Fredholm theory, singular integrals and Tb-theorems

The theme of the course is integral equations, and consists of two parts

Part I: Fredholm operator theory.

Part II: Singular integral operators.

Introduction:

Notation: $X, Y, ..$ Banach spaces (B-spaces)

$H_1, H_2 ..$ Hilbert spaces (H-spaces)

$L(X,Y) = \{T : X \to Y ;\ T \text{ is linear and bounded}\}$

We shall only consider real B-spaces.

Ex. 1:1: The classical B-spaces of functions on a given measure space (Ω, μ) are the L_p-spaces:

$$\|f\|_{L_p(\Omega,\mu)} := \left(\int_\Omega |f(x)|^p d\mu(x)\right)^{1/p}, \quad 1 \le p < \infty$$

$$\|f\|_{L_\infty(\Omega,\mu)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|$$

For L_∞, only the zero sets of μ are needed.

Only L_2 is a H-space.

Ex 1:2: The classical B-spaces of functions on a metric space (Ω, d) are:

- the space of continuous functions ($\alpha = 0$)

$$\|f\|_{C(\Omega,d)} := \sup_{x\in\Omega} |f(x)|, \quad f : \Omega \to \mathbb{R} \text{ continuous.}$$

- the Hölder spaces ($0 < \alpha < 1$):

$$\|f\|_{C^\alpha(\Omega,d)} := \max\left\{ \sup_{\substack{x\neq y\\ x,y\in\Omega}} \frac{|f(x)-f(y)|}{d(x,y)^\alpha}, \ \sup_{x\in\Omega} |f(x)| \right\}$$

- the Lipschitz space ($\alpha = 1$)

$$\|f\|_{C^{0,1}(\Omega,d)} := \max\left\{ \sup_{\substack{x\neq y\\ x,y\in\Omega}} \frac{|f(x)-f(y)|}{d(x,y)}, \ \sup_{x\in\Omega} |f(x)| \right\}$$

For $C(\Omega,\mu) = C^0(\Omega,\mu)$, only the topology of d is needed.

In spaces where differentiation is possible, more spaces are available: Sobolev spaces $W_p^k(\Omega,\mu)$, where derivatives up to order k lie in $L_p(\Omega,\mu)$, and higher Hölder spaces $C^{k,\alpha}(\Omega,\mu)$, where derivatives up to order k lie in $C^{0,\alpha}(\Omega,\mu)$. $(C^{0,0} = C^0,\ C^{0,\alpha} = C^\alpha)$

<u>Main problem for part I</u>:

Let $T \in L(X,Y)$ and $y \in Y$. We want to solve the equation

$$Tx = y \quad \text{for } x \in X.$$

Ideally we would like to have

(1) existence, i.e. range $R(T) = Y$

(2) uniqueness, i.e. null space $N(T) = \{0\} \subset X$

(3) continuous dependence of the solution x on the datum y.

As X and Y are **complete** normed spaces, we have (1),(2) $\Rightarrow$ (3).

Recall the two most fundamental theorems in functional analysis.

<u>Thm 1.3</u> (<u>B</u>ounded <u>i</u>nverse <u>t</u>heorem, BIT)

If $T \in L(X,Y)$ is bijective, then $T^{-1} \in L(Y,X)$ is bounded.

The key point: many $T \in L(X,Y)$ have dense range. The BIT hypothesis is that $R(T) = Y$ (T is surjective), not only that $\overline{R(T)} = Y$.

<u>Defn 1.4</u> (Dual space)

If X is a B-space, then its <u>dual space</u> is $X^* := L(X,\mathbb{R})$ with norm

$$\|f\|_{X^*} := \sup_{\substack{x \neq 0 \\ x \in X}} \frac{|f(x)|}{\|x\|_X}.$$

Thm 1.5 (Hahn-Banach's theorem, HB)
If Z is a B-space and $X \subset Z$ a subspace, then any $f \in X^*$ can be extended to some $\tilde{f} \in Z^*$, such that $\tilde{f}|_X = f$ and $\|\tilde{f}\|_{Z^*} = \|f\|_{X^*}$.

HB is very different from BIT. Completeness of the spaces is essential in BIT, whereas no completeness assumption is needed in HB. On the other hand HB make use of the axiom of choice (but BIT does not need this). Throughout this course, we shall assume this axiom holds.

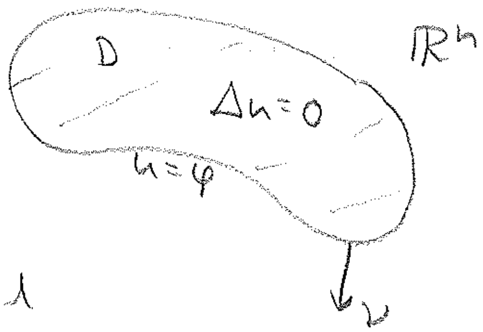
Part I of the course will be studied from the abstract point of view of operator theory (as compared to part II which will be more concrete, but more technical). We start with a classical example of an integral equation.

The double layer potential:

We wish to solve the Dirichlet boundary value problem for the Laplace equation:
Let $D \subset \mathbb{R}^n$ be a (bounded, smooth say) domain. Given $\varphi : \partial D \to \mathbb{R}$, find $u : \overline{D} \to \mathbb{R}$ such that

$$\begin{cases} \Delta u = 0 & \text{in } D \quad (\text{i.e. } u \text{ is a harmonic function}) \\ u = \varphi & \text{on } \partial D \end{cases}$$

The Laplace operator is
$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$



This Dirichlet problem is central in the theory of elliptic partial differential equations.

(i) Fundamental solution $\Gamma(y,x)$ to Δ in $\mathbb{R}^n$:

$$\Gamma(y,x) := \begin{cases} \dfrac{-1}{(n-2)\sigma_{n-1}|y-x|^{n-2}} & , n \geq 3 \\ \dfrac{1}{4\pi} \ln|y-x| & , n = 2 \end{cases}$$

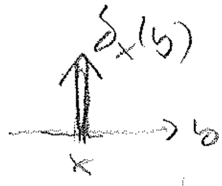
(σ_{n-1} := area of unit sphere S^{n-1} in $\mathbb{R}^n$.)

We have

$$E(y,x) := \nabla_y \Gamma(y,x) = \frac{y-x}{\sigma_{n-1}|y-x|^n} \quad , n \geq 2$$

$$\Delta_y \Gamma(y,x) = \operatorname{div}_y E(y,x) = \delta_x(y)$$

(Dirac delta distribution at x)



(ii) Green function $G(y,x)$ for Δ in D:

$$G(y,x) := \Gamma(y,x) - g_x(y),$$

where g_x is such that $\begin{cases} \Delta g_x = 0 & \text{in } D \\ g_x = \Gamma(\cdot,x) & \text{on } \partial D \end{cases}$

(Thus the definition of the Greens function uses solvability of the Dirichlet problem, but only for some special data φ.)

We have $\begin{cases} \Delta_y G(\cdot,x) = \delta_x & \text{in } D \\ G(\cdot,x) = 0 & \text{on } \partial D \end{cases}$

Now apply Green's identity:

outward normal derivatives

$$\iint_D u(\Delta f) - \underbrace{\cancel{(\Delta u)}}_{0} f = \int_{\partial D} u \frac{\partial f}{\partial \nu} - \frac{\partial u}{\partial \nu} f$$

with $f = G(\cdot,x)$ $\Rightarrow$

surface measure

$$u(x) = \iint_D u(y) \underbrace{\delta_x(y)}_{\Delta_y G(y,x)} dy = \int_{\partial D} \varphi(y) \underbrace{\frac{\partial G(y,x)}{\partial \nu(y)} d\sigma(y)}_{=: d\omega^x(y)}$$

$\dfrac{\partial G}{\partial \nu}$ =: Poisson kernel for D

$d\omega^x$ =: harmonic measure for D (at x)

This calculates the solution u for any given datum φ, provided the solutions g_x with data $\Gamma(\cdot,x)|_{\partial D}$ are known.

To avoid using the (typically unknown) Green's function, we instead apply Green's identity with $f = \Gamma(\cdot, x)$ $\Rightarrow$

$$u(x) = \iint_D u(y)\,\delta_x(y)\,dy$$
$$= \underbrace{\int_{\partial D} E(y,x)\cdot\nu(y)\,\varphi(y)\,d\sigma(y)}_{=:\text{ double layer potential of } \varphi} - \underbrace{\int_{\partial D} \Gamma(y,x)\,\frac{\partial u}{\partial \nu}(y)\,d\sigma(y)}_{=:\text{ single layer potential of } \frac{\partial u}{\partial \nu}}$$

This is the analogue of the Cauchy integral for analytic function, for harmonic functions: it calculates u in D from its Dirichlet ($u=\varphi$) and Neumann data ($\frac{\partial u}{\partial \nu}$) on ∂D.

<u>Remark 1.6</u>: In dimension $n=3$, $-E(y,\cdot)$ is the electric field from a point charge at y, and $\Gamma(y,\cdot)$ is the corresponding potential. Thus the electric potential from charges on ∂D with density $\psi(y)$ will be $\int \Gamma(y,x)\,\psi(y)\,d\sigma(y)$.
The electric potential from a dipole at y in direction $\nu(y)$ will be $\frac{\partial \Gamma(y,x)}{\partial \nu(y)} = E(y,x)\cdot\nu(y)$, and that from a dipoles on ∂D with density $\varphi(y)$ will be $\int E(y,x)\cdot\nu(y)\,\varphi(y)\,d\sigma(y)$.
This is the physical origin of the names.

The boundary integral equation method for solving the Dirichlet problem is as follows.

Start with an auxiliary function $h : \partial D \to \mathbb{R}$ and make the ansatz

$$u(x) := \int_{\partial D} E(y,x)\cdot\nu(y)\, h(y)\, d\sigma(y) \quad , x \in D$$

for the solution.

Since $x \mapsto E(y,x)\cdot\nu(y)$ is harmonic in D for each $y \in \partial D$, we will always have $\Delta u = 0$.

It remains to choose h such that $u|_{\partial D} = \varphi$

<u>Trace calculation:</u> Fix $x_0 \in \partial D$.

Assume that h and ∂D are smooth at x_0, so that $E(y,x_0)\cdot\nu(y)\,(h(y)-h(x_0))$ is integrable on ∂D. Assume for simplicity that $x \to x_0 \in \partial D$ along $\operatorname{span}\{\nu(x_0)\}$.

Dominated convergence theorem $\Rightarrow$

$$\lim_{\substack{x\to x_0\\ x\in D}} \int_{\partial D} E(y,x)\cdot\nu(y)\,(h(y)-h(x_0))\,d\sigma(y) = \int_{\partial D} E(y,x_0)\cdot\nu(y)\,(h(y)-h(x_0))\,d\sigma(y)$$

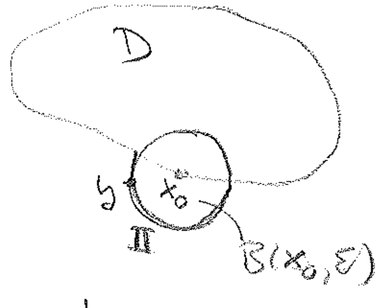
$$\text{LHS} = \lim_{\substack{x\to x_0\\ x\in D}} u(x) - h(x_0) \quad \text{since}$$

$$\int_{\partial D} E(y,x)\cdot\nu(y) = 1 \text{ by Gauss' theorem.}$$

$$\text{RHS} = \lim_{\varepsilon\to 0} \int_{\partial D\setminus B(x_0,\varepsilon)} E(y,x_0)\cdot\nu(y)\,h(y)\,d\sigma(y)$$

$$-\lim_{\varepsilon\to 0}\Big\{ \underbrace{\int_{\partial(D\cup B(x_0,\varepsilon))} E(y,x_0)\cdot\nu(y)\,h(x_0)}_{=:\mathrm{I}} - \underbrace{\int_{\partial B(x_0,\varepsilon)\setminus D} E(y,x_0)\cdot\nu(y)\,h(x_0)}_{=:\mathrm{II}} \Big\}$$

Gauss $\Rightarrow$ $\mathrm{I} = h(x_0)$



II: on the outer half-sphere we have

$$E(y,x_0)\cdot\nu(y) = \frac{y-x_0}{\sigma_{n-1}|y-x_0|^n}\cdot\frac{y-x_0}{|y-x_0|} = \frac{1}{\sigma_{n-1}\varepsilon^{n-1}} = |\partial B(x_0,\varepsilon)|$$

$$\Rightarrow \mathrm{II} = \frac{|\partial B(x_0,\varepsilon)\setminus D|}{|\partial B(x_0,\varepsilon)|} \longrightarrow \frac{1}{2} \quad , \varepsilon \to 0$$

In total we have proved:

<u>Prop. 1.7</u>: The trace of the double layer potential
$u(x) = \int_{\partial D} E(y,x)\cdot \nu(y) h(y)$, $x \in D$, of h
is
$$\lim_{\substack{x \to x_0 \\ x \in D}} u(x) = \tfrac{1}{2} h(x_0) + \text{p.v.} \int_{\partial D} E(y,x_0)\cdot \nu(y) h(y)\, d\sigma(y).$$
("principal value", i.e. $\lim_{\varepsilon \to 0} \int_{\partial D \setminus B(x_0,\varepsilon)}$)

<u>Defn. 1.8</u>: The (principal value) <u>double layer potential</u> is integral operator
$$Kh(x) := \frac{2}{\sigma_{n-1}} \text{p.v.} \int_{\partial D} \frac{y-x}{|y-x|^n} \cdot \nu(y) h(y)\, d\sigma(y), \quad x \in \partial D$$
acting on functions on ∂D

- The conclusion is that the Dirichlet problem
$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \varphi & \text{on } \partial D \end{cases}$ can be solved with the following boundary integral equation method:
  1. Solve the integral equation on ∂D
  $$\tfrac{1}{2}(h + Kh) = \varphi \quad \text{for } h: \partial D \to \mathbb{R}$$
  2. The solution to the Dirichlet problem is then
  $$u(x) = \int_{\partial D} E(y,x)\cdot \nu(y) h(y)\, d\sigma(y), \quad x \in D.$$

The equation $Th = \varphi$ for $T = \frac{1}{2}(I + K)$ will be our main example in Part I. Of course, to solve $Th = \varphi$, we need to choose a function space on ∂D to solve it in. We shall return to this later.

Integral operators:

Taking K as an example, we make the following definition.

Defn 1.9: An *integral operator* between spaces of functions on measure spaces (Ω_1, μ_1), (Ω_2, μ_2), is an operator T given by

$$Tf(x) = \int_{\Omega_2} k(x,y) f(y) \, d\mu_2(y) \quad , \quad x \in \Omega_1$$

for some *kernel* $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$.

Note that when Ω_i are finite discrete spaces, k is the matrix for the finite dimensional operator T. Thus the kernel can be viewed as a "continuous matrix".

To bound the norm of integral operators on L_p-spaces, we have the following useful standard technique.

