Von Neumann's and Crouzeix's theorems do not apply, but still $\|\phi(A)\| \le C \sup\{|\phi(\lambda)| \ ; \ |\lambda - 3| < 1 \text{ or } |\lambda - 7| < 1\}$ is clear from the Dunford integral!

Partial differential equations (PDEs)

Applications to PDEs typically involves functional calculus of unbounded differential operators D on a space like $\mathcal{H} = L_2(\mathbf{R}^n)$ (rather than bounded operators T).
Consider the positive Laplace operator $-\Delta = -\sum_1^n \partial_k^2$, with spectrum $\sigma(-\Delta) = \overline{\mathbf{R}_+}$.

- The heat equation $\partial_t f_t - \Delta f_t = 0$ has solution

$$f_t = e^{-t(-\Delta)} f_0, \qquad t > 0.$$

- The wave equation $\partial_t^2 f_t - \Delta f_t = 0$ has solution

$$f_t = \cos(t\sqrt{-\Delta}) f_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} (\partial_t f)_0, \qquad t \in \mathbf{R}.$$

For functions $f_t(x) = f(t, x)$ we write $x \in \mathbf{R}^n$ for the space variable and t for the time/evolution variable.

Integral formulae

Calculating functions $\phi(-\Delta)$ of the self-adjoint unbounded operator $-\Delta$ with the Fourier transform $\mathcal{F}$ as $\phi(-\Delta) = \mathcal{F}^{-1} M_{\phi(|\xi|^2)} \mathcal{F}$ gives well known integral formulae for solutions.

- For the heat equation in $\mathbf{R}^n$, we have

$$(e^{-t(-\Delta)} f)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} e^{-|y-x|^2/(4t)} f(y) dy.$$

- For the wave equation in $\mathbf{R}^3$ (n=3), we have

$$(\cos(t\sqrt{-\Delta}) f)(x) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{|y-x|=t} f(y) d\sigma(y) \right),$$

$$((-\Delta)^{-1/2} \sin(t\sqrt{-\Delta}) f)(x) = \frac{1}{4\pi t} \int_{|y-x|=t} f(y) d\sigma(y),$$

where $d\sigma$ is surface measure on the sphere $|y-x| = t$.

Cauchy–Riemann's equations (CR) and Hardy spaces

The CR system of equations for an analytic function $f = u + iv$ of one complex variable $z = x + iy$ can be written

$$\partial_y f_y + D f_y = 0,$$

where $D := -i\partial_x$ acts on one-variable functions $x \mapsto f_y(x) = f(x, y)$ for fixed $y > 0$.

- Problem: e^{-yD} is not bounded for any $y \neq 0$. This $D = \mathcal{F}^{-1} M_\xi \mathcal{F}$ is a two-sided unbounded self-adjoint operator in $\mathcal{H} = L_2(\mathbf{R})$:

$$\sigma(D) = (-\infty, \infty).$$

- Apply $\chi^+ := \chi_{(0,\infty)}$ and $\chi^- := \chi_{(-\infty,0)}$ to get the spectral projections

$$P^\pm = \chi^\pm(D) = \mathcal{F}^{-1} M_{\chi^\pm(\xi)} \mathcal{F}.$$

- The Hilbert space splits orthogonally $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm := \mathrm{R}(P^\pm)$. The *Hardy subspaces* $\mathcal{H}^\pm$ are invariant under D and $D^\pm := D|_{\mathcal{H}^\pm}$ have spectra

$$\sigma(D^+) = [0, \infty) \qquad \text{and} \qquad \sigma(D^-) = (-\infty, 0].$$

Functional calculus and the Cauchy integral

- Solution to CR for $y > 0$ with boundary data $f_0 \in \mathcal{H}^+$ at $y = 0$ is

$$f(x+iy) = (e^{-yD^+} f_0)(x), \qquad y > 0.$$

Calculating $e^{-yD^+} = \mathcal{F}^{-1} M_{e^{-y\xi}\chi^+(\xi)} \mathcal{F}$ with the Fourier transform $\mathcal{F}$ gives the Cauchy integral

$$(e^{-yD^+} f_0)(x) = \frac{1}{2\pi i} \int_{\mathbf{R}} \frac{f_0(t)}{t-(x+iy)} dt, \qquad y > 0, f \in \mathcal{H}^+.$$

- Solution to CR for $y < 0$ with boundary data $f_0 \in \mathcal{H}^-$ at $y = 0$ is

$$f(x+iy) = (e^{-yD^-} f_0)(x), \qquad y < 0.$$

Calculating $e^{-yD^-} = \mathcal{F}^{-1} M_{e^{-y\xi}\chi^-(\xi)} \mathcal{F}$ with the Fourier transform $\mathcal{F}$ gives the Cauchy integral

$$(e^{-yD^-} f_0)(x) = \frac{1}{2\pi i} \int_{\mathbf{R}} \frac{f_0(t)}{t-(x+iy)} dt, \qquad y < 0, f \in \mathcal{H}^-.$$

The Kato conjecture for spectral projections

Note for CR that $\|\chi^{\pm}(D)\|_{L_2 \to L_2} = 1 < \infty$ as a consequence of self-adjointness of $D = -i\partial_x$. Consider the following natural generalization to "variable coefficients" B.

- D is a self-adjoint operator in a Hilbert space so that $\sigma(D) \subset (-\infty, \infty)$.
- $B : \mathcal{H} \to \mathcal{H}$ is a bounded operator with numerical range $\overline{W(B)}$ being compactly contained in the right half plane $\operatorname{Re} \lambda > 0$.
- Then BD is a closed operator in $\mathcal{H}$ with spectrum

  $$\sigma(BD) \subset S_\omega := \{|\arg \lambda| < \omega\} \cup \{|\arg(-\lambda)| < \omega\}$$

  contained in a double sector S_ω around $\mathbf{R}$, for some $\omega \in (0, \pi/2)$ depending on B.

Let $\chi^{\pm}$ be the characteristic function for the right/left half plane. Can

$$\chi^{\pm}(BD)$$

be defined through Dunford functional calculus as bounded linear projections on $\mathcal{H}$?

Positive answer to a restricted Kato conjecture

The classical Kato conjecture for square roots was posed by T. Kato (*Fractional powers of dissipative operators.* J. Math. Soc. Japan, 1961). This famous conjecture was solved by P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and P. Tchamitchian (*The solution of the Kato square root problem for second order elliptic operators on* $\mathbf{R}^n$. Annals of Math., 2002). Extending this result, the following solution to the Kato conjecture for spectral projections was found.

Theorem (A. Axelsson, A. McIntosh, S. Keith, Invent. Math. 2006)

Assume furthermore that $D = \sum_{k=1}^n a_k \partial_k$ is a first order partial differential operator with constant coefficients in $\mathcal{H} = L_2(\mathbf{R}^n)$ and that $B = M_b$ is a multiplication operator. Then $\|\chi^\pm(BD)\|_{L_2 \to L_2} < \infty$.

This theorem says that we can cut through the spectrum at 0 and ∞, and obtain two bounded spectral projections $P^\pm$, even though the symbols $\chi^\pm$ are not analytic on a neighbourhood of $\sigma(BD)$.

A counterexample for the general Kato conjecture

The proof of $\|\chi^\pm(BD)\| < \infty$ uses harmonic analysis techniques that require D to be a differential operator and B to be a multiplication operator. A counter example to the Kato conjecture was found by A. McIntosh (*On the comparability of $A^{1/2}$ and $A^{*1/2}$*. Proc. Amer. Math. Soc., 1972). A variant of it is the following.

Example

Let $\mathcal{H} = \ell_2(\mathbf{Z}) = \text{span}\{e_k\}$ and $\zeta \in \mathbf{C}$. Define

$$De_k := 2^{|k|} e_{-k} \qquad \text{and} \qquad B_\zeta e_k := e_k + \zeta \sum_{j \neq 0} \frac{1}{j} e_{k+j}.$$

Then there are $\zeta \approx 0$ such that $\chi^\pm(B_\zeta D)$ is not bounded.

Note that using Fourier series, D can be viewed as a differential operator of "infinite order" and B as a multiplication operator.
One finds that $\sigma(D) = \{1\} \cup \{\pm 2^k \ ; \ k = 1, 2, 3, \dots\}$ and $W(B)$ is the straight line from $1 - i\zeta$ to $1 + i\zeta$.

Inhomogeneous linear PDEs

Consider the heat equation with sources

$$\partial_t f_t + D f_t = g_t,$$

where $D = -\Delta$ is the Laplace operator on $\mathbf{R}^n$ as above. It is important that $\sigma(D) = [0, \infty) \subset \{\mathrm{Re}\,\lambda > 0\}$ so that the semigroup $\{e^{-tD}\}_{t>0}$ is uniformly bounded. Solving with integrating factor $e^{-(t-s)D}$ for $0 < s < t$:

$$\partial_s(e^{-(t-s)D} f_s) = D e^{-(t-s)D} g_s,$$

$$f_t = e^{-tD} f_0 + \int_0^t e^{-(t-s)D} g_s \, ds.$$

One says that the equation has *maximal regularity* in the function space $L_2(\mathbf{R}_+; L_2(\mathbf{R}^n)) = L_2(\mathbf{R}_+ \times \mathbf{R}^n)$ if (with initial data $f_0 = 0$)

$$g_t \mapsto D f_t = \int_0^t D e^{-(t-s)D} g_s \, ds$$

is bounded on this space of functions $g_t(x) = g(t, x)$ in $\mathbf{R}^{1+n}_+ := \mathbf{R}_+ \times \mathbf{R}^n$.