Prop. 1.10 (Schur estimates)

$$\|T\|_{L_\infty(\mu_2) \to L_\infty(\mu_1)} = \operatorname*{ess\,sup}_{x \in \Omega_1} \int_{\Omega_2} |k(x,y)| \, d\mu_2(y) =: M_\infty$$

$$\|T\|_{L_1(\mu_2) \to L_1(\mu_1)} = \operatorname*{ess\,sup}_{y \in \Omega_2} \int_{\Omega_1} |k(x,y)| \, d\mu_1(x) =: M_1$$

For $1 < p < \infty$, we have the Schur estimate

$$\|T\|_{L_p(\mu_2) \to L_p(\mu_1)} \leq M_1^{1/p} M_\infty^{1/q} \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

Proof: For L_∞:

$$|Tf(x)| \leq \int_{\Omega_2} |k(x,y)| \cdot |f(y)| \, d\mu_2(y)$$

$$\leq \underbrace{\left(\int_{\Omega_2} |k(x,y)| \, d\mu_2(y)\right)}_{\leq M_\infty} \operatorname*{ess\,sup}_{y \in \Omega_2} |f(y)| \quad , \text{a.e. } x \in \Omega_1$$

For L_1:

$$\int_{\Omega_1} |Tf(x)|\, d\mu_1(x) \underset{\text{Fubini}}{\leq} \int_{\Omega_2} \underbrace{\left(\int_{\Omega_1} |k(x,y)|\, d\mu_1(x)\right)}_{\leq M_1} |f(y)|\, d\mu_2(y)$$

In each case one pick appropriate (x and) f to prove $=$. (which?)

For L_p:

Apply Hölder's inequality:

$$|Tf(x)|^p \leq \left(\int_{\Omega_2} |k(x,y)|^{1/q} \left(|k(x,y)|^{1/p} |f(y)|\right) d\mu_2(y)\right)^p$$

$$\leq \underbrace{\left(\int_{\Omega_2} |k(x,y)|\, d\mu_2(y)\right)^{p/q}}_{\leq M_\infty} \left(\int_{\Omega_2} |k(x,y)| \cdot |f(y)|^p\, d\mu_2(y)\right)$$

$$\Rightarrow \int |Tf(x)|^p\, d\mu_1(x) \leq M_\infty^{p/q} \int_{\Omega_2} \underbrace{\left(\int_{\Omega_1} |k(x,y)|\, d\mu_1(x)\right)}_{\leq M_1} |f(y)|^p\, d\mu_2(y) \qquad \blacksquare$$

Note that the above Schur estimate from an abstract point is nothing but interpolation

$L_1, L_\infty \to L_p$.

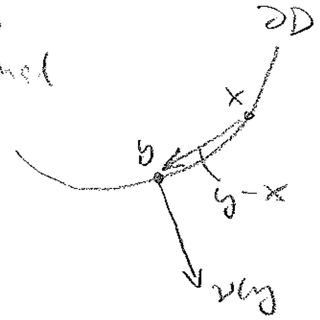
<u>Problem 1.11</u>: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2/2-matrix, defining an operator on the 2D Hilbert space. Compute the exact formula for $\|A\|_{L_2 \to L_2}$. (It contains two square roots.) Compare with the Schur estimate. Give non-trivial examples where the estimate is sharp / is not sharp.

<u>Example 1.12</u>:

Assume that D is a bounded domain with a $C^{1,\alpha}$ smooth boundary (so the normal vector field $\nu(y)$ is C^α α-Hölder continuous ($\alpha > 0$)). For the double layer potential K, in Defn. 1.9 we have

$(\Omega_i, \mu_i) = (\partial D, \sigma)$ (n-1-dimensional) and

$$k(x,y) = E(y,x)\cdot\nu(y) = \underbrace{\frac{2}{\sigma_{n-1}}\frac{y-x}{|y-x|^n}}_{\substack{\approx \text{ tangential} \\ \text{when } y\approx x}} \cdot \underbrace{\nu(y)}_{\text{normal}}$$



If ∂D is $C^{1,\alpha}$ one can prove the improved estimate

$$|(y-x)\cdot\nu(y)| \le C\,|y-x|^{1+\alpha} \quad \text{for some } C<\infty$$

(exercise!)

$$\Rightarrow |k(x,y)| \le \frac{2C}{\sigma_{n-1}} \frac{1}{|y-x|^{n-1-\alpha}}$$

From this $M_1, M_\infty < \infty$ clearly follows in Prop. 1.10. Thus K is a bounded operator on $L_p(\partial D, \sigma)$ for any $1 \le p \le \infty$.

<u>Problem 1.13</u>: Assume

$$\int_{\Omega_2} |k(x,y)|^{\beta q}\, \omega_2^q(y)\, d\mu_2(y) \le \tilde{M}_\infty\, \omega_1^q(x), \quad \text{a.e. } x\in\Omega_1$$

$$\int_{\Omega_1} |k(x,y)|^{\alpha p}\, \omega_1^p(x)\, d\mu_1(x) \le \tilde{M}_1\, \omega_2^p(y), \quad \text{a.e. } y\in\Omega_2$$

for some $\alpha+\beta=1$, $\frac{1}{p}+\frac{1}{q}=1$ and functions $0<\omega_i<\infty$ on Ω_i.

Prove the weighted Schur estimates

$$\|T\|_{L_p(\mu_2)\to L_p(\mu_1)} \le \tilde{M}_1^{1/p}\, \tilde{M}_\infty^{1/q} \quad \text{for } 1<p<\infty.$$

Compact operators on Banach spaces

For a metric space M, we recall that the following are equivalent:

(i) M is compact. (Each open cover of M has a finite subcover.)

(ii) M is sequentially compact. (Every sequence in M has a subsequence converging to a point in M.)

(iii) M is totally bounded (it can be covered by finitely many balls of radii ε, for any $\varepsilon > 0$) and complete (every Cauchy sequence converges).

Let now M be a subset of a complete metric space X (i.e. a B-space). Then M is complete iff M is closed. Dropping the assumption of closedness, we have equivalences:

M precompact ($\bar{M}$ compact) $\Leftrightarrow$
every sequence in M has subseq. converging (in X) $\Leftrightarrow$
M is totally bounded.

Ex 2:1: $\overline{B}_X$ = closed unit ball in a B-space X.

- If $\dim X < \infty$, then $\overline{B}_X$ is compact. It takes $\sim \varepsilon^{-n}$ ε-balls to cover $\overline{B}_X$. (why?)
- If $\dim X = \infty$, then $\overline{B}_X$ is not compact. The standard proof uses:

Lemma 2:2 (F. Riesz' lemma)

If Y is a closed proper subspace of a B-space X, then for any $\varepsilon > 0$, there exists $u \in X$ such that $\|u\| = 1$, $\operatorname{dist}(u, Y) \geq 1 - \varepsilon$.
If $\dim Y < \infty$, then this holds for $\varepsilon = 0$.

If X is a H-space, this is trivial: take $u \in B_{Y^\perp}$.

Defn. 2.3 Let X, Y be B-spaces and $T \in L(X,Y)$.

(i) T has finite rank if $\dim R(T) < \infty$.
$FR(X,Y) := \{\text{such operators}\} \subset L(X,Y)$

(ii) T is compact if $T(B_X)$ is precompact, i.e. totally bounded.
$C(X,Y) := \{\text{such operators}\} \subset L(X,Y)$.

Here and elsewhere $B_X := \{x \in X ; \|x\| < 1\}$

Elementary properties:

Prop. 2.4: Let X, Y, Z be B-spaces.

(i) $FR(X,Y) \subset C(X,Y) \subset L(X,Y)$

(ii) $FR(X,Y)$ and $C(X,Y)$ are linear subspaces

(iii) $T \in L(Y,Z)$, $K \in C(X,Y) \Rightarrow TK \in C(X,Z)$
$T \in L(X,Y)$, $K \in C(Y,Z) \Rightarrow KT \in C(X,Z)$
This holds also when $C(\cdot,\cdot)$ is replaced by $FR(\cdot,\cdot)$
(When $X=Y=Z$ we say that $C(X,X)$ and $FR(X,X)$ are ideals in $L(X,X)$)

(iv) If $T_n \in C(X,Y)$ and $T_n \to T \in L(X,Y)$, then $T \in C(X,Y)$
(Thus $C(X,X)$ is a closed ideal in $L(X,X)$.)

Proof: We leave as exercise everything but
(iv): Need to show that $T(B_X)$ is totally bounded.
Let $\varepsilon > 0$ be given. Choose n so that
$\|T_n(x) - T(x)\| < \frac{\varepsilon}{2}$ when $x \in B_X$.
Since $T_n \in C(X,Y)$, we have $T_n(B_X) \subset \bigcup_{i=1}^{N} B(y_i; \frac{\varepsilon}{2})$
for some balls and $N < \infty$.
Thus $T(B_X) \subset \bigcup_{i=1}^{N} B(y_i; \varepsilon)$ as desired ∎

Problem 2.5 Let $q \geq p$. Show that the inclusion map
$L_q([0,1], dx) \longrightarrow L_p([0,1], dx)$ is continuous but
not compact (i.e. L_q is not compactly embedded in L_p).

The following classical theorem describes the compact subsets in the space of continuous functions.

Thm 2:6 (Arzelà - Ascoli)
Let (Ω, d) be a compact metric space.
Then $M \subset C(\Omega)$ is precompact iff
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall f \in M : d(x,y) \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon$
(i.e. M is equicontinuous) and M is bounded.

Ex 2:7: Ω as above. Then for $\alpha > 0$,
$C^\alpha(\Omega)$ is compactly embedded in $C(\Omega)$.
To prove this, apply Arzelà - Ascoli:
we need to show $B_{C^\alpha(\Omega)} \subset C(\Omega)$ is equicontinuous. But this is clear:
$\forall f \in B_{C^\alpha(\Omega)} \; \forall x, y \in \Omega : |f(x) - f(y)| \leq 1 \cdot \mathrm{dist}(x,y)^\alpha$.

Note from 2.5 and 2.7 that it seems that regularity rather than size give compactness.

Problem 2:8:
Let $D \subset \mathbb{R}^n$ be domain with compact and $C^{1,\alpha}$-regular boundary for some $\alpha > 0$.
(a) Show that an integral operator with continuous kernel,
$$Tf(x) = \int_{\partial D} k(x,y) f(y) \, d\sigma(y), \quad x \in \partial D, \quad \text{on } \partial D,$$
defines a compact operator on $C(\partial D)$.
(b) Show that the double layer potential operator on ∂D is compact on $C(\partial D)$.

We next prove that compactness is preserved under taking adjoints.
For $f \in X^*$ and $x \in X$, it is convenient to use the notation
$$\langle f, x \rangle := f(x).$$
Thus $\langle f, Tx \rangle = \langle T^* f, x \rangle$ (rather than $f(Tx) = (T^* f)(x)$).

Thm 2.9:

Let X, Y be B-spaces and $T \in L(X,Y)$.
Then $T \in C(X,Y)$ iff $T^* \in C(Y^*, X^*)$.

Proof: We prove $\Rightarrow$. The implication $\Leftarrow$ can be proved similarly and is left as exercise.

Assume $T \in C(X,Y)$.
We need to prove that $T^*(B_{Y^*}) \subset X^*$ is totally bounded.
To this end, let $\Omega := \overline{T(B_X)} \subset Y$ and
$$M := \{\Omega \to \mathbb{R}\,;\, y \mapsto \langle f, y\rangle \,;\, f \in B_{Y^*}\} \subset C(\Omega)$$
We see that Ω is compact and M is bounded.
Moreover, M is equicontinuous:
$$|\langle f, y_1\rangle - \langle f, y_2\rangle| = |\langle f, y_1 - y_2\rangle| \leq \|y_1 - y_2\|_Y$$
$$= d_\Omega(y_1, y_2) \quad , \forall y_1, y_2 \in \Omega \,, f \in B_{Y^*}.$$
Arzelà-Ascoli shows that M is totally bounded.
Thus, given $\varepsilon > 0$, there exists $f_1, \dots, f_N \in B_{Y^*}$ such that
$$M \subset \bigcup_{i=1}^{N} B_{C(\Omega)}(f_i, \varepsilon) \quad \text{, i.e.}$$
$$\forall f \in B_{Y^*} \; \exists i \in \{1, \dots, N\} : |\langle f, y\rangle - \langle f_i, y\rangle| \leq \varepsilon \quad , \forall y \in \Omega.$$
This gives in particular, since $T(B_X) \subset \Omega$, that
$$\|T^* f - T^* f_i\|_{X^*} = \sup_{\|x\| \leq 1} \langle T^* f - T^* f_i, x\rangle$$
$$= \sup_{\|x\| \leq 1} |\langle f, Tx\rangle - \langle f_i, Tx\rangle| \leq \varepsilon$$
$$\therefore \; T^*(B_{Y^*}) \subset \bigcup_{i=1}^{N} B_{X^*}(T^* f_i, \varepsilon) \quad \blacksquare$$

$$X^* \xleftarrow{T^*} Y^* \quad f$$

$$x \quad X \xrightarrow{T} Y \supset \Omega \quad y$$

Compact operators on Hilbert spaces

It is natural to ask if the converse of Prop. 2.4(iv) holds: is any compact operator the uniform limit of finite rank operators? $(\overline{FR(X,Y)} = C(X,Y)?)$

For general B-spaces, the answer is no.
A counterexample was found by Per Enflo (Acta Math. 1973).
However, the following positive result holds:

Prop 2.10:
If $T \in C(X,Y)$ and either X or Y is a H-space, then there exists $T_n \in FR(X,Y)$ such that $\|T_n - T\| \to 0$, $n \to \infty$.

Proof:
(i) Assume first that Y is a H-space, and $T \in C(X,Y)$.
Given $\varepsilon > 0$, take $x_1, \dots, x_N \in B_X$ such that
$\forall x \in B_X \; \exists i \in \{1, \dots, N\} : \|Tx - Tx_i\| < \varepsilon$.
Let $V := \operatorname{span}\{Tx_1, \dots, Tx_N\}$ and $P \in L(Y,Y)$ be orthogonal projection onto V. Then

$$\|T - PT\|_{X \to Y} = \sup_{\|x\| \le 1} \underbrace{\|Tx - P(Tx)\|_Y}_{\operatorname{dist}_Y(Tx, V)} \le \varepsilon$$

$$\le \operatorname{dist}_Y(Tx, \{x_1, \dots, x_N\}) \le \varepsilon$$

and $R(PT) \subset R(P)$ is finite-dimensional.

(ii) Assume next that X is a H-space.
Then $T^* \in C(Y^*, X)$, where $X^* = X$.
As in (i), there is an orthogonal projection $P \in FR(X,X)$ such that

$$\varepsilon \ge \|T^* - PT^*\| = \|(T - TP)^*\| = \|T - TP\|.$$

This completes the proof since
$\dim(TP) \le \dim(P) < \infty$ ∎

On H-spaces, there is a useful subclass of compact operators, which we now study.
Let $X = H_1$, $Y = H_2$ be H-spaces. For simplicity assume H_1 and H_2 are seperable with ON-bases $\{e_i\}_{i=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ respectively.
For $T \in L(H_1, H_2)$, we have
$$\sum_{i=1}^{\infty} \|Te_i\|_{H_2}^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_j, Te_i\rangle|^2 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |\langle T^* f_j, e_i\rangle|^2$$
$$= \sum_{j=1}^{\infty} \|T^* f_j\|_{H_1}^2 .$$
In particular LHS is independent of choice of ON-basis for H_1, and RHS is independent of choice of ON-basis for H_2.

<u>Defn 2.11</u>: Let H_i be H-spaces and $T \in L(H_1, H_2)$.
T is a <u>Hilbert-Schmidt</u> operator if
$\sum_{i=1}^{\infty} \|Te_i\|^2 < \infty$ for ON-basis $\{e_i\}_{i=1}^{\infty}$.
$HS(H_1, H_2) := \{\text{such operators}\} \subset L(H_1, H_2)$
The Hilbert-Schmidt inner product
$\langle T_1, T_2\rangle_{HS} := \sum_{i=1}^{\infty} \langle T_1 e_i, T_2 e_i\rangle$ makes $HS(H_1, H_2)$
a H-space of operators, with norm $\|T\|_{HS} := \sqrt{\langle T, T\rangle_{HS}}$.

Note that the HS-norm is
$\|T\|_{HS} = \left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\langle f_j, Te_i\rangle|^2 \right)^{1/2}$, i.e.
the summed squares of T:s matrix elements in any ON-bases.

Prop. 2.12: Let H_1, H_2 be H-spaces.

(i) $FR(H_1,H_2) \subset HS(H_1,H_2) \subset C(H_1,H_2) \subset L(H_1,H_2)$

and $\overline{FR(H_1,H_2)} = \overline{HS(H_1,H_2)} = C(H_1,H_2)$

(ii) $HS(H_1,H_2)$ is a linear space

(iii) $T \in L(H_1,H_2)$, $S \in HS(H_2,H_3) \Rightarrow ST \in HS(H_1,H_3)$

$T \in L(H_2,H_3)$, $S \in HS(H_1,H_2) \Rightarrow TS \in HS(H_1,H_3)$

Proof: We prove (i) and leave (ii)-(iii) to the reader.

- If $T \in FR(H_1,H_2)$, then $\dim N(T)^\perp < \infty$.
  Pick ON-basis $\{e_i\}_{i=1}^\infty$ for H_1 such that $\{e_1, \dots, e_N\}$ is ON-basis for $N(T)^\perp$.

  $\Rightarrow \|T\|_{HS}^2 = \sum_{i=1}^N \|Te_i\|^2 < \infty$.

  Thus $FR(H_1,H_2) \subset HS(H_1,H_2)$.
- If $T \in HS(H_1,H_2)$ and $\{e_i\}_{i=1}^\infty$ ON-basis for H_1, let $T_k \in FR(H_1,H_2)$ be such that

  $$T_k e_i = \begin{cases} Te_i & ; i \le k \\ 0 & ; i > k. \end{cases}$$

  Then for all $x = \sum_{i=1}^\infty x_i e_i \in X$ we have

  $$\|(T-T_k)x\| \le \sum_{i=1}^\infty |x_i| \, \|(T-T_k)e_i\| = \sum_{i>k} |x_i| \, \|Te_i\|$$

  $$\le \Big(\sum_{i>k} |x_i|^2\Big)^{1/2} \Big(\sum_{i>k} \|Te_i\|^2\Big)^{1/2} \le \underbrace{\Big(\sum_{i>k} \|Te_i\|^2\Big)^{1/2}}_{\to 0 \text{ as } k \to \infty} \|x\|$$

  $\therefore HS(H_1,H_2) \subset \overline{FR(H_1,H_2)}$

  $= C(H_1,H_2)$

  by Prop. 2.10. ∎

By the Schwarz kernel theorem, basically any linear operator on functions on $\mathbb{R}^n$ can be represented as a formal integral operator, but formal in this sense that the kernel is in general a distribution only. If we require that the kernel should be a function, we have, for Hilbert-Schmidt operators, the following result.

<u>Thm 2.13</u> (Hilbert-Schmidt kernel theorem)

Let $(\Omega_1, \mu_1), (\Omega_2, \mu_2)$ be measure spaces, and $H_1 := L_2(\Omega_1, \mu_1)$, $H_2 := L_2(\Omega_2, \mu_2)$.

Consider $T \in L(H_2, H_1)$. Then $T \in HS(H_1, H_2)$ iff there is a kernel $k \in L_2(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ such that

$$Tf(x) = \int_{\Omega_2} k(x,y) f(y) \, d\mu_2(y) \quad , f \in H_2 \, , \text{a.e. } x \in \Omega_1$$

In this case $\|T\|_{HS}^2 = \iint_{\Omega_1 \Omega_2} |k(x,y)|^2 d\mu_1(x) \, d\mu_2(y)$.

<u>Proof</u>: Let $\{e_i^{(1)}\}_{i=1}^\infty$ and $\{e_j^{(2)}\}_{j=1}^\infty$ be ON-bases for H_1, H_2 respectively. Using Fubini's theorem one shows that $\{\tilde{e}_{ij}\}_{i,j=1}^\infty$ forms an ON-basis for $H_1 \otimes H_2 = L_2(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$, where $\tilde{e}_{ij}(x,y) := e_i^{(1)}(x) e_j^{(2)}(y)$.

<u>$\Leftarrow$</u>: Note that

$$\langle e_i^{(1)}, Te_j^{(2)} \rangle = \iint_{\Omega_1 \Omega_2} e_i^{(1)}(x) k(x,y) e_j^{(2)}(y) d\mu_1(x) d\mu_2(y) = \langle k, \tilde{e}_{ij} \rangle$$

Thus $k \in L_2 \Rightarrow T \in HS$ and $\|T\|_{HS} = \|k\|_{L_2}$.

<u>$\Rightarrow$</u>: If $T \in HS$, let $k := \sum_{i,j} \langle e_i^{(1)}, Te_j^{(2)} \rangle \tilde{e}_{ij}$,

with convergence in $L_2(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$.

By Fubini $\int_{\Omega_2} k(x,y) f(y) d\mu_2(y)$ exists for a.e. $x \in \Omega_1$ if $f \in H_2$, and $\int_{\Omega_1} g(x) \left(\int_{\Omega_2} k(x,y) f(y) d\mu_2(y) \right) d\mu_1(x) = \langle k, g \otimes f \rangle$ if $g \in H_1$.

But $\langle k, g\otimes f\rangle = \sum_{i,j} \langle e_i^{(1)}, T e_j^{(2)}\rangle \langle e_i^{(1)}, g\rangle \langle e_j^{(2)}, f\rangle$

$= \langle \sum_i \langle e_i^{(1)}, g\rangle e_i^{(1)}, T(\sum_j \langle e_j^{(2)}, f\rangle e_j^{(2)}) \rangle = \langle g, Tf\rangle$

Thus $Tf(x) = \int_{\Omega_2} k(x,y) f(y)\, d\mu_2(y)$ for a.e. $x \in \Omega_1$ ∎

Just like Arzelà-Ascoli's theorem is convenient for proving compactness of operators on $C(\Omega)$, the Hilbert-Schmidt kernel theorem is a standard tool for proving compactness of H-space operators. The following result shows how it can be applied.

Prop. 2.14:

Let $D \subset \mathbb{R}^n$ be a domain with compact and $C^{1,\alpha}$-regular boundary for some $\alpha > 0$. Consider an integral operator

$$Tf(x) = \int_{\partial D} k(x,y) f(y)\, d\sigma(y) \quad , x \in \partial D,$$

with kernel bounds $|k(x,y)| \le \frac{C}{|x-y|^{n-1-\alpha}}$.

Then T is a compact operator on $L_2(\partial D)$.

Proof: By Example 1.12 T is bounded on $L_2(\partial D)$.

Now truncate the kernel:

$$k_m(x,y) := \begin{cases} k(x,y) & ; |x-y| > \frac{1}{m} \\ 0 & ; |x-y| \le \frac{1}{m} \end{cases} \quad \text{and}$$

consider $T_m f(x) := \int_{\partial D} k_m(x,y) f(y)\, d\sigma(y)$.

Apply Schur's estimate 1.10 to $T - T_m$:

$$\|T - T_m\|_{L_2(\partial D)\to L_2(\partial D)} \underset{\substack{\uparrow \\ M_1 = M_\infty!}}{\le} \sup_{x\in\partial D} \int_{|y-x|\le \frac{1}{m}} C \cdot \frac{dy}{|x-y|^{n-1-\alpha}}$$

$$\le C_2 \cdot \int_0^{1/m} \frac{1}{t^{n-1-\alpha}} \cdot t^{n-2}\, dt = C_2 \frac{1}{m^\alpha} \to 0,\ m \to \infty.$$

Since k_m and ∂D are bounded, $T_m \in HS(L_2(\partial D), L_2(\partial D))$ by Thm 2.13. In particular T_m is compact, and therefore also $T = \lim_{m\to\infty} T_m$. ∎

<u>Ex 2.15</u>. Let ∂D be compact and $C^{1,\alpha}$-regular. Then the double layer potential operator on ∂D from Defn. 1.8 is compact on $L_2(\partial D)$.

<u>Ex 2.16</u>: There are compact operators on a H-space which are not Hilbert-Schmidt operators.
$\left(HS(H_1, H_2) \neq C(H_1, H_2) = \overline{HS(H_1, H_2)} \right)$
Here is an example:

$$H_1 = H_2 = L_2([0,1]), \quad \gamma \in (0,1)$$

Let $Tf(x) := \int_0^1 \frac{f(y)}{|x-y|^\gamma} dy$

By Thm 2.13 $T \in HS(H_1, H_2)$ iff $\gamma < \frac{1}{2}$
(i.e. $\int_0^1 \int_0^1 \frac{dx\,dy}{|x-y|^{2\gamma}} < \infty$).

However, the techniques from Prop. 2.14 apply whenever $\gamma < 1$ to show $T \in C(H_1, H_2)$.

Complements and projections

Let X be any vector space, with a subspace Y, finite or infinite dimensional. Algebraically, there is always a second subspace $Z \subset X$ (by no means unique) such that

$$X = Y + Z \quad \text{and} \quad \{0\} = Y \cap Z.$$

This is a consequence of the axiom of choice. However, to be able to do analysis, X need to be a B-space, and Y and Z closed subspaces. In this case, we shall see that existence of a closed complementary subspace Z is not always true.

Defn. 3.1:

(i) Let X be a B-space, and let $Y \subset X$ be a closed subspace. If there exists a second closed subspace $Z \subset X$ such that

$$X = Y + Z = \{y + z \,;\, y \in Y, z \in Z\} \quad \text{and}$$
$$Y \cap Z = \{0\},$$

then Z is called a (topological) complement of Y in X, and Y is called a complemented subspace. In this case we say that we have an (intrinsic) direct sum $X = Y \oplus Z$

(ii) If Y and Z are B-spaces, then their (extrinsic) direct sum $Y \hat{\oplus} Z$ is the B-space

$Y \hat{\oplus} Z := \{(y, z) \in Y \times Z \,;\, y \in Y, z \in Z\}$ with norm $\|(y,z)\| := \|y\| + \|z\|$ (or any other equivalent norm like $\max(\|y\|, \|z\|)$, $(\|y\|^2 + \|z\|^2)^{1/2}, \ldots$)

Lemma 3.2: Let X be a B-space with complementary closed subspaces $Y, Z \subset X$, so that $X = Y \oplus Z$.
Then
$$I : Y \hat{\oplus} Z \longrightarrow X : (y,z) \longmapsto y+z$$
is a topological isomorphism. In particular, there is $C < \infty$ such that
$$\forall y \in Y, z \in Z : \|y+z\|_X \leq \|y\|_X + \|z\|_X \leq C \cdot \|y+z\|_X .$$
Proof: Apply BIT. ∎

Defn 3.3: Let X be a B-space and $P \in L(X,X)$.
Then P is called a (bounded) projection if $P^2 = P$.

Lemma 3.3: Let P be a projection in a B-space X.
Then $Y := R(P)$ is a closed subspace and $\forall y \in Y : Py = y$. Conversely, any bounded extension $P \in L(X,Y)$ of the identity map $I \in L(Y,Y)$ is a projection with range Y.

Proof: The range is closed since $R(P) = N(I-P)$.
The latter is verified using $P^2 = P$.
Details are left as an exercise ∎

Complements and projections are related as follows.

Prop. 3.4: Let X be a B-space, and let $Y \subset X$ be a closed subspace. Then the following are equivalent:

(i) Y is a complemented subspace.

(ii) There exists a bounded projection P onto Y (i.e. with $R(P) = Y$)

(iii) Any $T \in L(Y,Y)$ can be extended to $\tilde{T} \in L(X,Y)$ such that $\tilde{T}|_Y = T$.

Proof: (i) $\Rightarrow$ (iii): Let $X = Y \oplus Z$, with Z being some topological complement of Y.
Then $\forall x \in X \; \exists!\, y \in Y, z \in Z : x = y + z$.

Given $T \in L(Y,Y)$, define

$$\tilde{T}x = \tilde{T}(y+z) := Ty.$$

Since $\|\tilde{T}x\| = \|Ty\| \leq c_1\|y\| \leq c_1(\|y\|+\|z\|) \leq c_1 C\|y+z\| = c_1 C\|x\|$

by lemma 3.2, $\tilde{T} \in L(X,Y)$.

(iii) $\Rightarrow$ (ii): Apply (iii) with $T=I$, and use lemma 3.3.

(ii) $\Rightarrow$ (i): Given P = projection onto Y, let

$Z := R(I-P) = N(P)$. Note that Z is closed.

$Y+Z = X$ since $x = \underbrace{(Px)}_{=:y} + \underbrace{(I-P)x}_{=:z}$

$Y \cap Z = \{0\}$ since $\left.\begin{array}{l} x\in Y \Leftrightarrow (I-P)x=0 \\ x\in Z \Leftrightarrow Px=0 \end{array}\right\} \Rightarrow x=0.$ ∎

Note the picture behind (i) $\Leftrightarrow$ (ii)

$$\begin{array}{ccccc} & & P & & I-P \\ & & \downarrow & & \downarrow \\ X & = & Y & \oplus & Z \end{array}$$

$P_1 := P$ and $P_2 := I-P$ are complementary projections:

$P_1+P_2 = I$, $P_1^2 = P_1$, $P_2^2 = P_2$, $P_1P_2 = 0 = P_2P_1$

Note also for (iii) that HB does not apply to general operators.

There is a dichotomy between H-spaces and B-space concerning complemented subspaces:

(H) In a Hilbert space H, any closed subspace $H_1 \subset H$ is complemented. Clearly 3.4(ii) holds, since we can take P = orthogonal projection onto H_1.

(B) Lindenstrauss and Tzafriri proved 1971 that in any B-space which is not isomorphic to a Hilbert space, there exists a closed subspace which is not complemented.

A concrete example: Phillips proved 1940 that

c_0 is not complemented in l_∞.

$\left(l_\infty = \{ (x_j)_{j=1}^\infty ; \sup_j |x_j| < \infty \} , c_0 = \{ (x_j)_{j=1}^\infty ; \lim_{j\to\infty} x_j = 0 \} \right)$

Note that it is a non-trivial task to prove that no bounded projection onto a given subspace exists!

In B-spaces, the only positive general result is the following:

<u>Prop. 3.5</u>: Let X be a B-space, and let $Y \subset X$ be a subspace.

(i) If $\dim Y < \infty$, then Y is closed and complemented.

(ii) If $\dim(X/Y) < \infty$ (where X/Y denotes the algebraic quotient vector space), then Y need not be closed. However, if it is closed then it is complemented.

<u>Proof</u>: (i) Let $\{v_1, \dots, v_N\}$ be a basis for Y, and let $\{v_i^*\}$ be the dual basis, i.e. $\langle v_i^*, v_j \rangle = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$.

Use HB to extend $v_i^* \in Y^*$ to $f_i \in X^*$.

Define $Px := \sum_{i=1}^{n} f_i(x) v_i$.