A useful abstract point of view

In $\int_0^t De^{-(t-s)D} g_s ds$, replace D by a spectral point λ:

$$\Phi(\lambda) : L_2(\mathbf{R}^{1+n}_+) \to L_2(\mathbf{R}^{1+n}_+) : g_t \mapsto \int_0^t \lambda e^{-(t-s)\lambda} g_s ds.$$

View $\lambda \mapsto \Phi(\lambda)$ as an $L_2(\mathbf{R}^{1+n}_+)$-operator-valued function for $\operatorname{Re}\lambda > 0$.

- For any $\mu < \pi/2$, it is holomorphic and uniformly bounded on the sector $|\arg\lambda| < \mu$.
- Since on functions $g(t,x)$, D acts in the x-variable and $\Phi(\lambda)$ acts in the t variable, we have

$$\Phi(\lambda)D = D\Phi(\lambda) \qquad \text{for all } \lambda.$$

Applying the operator-valued function $\lambda \mapsto \Phi(\lambda)$ to the operator D yields

$$\Phi(D) : g_t \mapsto \int_0^t De^{-(t-s)D} g_s ds.$$

This generalization of functional calculus, using operator-valued symbols $\Phi(\lambda)$, we refer to as *operational calculus*.

Joint functional calculus of two commuting operators

For the solution f_t to $\partial_t f_t + Df_t = g_t$ with boundary conditions $f_0 = 0$, we have

$$Df = \Phi(D)f = \frac{D}{\partial_t + D} g.$$

The operator $D/(\partial_t + D)$ is defined through functional calculus of the two commuting unbounded operators D and ∂_t (the domain of ∂_t being functions with zero boundary condition at $t = 0$). Thus

$$\Phi(\lambda) = \lambda(\lambda I + \partial_t)^{-1}, \qquad \text{for } \operatorname{Re}\lambda > 0.$$

References for operational calculus and functional calculus of commuting operators:

D. Albrecht (*Functional calculi of commuting unbounded operators*. PhD thesis, Monash Univ., 1994)

D. Albrecht, E. Franks, A. McIntosh, (*Holomorphic functional calculus and sums of commuting operator*. Bull. Austral. Math. Soc., 1998)

A Duhamel formula for bi-sectorial operators

Replace the positive self-adjoint operator $D = -\Delta$ by a bi-sectorial differential operator BD with bounded coefficients $B \in L_\infty(\mathbf{R}^n)$, as above for the Kato problem, and consider the maximal regularity question for

$$\partial_t f_t + BD f_t = g_t.$$

Apply the two spectral projections to get $f_t^\pm := \chi^\pm(BD) f_t$, with $f_t = f_t^+ + f_t^-$. Integrating each of the equations

$$\partial_t f_t^+ + (BD)^+ f_t^+ = g_t^+ \qquad \text{and} \qquad \partial_t f_t^- + (BD)^- f_t^- = g_t^-,$$

we get the solution formula

$$f_t = e^{-t(BD)^+}\chi^+(BD) f_0 + \int_0^t e^{-(t-s)(BD)^+}\chi^+(BD) g_s \, ds \\ - \int_t^\infty e^{(s-t)(BD)^-}\chi^-(BD) g_s \, ds.$$

New maximal regularity results for elliptic equations

Recent joint work with P. Auscher:
A. Axelsson, P. Auscher (*Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I*. To appear in Invent. Math.)
A. Rosén, P. Auscher (*Weighted maximal regularity estimates and solvability of non-smooth elliptic systems II*. Preprint)

- We here prove maximal regularity estimates for $\partial_t f_t + BDf_t = g_t$ in weighted spaces $L_2(\mathbf{R}^{1+n}_+; t^\alpha dtdx)$. Such hold for $-1 < \alpha < 1$ and are proved through operational calculus of BD very similarly to the proof of the boundedness of the spectral projections $\chi^\pm(BD)$ in the solution of the Kato conjecture.
- Maximal regularity does not hold for $\partial_t f_t + BDf_t = g_t$ in the endpoint spaces $L_2(\mathbf{R}^{1+n}_+; tdtdx)$ and $L_2(\mathbf{R}^{1+n}_+; t^{-1}dtdx)$, but we adapt the techniques and obtain perturbation results for Dirichlet and Neumann boundary value problems, for second order divergence form elliptic equations with non-smooth coefficients, with $L_2(\mathbf{R}^n)$ data.