P is seen to be a bounded projection onto Y.

(ii) Assume Y is closed and has finite codimension. Let $\{w_j + Y\}_{j=1}^M$ be a basis for X/Y. Take $Z := \operatorname{span}\{w_j\}$. Then it is seen that $X = Y \oplus Z$.

- An example of a non-closed $Y \subset X$ with $\dim X/Y = 1$: By the axiom of choice, there is an unbounded linear functional $f: X \to \mathbb{R}$ defined on all X, for any infinite dimensional B-space. Let $Y := N(f)$

Take $v \in X$ such that $f(v) = 1$, and let $Z := \operatorname{span}\{v\}$. Then $Y + Z = X$ and $Y \cap Z = \{0\}$

If Y were closed, then

$$|\lambda| + \|y\| \leq C \|\lambda v + y\| \;, \quad \forall \lambda \in \mathbb{R}, \; y \in Y$$

by lemma 3.2.

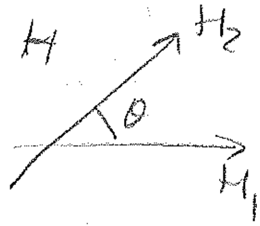
$\Rightarrow |f(\lambda v + y)| = |\lambda| \leq C \|\lambda v + y\|$ contradicting the unboundedness of f. Hence Y is not closed. ∎

Problem 3.6: Let $X_1, X_2 \subset X$ be closed subspaces of B-space X such that $X_1 \cap X_2 = \{0\}$.
Show that $X_1 + X_2$ is closed iff there is $C < \infty$ such that $\forall x_1 \in X_1, x_2 \in X_2 : \|x_1\| + \|x_2\| \leq C \|x_1 + x_2\|$.
Give example of X_i where this fails.

Problem 3.7 Let $H_1, H_2 \subset H$ be closed subspaces of a H-space H, such that $H_1 \cap H_2 = \{0\}$
Show that $H_1 + H_2$ is closed iff there is $\theta \in (0, \frac{\pi}{2}]$ such that

$$\forall x_1 \in H_1, x_2 \in H_2 : |\langle x_1, x_2 \rangle| \leq \cos\theta \, \|x_1\| \cdot \|x_2\|$$

Note the characterization of $H = H_1 \oplus H_2$ from problem 3.7: this means that H_1 and H_2 span H and are transverse with minimum angle θ.



Fredholm operators

Recall that for any $T \in L(X,Y)$, the nullspace $N(T) \subset X$ is always closed, but not necessarily the range $R(T) \subset Y$. We always have a bijective and bounded operator

$$\tilde{T} : X/N(T) \longrightarrow R(T).$$

If $R(T)$ is closed, then by BIT
$\tilde{T}^{-1} : R(T) \to X/N(T)$ is bounded.
If $\dim N(T) < \infty$, then by Prop. 3.5(i) there is a closed subspace $X_1 \subset X$ such that $X = N(T) \oplus X_1$.
In this case, we may replace $X/N(T)$ by X_1 above, and obtain the same result for $\tilde{T} : X_1 \to R(T)$.
Prop. 3.5(ii) has the following twin:

<u>Prop 3.8</u>: Let X and Y be B-spaces and $T \in L(X,Y)$. If $\dim(Y/R(T)) < \infty$, then $R(T)$ is a closed and complemented subspace in Y.

Note that this does not contradict Prop. 3.5(ii); it just shows that no non-closed Y appearing in Prop. 3.5(ii) can be the range of a bounded linear operator from some other B-space!

<u>Proof</u>: By Prop 3.5(ii) it suffices to show that $R(T)$ is closed. As before, let $\{y_j + R(T)\}_{j=1}^N$ be a basis for $Y/R(T)$ and define $Z := \operatorname{span}\{y_j\} \subset Y$.
Consider the bounded linear map

$$X/N(T) \hat{\oplus} Z \longrightarrow Y : (x + N(T), z) \longmapsto Tx + z$$

This is seen to be bijective, and BIT shows that there is $C<\infty$ such that

$$\|x+N(T)\|_{X/N(T)} + \|z\|_Z \leq C\|Tx+z\|_Y$$

Letting $z=0$, this shows in particular that $\tilde{T}: X/N(T) \to R(T)$ has a bounded inverse. Thus $R(T)$ is closed. ∎

Note that the proof above generalizes as follows: If $R(T)$ has a closed algebraic complement Z, then $R(T)$ is closed and Z is a topological complement.

Defn 3.9: Let X, Y be B-spaces and let $T \in L(X,Y)$. Define $\alpha(T) := \dim N(T)$, $\beta(T) := \dim(Y/R(T))$

(i) T is a Fredholm operator if $R(T)$ is closed, $\alpha(T)<\infty$ and $\beta(T)<\infty$.

(ii) T is a semi-Fredholm operator if $R(T)$ is closed and $\alpha(T)$ or $\beta(T)$ is finite

(iii) T is an upper semi-Fredholm operator if $R(T)$ is closed and $\alpha(T)<\infty$

(iv) T is a lower semi-Fredholm operator if $R(T)$ is closed and $\beta(T)<\infty$.

The classes (i)-(iv) are denoted $F(X,Y)$, $SF(X,Y)$, $SF_+(X,Y)$ and $SF_-(X,Y)$ respectively.

Note that $F(X,Y) = SF_+(X,Y) \cap SF_-(X,Y)$

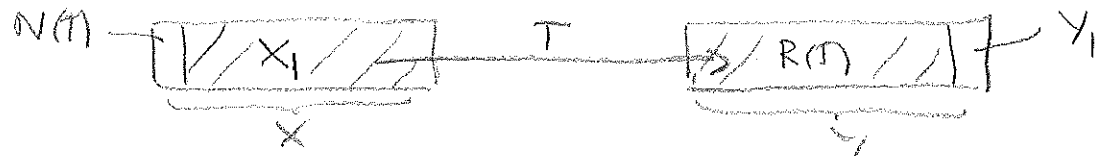
$SF(X,Y) = SF_+(X,Y) \cup SF_-(X,Y)$.

Note that by Prop. 3.8, the requirement $R(T)$ closed is redundant for $T \in SF_-(X,Y)$.
In practise however, $\beta(T)<\infty$ is typically deduced after $R(T)$ closed has been established.

A Fredholm operator is "an isomorphism modulo finite dimensional subspaces":

By Prop. 3.5(i), 3.8 we have

$X = N(T) \oplus X_1$ and $Y = R(T) \oplus Y_1$, where $X_1 \subset X$, $Y_1 \subset Y$ are closed subspaces and $\dim N(T) < \infty$ and $\dim Y_1 < \infty$. Moreover, $\tilde{T}: X_1 \to R(T)$ is a topological isomorphism.



For semi-Fredholm operators in B-spaces, this picture may not be possible since $N(T)$ or $R(T)$ may not be complemented. In H-spaces there is of course no problems.

Main problem for part I, second version

Let $T \in L(X,Y)$ and $y \in Y$. We want to solve the equation

$$Tx = y \quad \text{for } x \in X.$$

We ask for

(1) Fredholm existence, i.e. $Y/R(T)$ is finite dimensional ($\beta(T) < \infty$)

(2) Fredholm uniqueness, i.e. $N(T)$ is finite dimensional ($\alpha(T) < \infty$)

(3) Fredholm stability, i.e. $\tilde{T}^{-1}$ bounded ($R(T)$ closed)

If we have this weaker type of well-posedness, "Fredholm well-posedness", then $Tx = y$ is solvable for "almost all" y, i.e. $y \in R(T)$, and the solution x is "almost unique", i.e. x is unique modulo $N(T)$.

Then standard procedure to prove $T \in F(X,Y)$ is as follows:

(A) First prove Fredholm uniqueness and stability, i.e. $T \in SF_+(X,Y)$.

(B) Then prove Fredholm existence, i.e. $\beta(T) < \infty$, using one of the following two standard methods: (B_1) Duality

(B_2) Method of continuity.

We shall return to (B) later, and consider first (A), which is where in application the "hard work lies". The main tool for proving (A) is the following.

<u>Prop. 3.10</u>: Let X, Y be B-spaces and let $T \in L(X,Y)$. Then $T \in SF_+(X,Y)$ iff there is a compact operator $K \in C(X,Z)$, Z being some auxiliary B-space, and $C < \infty$ such that

$$\forall x \in X: \ \|x\|_X \leq C \cdot \|Tx\|_Y + \|Kx\|_Z .$$

<u>Proof</u>: ($\Rightarrow$) Assume $T \in SF_+(X,Y)$ so that $\alpha(T) < \infty$.

Then $X = N(T) \oplus X_1$ by Prop. 3.5(i).

! Let $K :=$ projection onto $N(T)$ along X_1. (see Prop. 3.4)

$\Rightarrow K \in FR(X,X) \subset C(X,X)$

Since $R(T)$ is closed, $T: X_1 \to R(T)$ is an isomorphism, so $\|x\| \leq C\|Tx\|$, $\forall x \in X_1$, for some $C < \infty$

$\therefore \forall x \in X: \|x\|_X \leq \|(1-K)x\|_X + \|Kx\|_X$

$$\leq C \|\underbrace{T(1-K)}_{=T \text{ since } R(K) \subset N(T)} x\|_Y + \|Kx\|_X$$

($\Leftarrow$) Assume $\|x\|_X \le C(\|Tx\|_Y + \|Kx\|_Z$, $\forall x \in X$.

- In particular $\|x\|_X \le \|Kx\|_Z$, $\forall x \in N(T)$.

  Thus $K: N(T) \to R(K|_{N(T)})$ is an isomorphism.

  Since K is compact, $N(T)$ must be finite dimensional by Ex 2.1.

- It remains to show that $R(T)$ is closed.

Take $y_n = Tx_n \in R(T)$ such that $y_n \to y \in Y$.

We may assume $x_n \in X_1$ = complement of $N(T)$

Need to show $y \in R(T)$

<u>Assume</u> first that $\|x_n\| \to \infty$, $n \to \infty$.

Let $\hat{x}_n := \frac{x_n}{\|x_n\|}$, so that $T\hat{x}_n = \frac{y_n}{\|x_n\|} \to \frac{y}{\infty} = 0$.

Since K is compact there is a subsequence $\{\hat{x}_{n_k}\}_{k=1}^\infty$ such that $\{K\hat{x}_{n_k}\}_{k=1}^\infty$ converges in Z.

$$\Rightarrow \|\hat{x}_{n_k} - \hat{x}_{n_\ell}\|_X \le \|T\hat{x}_{n_k} - T\hat{x}_{n_\ell}\|_Y + \|K\hat{x}_{n_k} - K\hat{x}_{n_\ell}\|_Z$$

$$\to 0, \quad k, \ell \to \infty.$$

Thus $\{\hat{x}_{n_k}\}_{k=1}^\infty$ is a Cauchy sequence, hence converges : $\hat{x}_{n_k} \to \hat{x}$, $k \to \infty$.

We have $T\hat{x} = \lim_{k\to\infty} T\hat{x}_{n_k} = 0$

But $x_n \in X_1$ = closed , so $\hat{x} \in X_1$. Thus $\hat{x} = 0$.

On the other hand

$\|\hat{x}\| = \lim_{k\to\infty} \|\hat{x}_{n_k}\| = 1$.

$\therefore$ Assumption leads to contradiction, so $\{x_n\}_{n=1}^\infty$ must contain a bounded subsequence.

Without loss of generality, we may assume that $\{x_n\}_{n=1}^\infty$ is bounded.

Repeating the argument above, we find a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $\{Kx_{n_k}\}_{k=1}^\infty$ converges, and from the assumed estimate, we deduce that $x_{n_k} \to x$, $k \to \infty$.

$\therefore$ $y = \lim_{k\to\infty} y_{n_k} = T(\lim_{k\to\infty} x_{n_k}) = Tx \in R(T)$.

This completes the proof. ∎

Central to Fredholm theory is perturbation theory. (Compare (B_2) above.) Here is our first perturbation result.

<u>Corollary 3.11</u>: Let X, Y be B-spaces.
If $T \in SF_+(X,Y)$ and $K \in C(X,Y)$, then

$$T+K \in SF_+(X,Y).$$

(That is, the class of upper semi-Fredholm operators is stable under compact perturbations.)

<u>Proof</u>: We use Prop. 3.10 twice:

$T \in SF_+(X,Y) \Rightarrow$

$\forall x \in X: \|x\|_X \le C\|Tx\|_Y + \|\tilde{K}x\|_Z$

$$\Rightarrow \forall x \in X: \|x\|_X \le C\|(T+K)x\|_Y + \underbrace{(C\|Kx\|_Y + \|\tilde{K}x\|_Z)}_{\|\hat{K}x\|_{Y\hat{\oplus}Z}}$$

where $\hat{K}: X \to Y\hat{\oplus}Z : x \mapsto (CKx, \tilde{K}x)$ is seen to be a compact operator.

Thus $T+K \in SF_+(X,Y)$ ∎

<u>Ex 3.12</u>: Consider the double layer potential operator K from Defn. 1.8 on a compact and $C^{1,\alpha}$-regular boundary ∂D.
We saw that solving the Dirichlet problem amounts to solving the integral equation

$$(I+K)h = 2\varphi$$

in some appropriate B-space X of functions
$h, \varphi : \partial D \to \mathbb{R}$.

Problem 2.8 : K is compact in $X = C(\partial D)$

Example 2.15 : K is compact in $X = L_2(\partial D)$.

Since I is an isomorphism, hence $I \in F(X,X) \subset SF_+(X,Y)$,
Cor. 3.11 shows that we have Fredholm uniqueness and stability for the equation

$$(I + K)h = 2\varphi.$$

Duality for Fredholm operators

The aim of this lecture is to develop the duality method (B₁) for proving Fredholm existence for a given operator. We start with the simpler case of H-spaces. Recall that, given a closed subspace $H_1 \subset H$ of a H-space H, we have the orthogonal direct sum $H = H_1 \oplus H_1^\perp$, where $H_1^\perp := \{x \in H ; \langle x, y\rangle = 0 \text{ for all } y \in H_1\}$ is the orthogonal complement of H_1. In particular

$$(H_1^\perp)^\perp = H_1$$

More generally, for any (possible non-closed) subspace $V \subset H$, we have $(V^\perp)^\perp = \overline{V}$.

<u>Thm 4.1</u> (Closed range thm for H-spaces)

Let H_1, H_2 be H-spaces, and let $T \in L(H_1, H_2)$. Consider the adjoint operator $T^* \in L(H_2, H_1)$.

Then
$$\left.\begin{aligned} N(T^*) &= R(T)^\perp \\ \overline{R(T)} &= N(T^*)^\perp \end{aligned}\right] \text{ in } H_2 = N(T^*) \oplus \overline{R(T)}$$

$$\left.\begin{aligned} N(T) &= R(T^*)^\perp \\ \overline{R(T^*)} &= N(T)^\perp \end{aligned}\right] \text{ in } H_1 = N(T) \oplus \overline{R(T^*)}$$

Moreover, $R(T)$ is closed iff $R(T^*)$ is closed.

<u>Proof</u>: Since $(T^*)^* = T$, it suffices to prove the splitting of H_2, i.e. that $N(T^*) = R(T)^\perp$.

We have, for any $x \in H_1$, $y \in H_2$, that

$$\langle x, T^* y\rangle = \langle Tx, y\rangle.$$

LHS $= 0$ for all $x \iff y \in N(T^*)$

RHS $= 0$ for all $x \iff y \in R(T)^\perp$

$\therefore N(T^*) = R(T)^\perp$.

- Again, since $T^{**} = T$, it remains to prove $R(T)$ closed $\Rightarrow R(T^*)$ closed.

Consider the restricted operators

$$\begin{array}{ccc} \overline{R(T^*)} & \xleftarrow{\;\tilde{T^*}\;} & N(T^*)^{\perp} \\ \| & & \| \\ N(T)^{\perp} & \xrightarrow{\;\tilde{T}\;} & R(T) \end{array}$$

It is seen that $\tilde{T^*}$ is the adjoint $(\tilde{T})^*$ of $\tilde{T}$, and that $R(\tilde{T}) = R(T)$, $R(\tilde{T^*}) = R(T^*)$.

We get

$R(T)$ closed $\overset{BIT}{\Longrightarrow}$ $\tilde{T}$ isomorphism $\Rightarrow (\tilde{T})^* = \tilde{T^*}$ isomorphism

$\Rightarrow R(T^*)$ closed. ∎

Cor 4.2: Let H_1, H_2 be H-spaces and $T \in L(H_1, H_2)$. Then $T \in SF(H_1, H_2)$ iff $T^* \in SF(H_2, H_1)$. In this case, $\alpha(T) = \beta(T^*)$ and $\beta(T) = \alpha(T^*)$.

Proof: $R(T)$ is closed $\Leftrightarrow$ $R(T^*)$ is closed by thm 4.1. Then

$N(T) = R(T^*)^{\perp}$ and $R(T)^{\perp} = N(T^*)$.

Checking the dimensions completes the proof. ∎

Ex 4.3: We continue Ex. 3.12 and consider the double layer potential operator K on a compact and $C^{1,\alpha}$-regular boundary ∂D.

For the integral equation

$$(I+K)h = 2\varphi,$$

we saw that $I+K \in SF_+(L_2(\partial D), L_2(\partial D))$.

Now consider $(I+K)^* = I+K^*$, where

$$K^* h(x) = \frac{2}{\sigma_{n-1}} \text{p.v.} \int_{\partial D} \nu(x) \cdot \frac{x-y}{|x-y|^n} h(y)\, d\sigma(y).$$

Using Ex 1.12, we see that Prop. 2.14 applies to K^* and shows that K^* is compact on $L_2(\partial D)$.

Thus $I+K^* \in SF_+(L_2(\partial D), L_2(\partial D))$ by Cor. 3.11.

In particular

$$\beta(I+K) = \alpha(I+K^*) < \infty,$$

so we have Fredholm existence for $(I+K)h = 2\varphi$, and $I+K \in F(L_2(\partial D), L_2(\partial D))$.
This illustrates the duality method (B$_1$):

$$T \in SF_-(H_1, H_2) \iff T^* \in SF_+(H_2, H_1),$$

where Prop. 3.10 can be applied to show $T^* \in SF_+(H_2, H_1)$.

Banach spaces:

We next want to generalize Thm 4.1 to general B-space. Here the situation is much more difficult when the B-space X is not reflexive. Consider a B-space X along with its dual space X^* from Defn. 1.4. We write

$$M^\perp := \{ f \in X^* ; \langle f, x \rangle = 0, \forall x \in M \} \text{ when } M \subset X,$$
$$^\perp N := \{ x \in X ; \langle f, x \rangle = 0, \forall f \in N \} \text{ when } N \subset X^*.$$

A simple first result is

Lem. 4.4: Let X, Y be B-spaces and $T \in L(X, Y)$. Consider the adjoint operator $T^* \in L(Y^*, X^*)$. Then

$$N(T^*) = R(T)^\perp \subset Y^* \quad (1)$$
$$\overline{R(T)} = {}^\perp N(T^*) \subset Y \quad (2)$$
$$N(T) = {}^\perp R(T^*) \subset X \quad (3)$$
$$\overline{R(T^*)} \subset N(T)^\perp \subset X^* \quad (4)$$

Proof: (1) and (3) are proved as in the proof of Thm 4.1. To prove (4), take $f = T^* g \in R(T^*)$. Then $\forall x \in N(T)$: $\langle f, x \rangle = \langle T^* g, x \rangle = \langle g, \underbrace{Tx}_{=0} \rangle = 0$.
Similarly $\overline{R(T)} \subset {}^\perp N(T^*)$ is proved for (2).
We now prove ${}^\perp N(T^*) \subset \overline{R(T)}$.
To this end, assume $y_0 \notin \overline{R(T)}$, and let $Z := \operatorname{span}\{y_0\} \subset Y$.

Then $Z + \overline{R(T)} \subset Y$ is closed. Indeed, by Prop. 3.10 the map

$$Z \hat{\oplus} \overline{R(T)} \to Y : (\lambda y_0, y) \mapsto \lambda y_0 + y$$

has closed range $(= Z + \overline{R(T)})$ since

$$\|(\lambda y_0, y)\| \leq \|y\| + \|\lambda y_0\|.$$

Thus the functional

$g(\lambda y_0 + y) := \lambda$ is bounded on $Z \oplus \overline{R(T)}$ (c.f. Probl. 3,6)

and can be extended to $g \in Y^*$ by HB.

Now $\langle g, y_0 \rangle = 1$ and

$\forall x \in X : \langle T^* g, x \rangle = \langle g, Tx \rangle = 0.$

This shows that $g \in N(T^*)$ and $y_0 \notin {}^\perp N(T^*)$, which completes the proof. ∎

<u>Ex 4.5</u>: In lemma 4.4, we do <u>not</u> in general have $\overline{R(T^*)} = N(T)^\perp$. Here is an example:

$$X = Y = \ell_1 = \{(x_j)_{j=1}^\infty ; \sum_{j=1}^\infty |x_j| < \infty\}.$$

$$\Rightarrow X^* = Y^* = \ell_\infty = \{(x_j)_{j=1}^\infty ; \sup_{j \in \mathbb{Z}^+} |x_j| < \infty\}$$

$$T : (x_j) \mapsto (\tfrac{1}{j} x_j)$$

$$\Rightarrow T^* : (y_j) \mapsto (\tfrac{1}{j} y_j)$$

Clearly $N(T) = \{0\}$, so $N(T)^\perp = \ell_\infty$, but

$$R(T^*) \subset c_0 = \{(x_j)_{j=1}^\infty ; \lim_{j \to \infty} x_j = 0\} \subsetneq \ell_\infty.$$

For $T \in SF(X,Y)$, $\alpha(T)$ and $\beta(T)$ quantify $N(T)$ and $Y/R(T)$. The following (non-integer) quantifies the closedness of $R(T)$.

<u>Defn 4.6</u>: Let X, Y be B-spaces and $T \in L(X,Y)$. Then the <u>reduced minimum modulus</u> of T is

$$\gamma(T) := \inf_{x \notin N(T)} \frac{\|Tx\|}{\operatorname{dist}(x, N(T))}.$$

We note that

$$\frac{1}{\gamma(T)} = \|\tilde{T}^{-1}\| \quad , \text{where } \tilde{T}: X/N(T) \to R(T).$$

In particular

$\gamma(T) > 0$ iff $R(T)$ is closed.

The main theorem of this lecture is the following.

Thm 4.7: (Closed range thm for B-spaces)

Let X, Y be B-spaces and let $T \in L(X,Y)$.
Consider the adjoint operator $T^* \in L(Y^*, X^*)$.
Then $\gamma(T) = \gamma(T^*)$. In particular
$R(T)$ is closed iff $R(T^*)$ is closed.
In this case $R(T^*) = N(T)^\perp$.

Note that when $R(T)$ is not closed, we may not even have $\overline{R(T^*)} = N(T)^\perp$. Ex 4.5 illustrates this (where $\gamma(T) = 0$).

Problem 4.8: Let $Z_1 \subset Z$ be a closed subspace of a B-space Z. Prove that the following maps are well defined isometric isomorphisms.

(i) $U: (Z/Z_1)^* \to Z_1^\perp : f \mapsto f \circ \pi$,
where $\pi: Z \to Z/Z_1 : z \mapsto z + Z_1$ is the quotient map.

(ii) $V: Z_1^* \to Z^*/Z_1^\perp : g \mapsto \tilde{g} + Z_1^\perp$,
where $\tilde{g} \in Z^*$ is a HB-extension of $g \in Z_1^*$.

Proof of thm 4.7:

(i) Consider the diagram

$$N(T)^\perp \xleftarrow{U} (X/N(T))^* \xleftarrow{(\tilde{T})^*} (\overline{R(T)})^* \xrightarrow{V} Y^*/\overline{R(T)}^\perp$$

$$X/N(T) \xrightarrow{\tilde{T}} \overline{R(T)}$$

where U and V are the isometries from Problem 4.8.

By Lem. 4.4, $\overline{R(T)}^\perp = R(T)^\perp = N(T^*)$
and $\overline{R(T^*)} \subset N(T)^\perp$.

Consider the restricted map
$\widetilde{T^*} : Y^*/N(T^*) \longrightarrow \overline{R(T^*)}$.

We claim that $\widetilde{T^*} = U(\tilde{T})^* V^{-1}$

Indeed,

$\forall g \in Y^*, x \in X : \langle U(\tilde{T})^* V^{-1}(g + \overline{R(T)}^\perp), x \rangle$

$= \langle (\tilde{T})^* V^{-1}(g + \overline{R(T)}^\perp), x + N(T) \rangle$

$= \langle V^{-1}(g + \overline{R(T)}^\perp), Tx \rangle = \langle g|_{\overline{R(T)}}, Tx \rangle = \langle g, Tx \rangle$

$= \langle T^* g, x \rangle = \langle \widetilde{T^*}(g + N(T^*)), x \rangle.$

Assume that the theorem has been proved for $\tilde{T} \in L(X/N(T), \overline{R(T)})$. Then

$$\gamma(T^*) = \gamma(\widetilde{T^*}) \underset{\substack{\uparrow \\ U,V \text{ isometries}}}{=} \gamma(\tilde{T}^*) = \gamma(\tilde{T}) = \gamma(T)$$

and $R(T^*) = R(\widetilde{T^*}) = U(R(\tilde{T}^*)) = U(N(\tilde{T})^\perp)$

$= U((X/N(T))^*) = N(T)^\perp$.

Thus, replacing T by $\tilde{T}$, we may assume that $N(T) = \{0\}$ and $R(T)$ is dense in Y.

By Lem. 4.4 $N(T^*) = \{0\}$.

However, note that we do not know at this stage that $R(T^*)$ is dense in X^*.

ii) Assume first $R(T)$ is closed, so that $\gamma(T) > 0$.

Then $R(T) = Y$ and $T^{-1} \in L(Y, X)$ exists.

Taking adjoints T^* is seen to be invertible with $(T^*)^{-1} = (T^{-1})^*$. In particular $R(T^*) = X^*$.

The proof of $\gamma(T) = \gamma(T^*)$ is similar to the standard one for $\|T\| = \|T^*\|$:

$$\forall g \in Y^*: \|g\|_{Y^*} = \sup_{x \neq 0} \frac{\langle g, Tx\rangle}{\underbrace{\|Tx\|}_{\geq \gamma(T)\|x\|}} \leq \frac{1}{\gamma(T)} \sup_{x\neq 0} \frac{\langle T^* g, x\rangle}{\|x\|}$$

$$\leq \frac{1}{\gamma(T)} \|T^* g\| \quad \Rightarrow \quad \gamma(T^*) \geq \gamma(T).$$

$$\forall x \in X: \|x\|_X = \sup_{g \neq 0} \frac{\langle T^* g, x\rangle}{\underbrace{\|T^* g\|}_{\geq \gamma(T^*)\|g\|}} \leq \frac{1}{\gamma(T^*)} \sup_{g \neq 0} \frac{\langle g, Tx\rangle}{\|g\|}$$

$$\leq \frac{1}{\gamma(T^*)} \|Tx\| \quad \Rightarrow \quad \gamma(T^*) \leq \gamma(T).$$

(iii) Assume now instead that $R(T^*)$ is closed, so that $\gamma := \gamma(T^*) > 0$

We need to prove $R(T)$ closed, i.e. T is an isomorphism. (We still assume $N(T) = \{0\}$, $\overline{R(T)} = Y$).

This will follow from

$T(B_X(0,1)) \supset B_Y(0,\gamma)$, which clearly shows that T is surjective.

The proof follows that of BIT, and is divided into:

Step 1: Show $B_Y(0,\gamma) \subset \overline{T(B_X(0,1))}$

Step 2: Improve the conclusion of step 1 to

$$B_Y(0,\gamma) \subset T(B_X(0,1))$$

As compared to BIT (where step 1 uses Baire's category theorem), we here prove step 1 using HB. Before this, let us recall the proof of step 2 from the proof of BIT.

Proof of Step 2: Assume $y \in B_Y(0,\gamma) \subset \overline{T(B_X(0,1))}$.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots > 0$.

By scaling, we have $B_Y(0, \varepsilon_k) \subset \overline{T(B_X(0, \frac{\varepsilon_k}{\gamma}))}$.

Using this inclusion, choose recursively vectors $x_1, x_2, x_3, \ldots \in X$:

x_1: $\|y - Tx_1\| < \varepsilon_1$, $\|x_1\| < 1$

x_2: $\|(y - Tx_1) - Tx_2\| < \varepsilon_2$, $\|x_2\| < \frac{\varepsilon_1}{\gamma}$

x_3 : $\|(y - Tx_1 - Tx_2) - Tx_3\| < \varepsilon_3$, $\|x_3\| < \frac{\varepsilon_2}{\gamma}$

....

We obtain $\sum_1^N \|x_h\| < \|x_1\| + \frac{1}{\gamma}\sum_1^{N-1} \varepsilon_h$

Choose $\{\varepsilon_h\}$ small enough so that $\sum_1^\infty \|x_h\| < 1$.

Then $\sum_1^N x_h \to x \in B_X(0,1)$ and

$\|y - T(\sum_1^N x_h)\| < \varepsilon_N \to 0$, so $y = Tx \in \overline{T(B_X(0,1))}$.

<u>Proof of step 1</u>: Let $y_0 \notin \overline{T(B_X)}$ and define

$\delta := \mathrm{dist}(y_0, \overline{T(B_X)})$ and

$\tilde{B}_\delta := \{ y \in Y ; \mathrm{dist}(y, \overline{T(B_X)}) < \delta \}$.

We see that:

- $\tilde{B}_\delta$ is convex and $-\tilde{B}_\delta = \tilde{B}_\delta$.
- $B_Y(0,\delta) \subset \tilde{B}_\delta \subset B_Y(0, \|T\|+\delta)$

Thus lemma 4.9. below applies and give us

$g \in Y^*$ such that

$\forall y \in \tilde{B}_\delta : \langle g, y\rangle < \langle g, y_0\rangle$

Since $\tilde{B}_\delta \supset T(B_X)$, we obtain

$$\|T^* g\| = \sup_{\|x\|=1} \underbrace{\langle T^* g, x\rangle}_{=\langle g, Tx\rangle} \leq \langle g, y_0\rangle \leq \|g\| \cdot \|y_0\|$$

By hypothesis

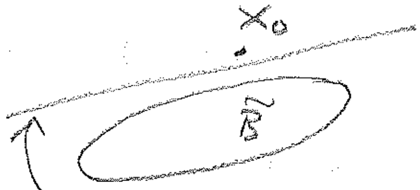
$\|T^* g\| \geq \gamma \|g\|$.

Since $g \neq 0$, we obtain $\|y_0\| \geq \gamma$, so $y_0 \notin B_Y(0,\gamma)$.

This completes the proof, modulo lemma 4.9. ∎

Lem. 4.9: Let X be a B-space, and let $\tilde{B} \subset X$ be a convex open set such that $-\tilde{B} = \tilde{B}$ and $B_X(0,a) \subset \tilde{B} \subset B_X(0,b)$ for some $0 < a < b < \infty$. Then, if $x_0 \notin \tilde{B}$, then there exists $f \in X^*$ such that

$$\forall x \in \tilde{B} : \langle f, x \rangle < \langle f, x_0 \rangle.$$



Proof: Define

$$\|x\|^{\sim} := \inf \{ \lambda > 0 ; \tfrac{x}{\lambda} \in \tilde{B} \} , \quad x \in X.$$

One proves that

- $\|x+y\|^{\sim} \le \|x\|^{\sim} + \|y\|^{\sim}$ (use $\tilde{B}$ convex)
- $\|\lambda x\|^{\sim} = |\lambda| \cdot \|x\|^{\sim}$ (use $-\tilde{B} = \tilde{B}$)
- $\frac{1}{b}\|x\| \le \|x\|^{\sim} \le \frac{1}{a}\|x\|$ (use $B_X(0,a) \subset \tilde{B} \subset B_X(0,b)$)

Thus $\|\cdot\|^{\sim}$ is a second norm on X, equivalent to $\|\cdot\|$, and $\tilde{B}$ is the unit ball in $\tilde{X} := (X, \|\cdot\|^{\sim})$, so that $\|x_0\|^{\sim} \ge 1$.

Apply HB to $\tilde{X}$: there is $f \in \tilde{X}^*$ such that $\|f\|_{\tilde{X}^*} = 1$ and $\langle f, x_0 \rangle = \|x_0\|^{\sim}$

$\Rightarrow \forall x \in \tilde{B} : \langle f, x \rangle \le 1 \cdot \|x\|^{\sim} < 1 \le \|x_0\|^{\sim} = \langle f, x_0 \rangle$

Identifying $\tilde{X} = X$ and $\tilde{X}^* = X^*$ as topological vector spaces, now gives the conclusion ∎

From theorem 4.7. we obtain the following extension of Cor. 4.2:

Cor. 4.10: Let X, Y be B-spaces and $T \in L(X,Y)$. Then $T \in SF(X,Y)$ iff $T^* \in SF(Y^*, X^*)$.

In this case
$$\alpha(T) = \beta(T^*)$$
$$\beta(T) = \alpha(T^*).$$

Proof: By thm 4.7 $R(T^*)$ is closed iff $R(T)$ is closed. In this case $N(T^*) = R(T)^\perp$ (Lem. 4.4)

$$N(T)^\perp = R(T^*) \text{ .}$$

As in Problem 4.8

$$N(T^*) = R(T)^\perp \approx (Y/R(T))^*$$

$$X^*/R(T^*) = X^*/N(T)^\perp \approx N(T)^*$$

This proves the corollary, since a B-space and it's dual have the same dimension. ■

Similar to Ex. 4.3 one can obtain Fredholm existence for

$$(I+K)h = 2\varphi$$

in the B-space $C(\partial D)$. More generally we have:

Cor. 4.10': Let X and Y be B-spaces and let $K \in C(X,Y)$ be compact. Then $I+K \in F(X,Y)$.

Proof: Cor. 3.11 shows $I+K \in SF_+(X,Y)$. Since $K^* \in C(Y^*,X^*)$ by Thm 2.9, it similarly follows that $I+K^* \in SF_+(Y^*,X^*)$. Thus $I+K \in F(X,Y)$ by Cor. 4.10. ■

Problem 4.11: Let X, Y be B-spaces and let $T \in L(X,Y)$. Show that $\alpha(T) \leq \beta(T^*)$ and $\alpha(T^*) \leq \beta(T)$. Give examples of T where the inequalities are strict.

We end this lecture by proving a version of the closed range theorem for pairs of subspaces.

Cor. 4.12: Let $X, Y \subset Z$ be closed subspaces in a B-space Z. Then $(X+Y)^\perp = X^\perp \cap Y^\perp$.

We have that

$X+Y$ is closed iff $X^\perp + Y^\perp$ is closed.

In this case $X^\perp + Y^\perp = (X \cap Y)^\perp$.

Proof: $(X+Y)^\perp = X^\perp \cap Y^\perp$ is straightforward to show.

Next, we apply thm 4.7 to

$T: Z \to Z/X \hat{\oplus} Z/Y \;:\; z \mapsto (z+X, z+Y)$.

Under the isomorphism U from problem 4.8, the adjoint operator is

$T^*: X^\perp \hat{\oplus} Y^\perp \to Z^* \;:\; (f;g) \mapsto f+g$,

and we get

$$X^\perp + Y^\perp = R(T^*) \overset{\text{thm 4.7}}{=} N(T)^\perp = (X \cap Y)^\perp.$$

It remains to show that $R(T)$ is closed iff $X+Y$ is closed.

($\Leftarrow$) Assume $X+Y$ is closed, and let

$Tz_n = (z_n + X, z_n + Y) \to (z^x + X, z^y + Y)$.

Thus $\exists x_n \in X, y_n \in Y$: $\begin{array}{l} z_n - x_n \to z^x \\ z_n - y_n \to z^y \end{array}$ in Z.

$\Rightarrow x_n - y_n \to z^y - z^x$.

Since $X+Y$ closed, $z^y - z^x \in X+Y$, and thus

$z^y + y = z^x + x$ for some $x \in X, y \in Y$.

$\therefore T(z^x + x) = T(z^y + y) = (z^x + X, z^y + Y)$.

($\Rightarrow$) Assume $R(T)$ closed, and let

$x_n + y_n \to z$ in Z, with $x_n \in X, y_n \in Y$.

Since $T(x_n - x_m) = (0, x_n - x_m + Y)$ and $\gamma(T) > 0$, we get

$$\begin{aligned} \operatorname{dist}(x_n - x_m, X \cap Y) &\le \gamma(T)^{-1} \| x_n - x_m + Y \| \\ &\le \gamma(T)^{-1} \| (x_n + y_n) - (x_m + y_m) \| \\ &\to 0, \quad n, m \to \infty \end{aligned}$$

Thus $\{x_n + X \cap Y\}$ converges in $Z / X \cap Y$

So there is $\{w_n^x\}_{n=1}^{\infty} \subset X \cap Y$ such that

$x_n - w_n^x \to z^x$, and $z^x \in X$ follows since X is closed.

Similarly $y_n - w_n^y \to z^y \in Y$ for some $w_n^y \in X \cap Y$.

$$\Rightarrow \underbrace{x_n + y_n}_{\to z} - (\underbrace{w_n^x + w_n^y}_{=: w_n}) \to z^x + z^y ,$$

so $w_n \to w$ and $w \in X \cap Y$ since $X \cap Y$ is closed.

$\therefore z = w + z^x + z^y \in X + Y$ ∎

Although we derived 4.12 from 4.7, cor. 4.12 can be thought of as a generalization of thm 4.7 in the sense that the result 4.7 for closed unbounded operators can be derived from 4.12. Indeed, if $T: X_1 \to X_2$ is an unbounded operator with domain $D(T) \subset X_1$, then one applies 4.12 with $Z := X_1 \oplus X_2$ = graph space,

$X := X_1 \oplus \{0\}$

$Y := G(T)$ = graph of T.

For more details, we refer to T. Kato "Perturbation theory for linear operators", IV, §5.2.

Fredholm inverses:

As commented earlier on, Fredholm operators are "isomorphisms modulo finite dimensional subspaces". The following result shows that Fredholm operators also are "invertible operators modulo compact operators". In view of Enflo's counterexample (see above 2.10), this might be a little surprising.

Thm 5.1: Let X, Y be B-spaces and $T \in L(X,Y)$.

(i) $T \in F(X,Y)$ iff there exists $S_1, S_2 \in L(Y,X)$ and $K_1 \in C(X,X)$, $K_2 \in C(Y,Y)$ such that

$$S_1 T = I - K_1 \quad \text{and} \quad T S_2 = I - K_2.$$

In this case, we can take $S_1 = S_2 =: S$ and K_1, K_2 to be finite rank projections.

(ii) $T \in SF_+(X,Y)$ and $R(T)$ is complemented iff $\exists S_1 \in L(Y,X)$, $K_1 \in C(X,X)$ such that

$$S_1 T = I - K_1$$

In this case, S_1 can be chosen so that K_1 is a finite rank projection.

(iii) $T \in SF_-(X,Y)$ and $N(T)$ is complemented iff $\exists S_2 \in L(Y,X)$, $K_2 \in C(Y,Y)$ such that

$$T S_2 = I - K_2.$$

In this case, S_2 can be chosen so that K_2 is a finite rank projection.

The proof of (ii) & (iii) uses the following generalization of Prop. 3.5.

Problem 5.2: Let $X_1 \subset X_2 \subset X$ be closed subspaces of a B-space X. Assume $\dim(X_2/X_1) < \infty$. Then X_1 is complemented in X iff X_2 is complemented in X.

We also recall from lecture 4 the following.

<u>Lemma 5.3</u>: Let X, Y be B-spaces and $T \in C(X,Y)$.
Then $I + K \in F(X,Y)$.

<u>Proof</u>: K and K^* are compact by thm 2.9.
$I+K$ and $I+K^*$ are ~~upper~~ semi-Fredholm by cor. 3.11.
$I+K$ and $I+K^*$ are Fredholm by cor. 4.10. ∎

<u>Proof of thm 5.1</u>:

(i) ($\Leftarrow$) We see that
$$\alpha(T) \le \alpha(1-K_1) < \infty$$
$$\beta(T) \le \beta(1-K_2) < \infty$$
Thus $T \in F(X,Y)$ by Prop. 3.8.

($\Rightarrow$) Use Prop. 3.5 to write
$$X = N(T) \oplus X_1 \quad , \quad Y = R(T) \oplus Y_1$$
Let $K_1 :=$ projection onto $N(T)$ along X_1.
$K_2 :=$ projection onto Y_1 along $R(T)$
$S := \tilde{T}^{-1}(1-K_2)$, where $\tilde{T} : X_1 \to R(T)$ is the restricted isomorphism.

Then K_1, K_2 are finite rank projections and
$$ST = \tilde{T}^{-1}(1-K_2)T = \tilde{T}^{-1}T = 1-K_1,$$
$$TS = T\tilde{T}^{-1}(1-K_2) = 1-K_2.$$

(ii) ($\Rightarrow$) as in (i), using the hypothesis to split Y.

($\Leftarrow$) If $S_1T = 1-K_1$, then
$$\forall x \in X : \|x\| = \|S_1Tx + K_1x\| \le \|S_1\| \cdot \|Tx\| + \|K_1x\|.$$
Thus $T \in SF_+(X,Y)$ by Prop. 3.10.
Next, we show that $R(T)$ is complemented.
$S_1T \in F(X,X)$ by lemma 5.3.
(i) $\Rightarrow$ $S(S_1T) = I - F$ for some $S \in L(X,X)$ and finite rank projection F.

Let $\tilde{X} := N(F)$

$\tilde{T} := T|_{\tilde{X}} : \tilde{X} \to Y$

$\tilde{S} := (1-F)SS_1 : Y \to \tilde{X}$

Then $X/\tilde{X}$ and $R(T)/R(\tilde{T})$ are finite dimensional, and $\tilde{S}\tilde{T} = I$ on $\tilde{X}$. By problem 5.2, it suffices to show $R(\tilde{T})$ complemented in Y.

Let $P := \tilde{T}\tilde{S}$. Then $P^2 = \tilde{T}(\tilde{S}\tilde{T})\tilde{S} = \tilde{T}\tilde{S} = P$ and $R(P) = R(\tilde{T})$ follows.

(iii) ($\Rightarrow$) as in (i), using the hypothesis to split X.

($\Leftarrow$) $T \in SF_-(X,Y)$ since $\beta(T) \le \beta(1-K_2)$ by Prop. 3.8.

The proof that $N(T)$ is complemented is similar to (ii, $\Leftarrow$):

$TS_2 \in F(Y,Y)$ by lem. 5.3.

$TS_2S = I - F$, with F = finite rank projection.

Define $\tilde{Y} := N(F)$, $\tilde{S} := S_2S|_{\tilde{Y}} : \tilde{Y} \to X$,

$\tilde{T} := (1-F)T : X \to \tilde{Y}$.

Then $Y/\tilde{Y}$ and $N(\tilde{T})/N(T)$ are finite dimensional, and $\tilde{T}\tilde{S} = I$ on $\tilde{Y}$. By problem 5.2, it suffices to show $N(\tilde{T})$ complemented in X.

Let $P := I - \tilde{S}\tilde{T}$. Then

$P^2 = I - 2\tilde{S}\tilde{T} + \tilde{S}\underbrace{\tilde{T}\tilde{S}}_{=I}\tilde{T} = P$, and

$R(P) = N(\tilde{T})$. This completes the proof. ∎

We express the result from thm 5.1 as follows:

(i) $T \in F(X,Y) \Leftrightarrow T$ has a two-sided Fredholm inverse S.

(ii) $T \in SF_+(X,Y) \Leftarrow T$ has a left Fredholm inverse S_1

(iii) $T \in SF_-(X,Y) \Leftarrow T$ has a right Fredholm inverse S_2.

In (ii) and (iii), the implication $\Rightarrow$ holds under the additional assumption that Y and X respectively is a Hilbert space, or more generally if

$R(T)$ and $N(T)$ are complemented.
When $X = Y$, (i) can be formulated in the following way.

Defn. 5.4. Let X be a B-space. Then the Calkin algebra is the quotient $Ca(X) := L(X,X)/C(X,X)$

Since $C(X,X)$ is a closed two-sided ideal by Prop. 2,4, it follows that $Ca(X)$ is a Banach algebra. Given $T \in L(X)$, consider
$[T] := T + C(X,X)$.
That T is invertible in $Ca(X)$ means
$\exists S \in L(X,X) : [T]\cdot[S] = [S]\cdot[T] = [I]$,
or equivalently $TS = I + K_1$
$ST = I + K_2$ for some $K_1, K_2 \in C(X,X)$,
i.e. T has a two-sided Fredholm inverse.
$\therefore T \in F(X,X) \iff [T]$ is invertible in $Ca(X)$.

Ex. 5.5: The set of invertible elements in any Banach algebra is an open set. Since the quotient map $T \mapsto [T]$ is continuous, this gives us a first perturbation result for Fredholm operators:
If $T \in F(X,X)$, then $\exists \delta > 0 : \tilde{T} \in F(X,X)$ whenever $\tilde{T} \in L(X,X)$ and $\|\tilde{T} - T\| < \delta$.
We shall soon generalize this result.

Note that, at least in H-spaces, $T \in SF_+(H,H)$ iff $[T]$ has a left inverse, and $T \in SF_-(H,H)$ iff $[T]$ has a right inverse.

The index:

The most fundamental quantity associated with a Fredholm operator is the following.

Defn 5.6: Let X, Y be B-spaces and let $T \in SF(X,Y)$. Then the *index* of T is

$$i(T) := \alpha(T) - \beta(T).$$

Note that either $\alpha(T) < \infty$ or $\beta(T) < \infty$ for $T \in SF(X,Y)$.

Clearly
$$T \in F(X,Y) \iff -\infty < i(T) < +\infty$$
$$T \in SF_+(X,Y) \iff i(T) < +\infty$$
$$T \in SF_-(X,Y) \iff i(T) > -\infty.$$

Ex 5.7: The importance of the index stems from the dimension theorem in linear algebra. If X, Y are finite dimensional and $T \in L(X,Y)$, then
$$\dim R(T) = \dim X - \dim N(T)$$
$$\iff \dim Y - \beta(T) = \dim X - \alpha(T)$$
$$\iff i(T) = \dim X - \dim Y.$$
Thus, all $T \in L(X,Y)$ have the same index. One also checks that $i(T) = \dim X - \dim Y$ more generally holds whenever at least one of X and Y is finite dimensional.

Ex. 5.8 When both X and Y are infinite dimensional, then all values $i(T) \in \mathbb{Z} \cup \{+\infty, -\infty\}$ are possible for $T \in SF(X,Y)$ in general, for fixed spaces. As an example, take $H = X = Y =$ H-space with ON-basis $\{e_i\}_{i=1}^{\infty}$. Define $T \in L(H,H)$ by

$$Te_i := e_{i+N}, \quad i = 1, 2, \dots$$

Then $T \in F(H,H)$ and $i(T) = -N$. On the other hand, if
$$Te_i := \begin{cases} e_{i-N} & ; i \geq N+1 \\ 0 & ; i \leq N \end{cases}, \quad \text{then}$$

$T \in F(H,H)$ and $i(T) = +\infty$.

A first algebraic result for the index is the following.

Thm 5.9 Let X, Y, Z be B-spaces and $T \in SF(X,Y)$ and $S \in SF(Y,Z)$. Assume that $(i(T), i(S)) \notin \{(+\infty, -\infty), (-\infty, +\infty)\}$. Then $ST \in SF(X,Z)$ and

$$i(ST) = i(S) + i(T).$$

The proof uses the following generalization of the dimension theorem (see Ex 5.7).

Lem. 5.10: Let $V_1, \dots, V_N$ be vector spaces, and consider linear maps

$$\{0\} \xrightarrow{T_0=0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \dots \xrightarrow{T_{N-1}} V_{N-1} \xrightarrow{T_N} V_N \xrightarrow{T_{N+1}=0} \{0\}$$

Assume that this is an exact complex, i.e assume $\forall k \in \{1, \dots, N\} : R(T_{k-1}) = N(T_k)$.

(In particular, T_1 is injective and T_N is surjective.)

Then: (i) For finite- or infinite dimensional V_j, we have

$$\forall j: \dim V_j \leq \dim V_{j+1} + \dim V_{j-1}$$

(ii) If all V_j are finite dimensional, then

$$\sum_{j=1}^{N} (-1)^j \dim V_j = 0$$

Proof: By the dimension theorem and hypothesis:

$$\begin{aligned} \dim V_j &= \dim N(T_j) + \dim R(T_j) \\ &= \dim R(T_{j-1}) + \dim R(T_j) \\ &= \dim \left(V_{k-1} / N(T_{j-1}) \right) + \dim R(T_j) \\ &\leq \dim V_{j-1} + \dim V_{j+1}, \text{ giving (i).} \end{aligned}$$

For (ii), we have the telescope sum

$$\sum (-1)^j \dim V_j = \sum (-1)^j \left(\dim N(T_j) + \dim N(T_{j+1}) \right). \quad \blacksquare$$

Proof of thm 5.9:

Consider maps

$$\{0\} \xrightarrow{0} N(T) \xrightarrow{I} N(ST) \xrightarrow{T} N(S) \xrightarrow{\pi}$$

$$\xrightarrow{\pi} Y/R(T) \xrightarrow{S} Z/R(ST) \xrightarrow{P} Z/R(S) \to \{0\},$$

where the maps are

$I: x \mapsto x$, $T: x \mapsto Tx = y$, $\pi: y \mapsto y + R(T)$,

$S: y + R(T) \mapsto \underbrace{Sy}_{=z} + R(ST)$, $P: z + R(ST) \mapsto z + R(S)$.

This is seen to be an exact complex, and the dimensions of the spaces are

$\alpha(T)$, $\alpha(ST)$, $\alpha(S)$, $\beta(T)$, $\beta(ST)$ and $\beta(S)$

respectively.

(a) Assume $\alpha(T) = \infty$. Then $\beta(T) < \infty$ since $T \in SF_-(X,Y)$.
By hypothesis $\beta(S) < \infty$,
Lem. 5.10 (i) $\Rightarrow$ $\beta(ST) < \infty$ and $\alpha(ST) = \infty$.
$\therefore$ $i(ST) = \infty$, $i(S) > -\infty$, $i(T) = \infty$, so $i(ST) = i(S) + i(T)$.

(b) Cases $\beta(T) = \infty$, $\alpha(S) = \infty$ or $\beta(S) = \infty$ are handled as in (a).

(c) Assume finally that $\alpha(T), \beta(T), \alpha(S), \beta(S)$ are finite. Lem. 5.10 (i) $\Rightarrow$ $\alpha(ST) < \infty$ and $\beta(ST) < \infty$.
Thus Lem. 5.10 (ii) applies and yields

$$-\alpha(T) + \alpha(ST) - \alpha(S) + \beta(T) - \beta(ST) + \beta(S) = 0,$$

i.e. $i(ST) = i(S) + i(T)$.

Note that the above argument is purely algebraic, and quotient spaces are regarded as algebraic such.
It remains to show $ST \in SF(X,Y)$. Two cases need to be considered:

- $T \in SF_-(X,Y)$ & $S \in SF_-(X,Y)$: From above $\beta(ST) < \infty$ follows, so Prop. 3.8 shows $ST \in SF_-(X,Y)$
- $T \in SF_+(X,Y)$ & $S \in SF_+(X,Y)$: Duality Cor. 4.10 reduces this case to the previous one. ∎

Problem 5.11: Deduce from Prop. 3.10 that

$T \in SF_+(X,Y)$, $S \in SF_+(Y,Z) \Rightarrow ST \in SF_+(X,Z)$.

Problem 5.12: Verify that the sequence of maps in the proof of thm 5.9 is an exact complex. (In particul, show that all maps are well defined.)

Perturbation theory for Fredholm operators

As for duality, perturbation theory is much simpler in H-spaces. The main result here is the following.

Thm 5.13: Let H_1, H_2 be H-spaces, and let $T \in SF(H_1, H_2)$. Assume that T has reduced minimum modulus $\gamma(T)$, see Defn. 4.6, and that $\tilde{T} \in L(H_1, H_2)$ satisfies $\|\tilde{T} - T\| < \gamma(T)$. Then

$$\tilde{T} \in SF(H_1, H_2), \quad i(\tilde{T}) = i(T),$$
$$\alpha(\tilde{T}) \leq \alpha(T) \text{ and } \beta(\tilde{T}) \leq \beta(T).$$

Proof: Split the spaces:

$$H_1 = N(T)^\perp \oplus N(T), \quad H_2 = R(T) \oplus R(T)^\perp$$

The operator matrix for T in these splittings is $\begin{bmatrix} T_{11} & 0 \\ 0 & 0 \end{bmatrix}$, where $T_{11} \in L(N(T)^\perp, R(T))$ is an isomorphism with $\|T_{11}\| = \|T\|$ and $\gamma(T_{11}) = \gamma(T)$. (This uses that the splittings are orthogonal.)

In these splittings, also write

$$\tilde{T} = \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix}.$$

Note that $\|\tilde{T}_{11} - T_{11}\| \leq \|\tilde{T} - T\| < \gamma(T) = \|T_{11}^{-1}\|^{-1}$.

Thus $\|I - T_{11}^{-1}\tilde{T}_{11}\| = \|T_{11}^{-1}(T_{11} - \tilde{T}_{11})\| < 1$, so $T_{11}^{-1}\tilde{T}_{11}$, and hence also $\tilde{T}_{11}$, is invertible.

Factorize $\tilde{T}$ as

$$\begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} \\ \tilde{T}_{21} & \tilde{T}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{T}_{11} & 0 \\ \tilde{T}_{21} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \tilde{T}_{22} - \tilde{T}_{21}\tilde{T}_{11}^{-1}\tilde{T}_{12} \end{bmatrix} \begin{bmatrix} I & \tilde{T}_{11}^{-1}\tilde{T}_{12} \\ 0 & I \end{bmatrix}$$

Here the first and last operator matrices are invertible, and the key operator is

$$A := \tilde{T}_{22} - \tilde{T}_{21}\tilde{T}_{11}^{-1}\tilde{T}_{12} : N(T) \to R(T)^{\perp},$$

Note that either $N(T)$ or $R(T)^{\perp}$ is finite dimensional, so the middle operator matrix belong to $SF(H_1, H_2)$, with index $= i(A)$. From thm 5.9 it follows that $\tilde{T} \in SF(H_1, H_2)$ and $i(\tilde{T}) = i(A) = \dim N(T) - \dim R(T)^{\perp}$
$= i(T)$.

It also follows that

$\alpha(\tilde{T}) = \alpha(A) \leq \dim N(T) = \alpha(T)$ and

$\beta(\tilde{T}) = \beta(A) \leq \dim R(T)^{\perp} = \beta(T)$. ∎

For B-space, the above proof presents two problems:

(1) For general semi-Fredholm operators, we may not have splittings since subspaces may be uncomplemented.

(2) Even if we do have splittings (as is the case for Fredholm operators), we can only be sure that $\tilde{T}_{11}$ is invertible when $\|\tilde{T} - T\| < \delta$, where $0 < \delta \leq \gamma(T)$ depends on the obliqueness of the splittings (i.e. the norm of the projections).

It is absolutely essential to have a perturbation theory for all $SF(X, Y)$, since this will be our main tool for proving $\beta(T) < \infty$, as discussed before Prop. 3.10. Hence (1) is a serious problem. Nevertheless, we prove the following main result:

Thm 5.14: Let X, Y be B-spaces, and let $T \in SF(X,Y)$. Assume that T has reduced minimum modulus $\gamma(T)$, see Defn. 4.6, and that $\tilde{T} \in L(X,Y)$ satisfies $\|\tilde{T} - T\| < \gamma(T)$.
Then $\tilde{T} \in SF(X,Y)$, $i(\tilde{T}) = i(T)$,
$\alpha(\tilde{T}) \le \alpha(T)$ and $\beta(\tilde{T}) \le \beta(T)$.

The proof uses the following lemmete:

Lemma 5.15:
Assume $X, Y \subset Z$ are subspaces of a finite-dimensional B-space Z, with $\dim Y > \dim X$.
Then there exists $y \in Y$ such that

$$\|y\| = 1 = \operatorname{dist}(y, X).$$

Note that the special case $Y \supset X$ follows from F. Riesz lemma 2.2. Likewise the case when Z is a H-space is trivial:
Let $P: Y \to X$ be orthogonal projection.
Take $y \in N(P)$, $\|y\| = 1$.
The extension of this idea of proof to B-spaces is non-trivial, since such nearest-point-map P will be non-linear. Even the existence of such P require in general the following modification of the B-space norm.

Problem 5.16: Let $(Z, \|\cdot\|)$ be a finite dimensional B-space
(i) Let $\{f_1, \dots, f_n\}$ be a basis for Z^*, and let $\delta > 0$.
Define

$$\|z\|_\delta := \|z\| + \delta \left(\sum_{i=1}^{n} |\langle f_i, z \rangle|^2 \right)^{1/2}.$$

Show that this is an equivalent (that is,

$\forall z: C_1\|z\| \le \|z\|_\delta \le C_2\|z\|$ for some $0<C_1\le C_2<\infty$) norm on Z which is strictly convex, i.e.

$\|x\|_\delta \le 1$, $\|y\|_\delta \le 1$ and $x \neq y \Rightarrow \|\frac{x+y}{2}\|_\delta < 1$

(ii) Let $X \subset Z$ be a subspace.
Show that in the metric $\|\cdot\|_\delta$, for all $z \in Z$, there exists a unique nearest point $x \in X$ to z, and that x depends continuously on z.

<u>Proof of lemma 5.15:</u>

Consider the nearest-point-in X map
$P_\delta : Z \longrightarrow X$,
with respect to the equivalent norm $\|\cdot\|_\delta$, from Problem 5.16. Restrict to the unit sphere S_Y^δ, in the norm $\|\cdot\|_\delta$, in Y:

$$P_\delta : S_Y^\delta \longrightarrow X.$$

Topologically, this is a continuous map $S^m \to \mathbb{R}^n$ for $m \ge n$. By the Borsuk-Ulam theorem 5.17 (see below), there exists $y_\delta \in S_Y^\delta$ such that $P_\delta(y_\delta) = P_\delta(-y_\delta)$. However, it is straightforward to see that for our map P_δ, we have $P_\delta(-z) = -P_\delta(z)$ for all $z \in Z$.
Hence $P_\delta(y_\delta) = 0$.
Note that

$$\|y_\delta\| \le \|y_\delta\|_\delta = 1 = \mathrm{dist}_{\|\cdot\|_\delta}(y_\delta, X). \quad (*)$$

By sequential compactness ($\dim Z < \infty$), a subsequence $\{y_{\delta_n}\}_{n=1}^\infty$ converges to some $y \in Y$.
Taking limits in $(*)$ gives

$$\|y\| = 1 = \mathrm{dist}(y, X). \blacksquare$$

We state the celebrated Borsuk-Ulam theorem from algebraic topology, which we used above.
For more details, see for example
Jiří Matoušek: "Using the Borsuk-Ulam theorem".

<u>Thm 5.17</u> (Borsuk-Ulam)
Assume that $f: S^n \to \mathbb{R}^n$ is continuous.
Then there exists $x \in S^n$ such that $f(x) = f(-x)$.

A classical application of this result, is that on earth, there are always two antipodal points where both temperature and pressure are the same. $(n=2)$.

<u>Lemma 5.18</u>: Let X, Y be B-spaces, and let $T \in L(X,Y)$ be injective with $\gamma(T) = 0$, i.e. $R(T)$ is not closed.
Then for all $N \in \mathbb{Z}_+$, $\varepsilon > 0$, there exists a subspace $V \subset X$ such that $\dim V = N$ and $\|T|_V\| \leq \varepsilon$.

<u>Proof</u>: Induction over N:

1. $N=1$: Since $\gamma(T) = 0$: $\forall \varepsilon > 0 \ \exists x \neq 0 : \|Tx\| \leq \varepsilon \|x\|$.
   Take $V = \text{span}\{x\}$.

2. Assume the lemma is proved for some $N \geq 1$ (and all T satisfying the hypothesis).
   Pick $x_0 \in X$ with $\|x_0\| = 1$, $\|Tx_0\| \leq \varepsilon$ as in 1.
   By HB, pick $f \in X^*$ with $\|f\| = 1$, $f(x_0) = 1$.
   $Px := f(x) x_0$ is projection onto $\text{span}\{x_0\}$ with $\|P\| = 1$.
   Now consider $T: N(P) \to Y$. Since $\dim X/N(P) = 1$, we have $\dim\left(R(T)/R(T|_{N(P)})\right) \leq 1$. Thus $R(T|_{N(P)})$ cannot be closed, so $T: N(P) \to Y$ satisfies the hypothesis. Thus, by assumption, we can pick $V \subset N(P)$ with $\dim V = N$ and $\|T|_V\| \leq \varepsilon$.
   We get for all $x \in \text{span}\{x_0, V\}$:
   $$\|Tx\| \leq \|T(Px)\| + \|T(1-P)x\|$$

$\leq \varepsilon \|Px\| + \varepsilon \|(1-P)x\| \leq 3\varepsilon \|x\|$.

This proves the lemma since $\varepsilon > 0$ is arbitrary and $\dim \operatorname{span}\{X_0, Y\} = N+1$ ∎

Proof of thm 5.14:

Assume first $\|\tilde{T} - T\| < \gamma(T)$, and consider the proof of $\tilde{T} \in SF(X,Y)$, $\alpha(\tilde{T}) \leq \alpha(T)$ and $\beta(\tilde{T}) \leq \beta(T)$.

(i) Consider first the case $T \in SF_+(X,Y)$.

We have for all $0 \neq x \in N(\tilde{T})$:

$\gamma(T)\, \operatorname{dist}(x, N(T)) \leq \|Tx\| \leq \|(T-\tilde{T})x\| < \gamma(T)\|x\|$.

Thus $\operatorname{dist}(x, N(T)) < \|x\|$. By lem 5.15, we must have $\alpha(\tilde{T}) \leq \alpha(T)$.

By Prop. 3.5(i), we have $X = N(\tilde{T}) \oplus \tilde{X}_1$,

Assume that $R(\tilde{T}) = R(\tilde{T}|_{\tilde{X}_1})$ is not closed.

Then lem. 5.18 shows existence of $V \subset \tilde{X}_1$ such that $\dim V = \alpha(T) + 1$ and $\|\tilde{T}|_V\| \leq \varepsilon$, where $\|T - \tilde{T}\| \leq \gamma(T) - 2\varepsilon$,

$\Rightarrow \forall x \in V:\ \gamma(T)\, \operatorname{dist}(x, N(T)) \leq \|Tx\| \leq$

$$\leq \|(T-\tilde{T})x\| + \|\tilde{T}x\| \leq (\gamma(T) - 2\varepsilon)\|x\| + \varepsilon\|x\|$$

so $\operatorname{dist}(x, N(T)) \leq (1 - \frac{\varepsilon}{\gamma(T)})\|x\|$ for all $x \in V$.

Again lem 5.15 shows $\dim V \leq \alpha(T)$, giving a contradiction to $\dim V = \alpha(T) + 1$.

It follows that $\tilde{T} \in SF_+(X,Y)$.

(ii) If $T \in SF_-(X,Y)$, then (i) applies to $T^* \in SF_+(Y^*, X^*)$, and shows $\tilde{T}^* \in SF_+(Y^*, X^*)$ and $\alpha(\tilde{T}^*) \leq \alpha(T^*)$.

Thus $\tilde{T} \in SF_-(X,Y)$ and $\beta(\tilde{T}) \leq \beta(T)$.

- Next we prove $i(\tilde{T}) = i(T)$ for sufficiently small $\|\tilde{T} - T\|$.

(iii) If $T \in F(X,Y)$, then the proof of thm 5.13 applies. However $\|\tilde{T} - T\| < \delta$ is needed, for some $\delta < \gamma(T)$ depending on the norms of the projections onto $N(T)$ and $R(T)$.

(iv) For more general $T \in SF(X,Y)$, we reduce the case $T \in SF_-(X,Y)$ to $T \in SF_+(X,Y)$ by duality (Cor. 4.10), since $i(T^*) = -i(T)$. Thus we may assume $\alpha(T) < \infty$, $\beta(T) = \infty$.

- Assume first $\alpha(T) = 0$, $\beta(T) = \infty$.
  We have $\forall x : \|Tx\| \geq \gamma(T)\|x\|$, so
  $\forall x : \|\tilde{T}x\| \geq \|Tx\| - \|(\tilde{T}-T)x\| \geq \frac{\gamma(T)}{2}\|x\|$
  if $\|\tilde{T}-T\| \leq \gamma(T)/2$.
  In this case $\gamma(\tilde{T}) \geq \frac{\gamma(T)}{2}$, and we obtain
  $\infty = \beta(T) \leq \beta(\tilde{T})$ from (iii) since $\|T-\tilde{T}\| \leq \gamma(\tilde{T})$.
  Thus $i(\tilde{T}) = -\infty = i(T)$.

- Assume next that $\alpha(T) < \infty$, $\beta(T) = \infty$.
  Write $X = N(T) \oplus X_1$ and consider
  $T_1 := T|_{X_1}$ and $\tilde{T}_1 := \tilde{T}|_{X_1}$.
  Then $T_1 \in SF_+(X_1, Y)$ with $\alpha(T_1) = 0$, $\beta(T_1) = \infty$.
  We have $\|\tilde{T}_1 - T_1\| \leq C\|\tilde{T}-T\|$ for some $C < \infty$ depending on the norm of the projection $X \to N(T)$. Thus, if $\|\tilde{T}-T\| < \frac{\gamma(T)}{2C}$, it follows from above that
  $\alpha(\tilde{T}_1) = 0$ and $\beta(\tilde{T}_1) = \infty$.
  This implies that
  $\beta(\tilde{T}) \geq \beta(\tilde{T}_1) - \alpha(T) = \infty$ since
  $R(\tilde{T}) = R(\tilde{T}_1) + R(\tilde{T}|_{N(T)})$, where
  $\dim R(\tilde{T}|_{N(T)}) \leq \alpha(T)$.
  Thus we again obtain $i(\tilde{T}) = -\infty = i(T)$.

(v) We have proved that $i(\tilde{T}) = i(T)$ if $\|\tilde{T}-T\| < \delta$ for some $0 < \delta \leq \gamma(T)$. That this in fact holds with $\delta = \gamma(T)$ follows from Thm 5.19 below (the proof of which only uses that $i(\tilde{T}) = i(T)$ for $\|\tilde{T}-T\| < \delta$ for some $\delta > 0$). ∎

Thm 5.19 (The method of continuity)

Let X, Y be B-spaces, and for $0 \leq s \leq 1$ let semi-Fredholms operators $T_s \in SF(X,Y)$ be given. If the map

$$[0,1] \to L(X,Y) : s \mapsto T_s$$

is continuous, then $i(T_1) = i(T_0)$.

Proof: The function

$[0,1] \to \mathbb{Z} : s \mapsto i(T_s)$ is continuous according to thm 5.14. Since $[0,1]$ is connected, it follows that $s \mapsto i(T_s)$ is a constant function. ∎

It is very important to note that in order to apply the method of continuity, we must know a priori that all T_s are semi-Fredholm operators. This is normally proved using Prop. 3.10.

A very common application of Thm 5.19 is the following.

Cor. 5.20:

Let X, Y be B-spaces and let $T \in SF(X,Y)$. If $K \in C(X,Y)$ is compact, then $T+K \in SF(X,Y)$ and $i(T+K) = i(T)$

Proof: By duality (Cor. 4.10), we may assume $T \in SF_+(X,Y)$.

Define $T_s := T + sK$. Since $sK \in C(X,Y)$, it follows that $T_s \in SF_+(X,Y)$ for $s \in [0,1]$, from Cor. 3.11.

Then apply Thm 5.19. ∎

Ex: 5.21: We return to Ex. 3:12 and the double layer potential operator K on a compact and $C^{1,\alpha}$-regular boundary ∂D.
There, from the compactness of K, Fredholm uniqueness and stability for
$$(I+K)h = 2\psi$$
was establish, both in $C(\partial D)$ and $L_2(\partial D)$.
Now we use Cor. 5.20 to prove Fredholm existence.
We get $i(I+K) = i(I) = 0$, i.e.
$$\beta(I+K) = \alpha(I+K) < \infty.$$
This holds both in $C(\partial D)$ and $L_2(\partial D)$.
For the latter, this improves Ex. 4.3.

We end with a discussion of the topology of $SF(X,Y)$.
Thm 5.14. shows that $SF(X,Y)$ is a open subset of $L(X,Y)$. Write
$$SF(X,Y) = \bigcup_\alpha SF^{(\alpha)}(X,Y), \text{ where}$$
$SF^{(\alpha)}(X,Y)$ are the connected components of $SF(X,Y)$.
Thm 5.19 shows that the index $i(T)$ is constant on each $SF^{(\alpha)}(X,Y)$. It is known, at least for X, Y = H-spaces and $k \in \mathbb{Z}$, that
$$\{T \in SF(X,Y);\ i(T) = k\} \text{ is connected.}$$

Note also from Thm 5.14, that for $T \in SF(X,Y)$, the distance from T to the boundary of $SF(X,Y)$ is at least $\gamma(T)$.

A regularity result for Fredholm operators

Let $T: X \to Y$ be an isomorphism, and let $\tilde{X} \hookrightarrow X$ and $\tilde{Y} \hookrightarrow Y$ be B-spaces densely embedded in X and Y respectively. Assume that T restricts to an operator $\tilde{T}: \tilde{X} \to \tilde{Y}$ which also is an isomorphism.

Then it is easy to see that

$$x \in X, \ Tx \in \tilde{Y} \Rightarrow x \in \tilde{X}.$$

Indeed, let $\tilde{y} := Tx$. Then $\tilde{y} = \tilde{T}\tilde{x}$ for a unique $\tilde{x} \in \tilde{X}$ since $\tilde{T}: \tilde{X} \to \tilde{Y}$ is an isomorphism. Now

$T(x - \tilde{x}) = \tilde{y} - \tilde{y} = 0$, so since $T: X \to Y$ is also an isomorphism, we conclude that $x = \tilde{x} \in \tilde{X}$.

This regularity result generalizes as follows to Fredholm operators.

<u>Prop. 6.1</u>: Let X, Y be B-spaces, and let $\tilde{X} \hookrightarrow X$ and $\tilde{Y} \hookrightarrow Y$ be dense embedded B-spaces. Assume that $T \in F(X, Y)$ restricts to an operator $\tilde{T} \in F(\tilde{X}, \tilde{Y})$. Define $N := i(T) - i(\tilde{T})$.

Then $N \geq 0$,

$$\alpha(T) - N \leq \alpha(\tilde{T}) \leq \alpha(T), \quad \beta(T) \leq \beta(\tilde{T}) \leq \beta(T) + N,$$
$$N(\tilde{T}) \subset N(T), \quad R(\tilde{T}) \subset R(T) \cap \tilde{Y}.$$

Furthermore, the following are equivalent:

(i) $N = 0$, i.e. $i(T) = i(\tilde{T})$

(ii) $\alpha(\tilde{T}) = \alpha(T)$ and $\beta(\tilde{T}) = \beta(T)$

(iii) $N(\tilde{T}) = N(T)$ and $R(\tilde{T}) = R(T) \cap \tilde{Y}$

(iv) $x \in X, \ Tx \in \tilde{Y} \Rightarrow x \in \tilde{X}$.

Note that Prop. 6.1 shows that the criteria for regularity, i.e. for (iv) to hold, is that the two operators should have same index.

<u>Proof</u>: Clearly $N(\tilde{T}) \subset N(T)$, and therefor $\alpha(\tilde{T}) \leq \alpha(T)$ and $\beta(\tilde{T}) = \alpha(\tilde{T}) - i(\tilde{T}) \leq \alpha(T) - i(\tilde{T}) = \beta(T) + N$ hold.

To derive the corresponding result for the cokernels, let $\{y_i + R(T)\}_{i=1}^k$ be a basis for $Y/R(T)$. Since $\tilde{Y}$ is dense in Y, we may assume that all $y_i \in \tilde{Y}$. As $R(\tilde{T}) \subset R(T)$, $\{y_i + R(\tilde{T})\}_{i=1}^k$ are linearly independent in $\tilde{Y}/R(\tilde{T})$. Thus $\beta(\tilde{T}) \geq \beta(T)$ and
$\alpha(\tilde{T}) = \beta(\tilde{T}) + i(\tilde{T}) \geq \beta(T) + i(\tilde{T}) = \alpha(T) - N$.
We see that $\alpha(T) - N \leq \alpha(\tilde{T}) \leq \alpha(T)$, so $N \geq 0$.
Also $R(\tilde{T}) \subset R(T) \cap \tilde{Y}$ is clear.

(i) $\Leftrightarrow$ (ii) : Clear.

(ii) $\Rightarrow$ (iii) : Since $N(\tilde{T}) \subset N(T)$ and $\alpha(\tilde{T}) = \alpha(T)$, it follows that $N(\tilde{T}) = N(T)$. It remains to see $R(\tilde{T}) \supset R(T) \cap \tilde{Y}$. Let $y \in R(T) \cap \tilde{Y}$. Since $y \in \tilde{Y}$, $y = \sum_{i=1}^k \lambda_i y_i + \tilde{T}(\tilde{x})$ for some $\tilde{x} \in \tilde{X}$. But since $y \in R(T)$ and $\{y_i + R(T)\}_{i=1}^k$ is a basis, we have $y = \tilde{T}(\tilde{x}) \in R(\tilde{T})$.

(iii) $\Rightarrow$ (iv) : If $Tx \in \tilde{Y}$, then $Tx = \tilde{T}\tilde{x}$ for some $\tilde{x} \in \tilde{X}$, since $R(T) \cap \tilde{Y} = R(\tilde{T})$. But $x - \tilde{x} \in N(T) = N(\tilde{T}) \subset \tilde{X}$, so $x \in \tilde{X}$.

(iv) $\Rightarrow$ (ii) : If $x \in N(T)$, then $Tx = 0 \in \tilde{Y}$, so $x \in \tilde{X}$ and thus $x \in N(\tilde{T})$. This shows $\alpha(\tilde{T}) = \alpha(T)$.
It remains to show that $\{y_i + R(\tilde{T})\}_{i=1}^k$ span $\tilde{Y}/R(\tilde{T})$.
Let $y \in \tilde{Y}$. Then $y = \sum_{i=1}^k \lambda_i y_i + Tx$, using that $\{y_i + R(T)\}_{i=1}^k$ is a basis for $Y/R(T)$. Since $Tx = y - \sum_{i=1}^k \lambda_i y_i \in \tilde{Y}$ we get $x \in \tilde{X}$, so $y + R(\tilde{T}) = \sum_{i=1}^k \lambda_i (y_i + R(\tilde{T}))$.
This completes the proof. ∎

It is important to note that $\tilde{T}$ being Fredholm cannot be deduced from T being Fredholm, or vice versa. However, if both have been shown to be Fredholm, then Prop. 6.1 gives information about the relation between their deficiency indices α and β.
Note also that it is important that $\tilde{Y}$ is dense in Y.

Ex 6.2: We continue our investigation of the double layer potential from Ex. 5.21. There we obtained Fredholm operators

$$I+K : L_2(\partial D) \to L_2(\partial D)$$

$$I+K : C(\partial D) \to C(\partial D),$$

both with index 0, and inclusion $C(\partial D) \hookrightarrow L_2(\partial D)$ is known to be dense. Thus, by Prop 6.1:

(ii) $\alpha(I+K; C(\partial D)) = \alpha(I+K; L_2(\partial D))$ and
$\qquad \beta(I+K; C(\partial D)) = \beta(I+K; L_2(\partial D))$,

(iii) $N(I+K; C(\partial D)) = N(I+K; L_2(\partial D))$ and
$\qquad R(I+K; C(\partial D)) = R(I+K; L_2(\partial D)) \cap C(\partial D)$, and

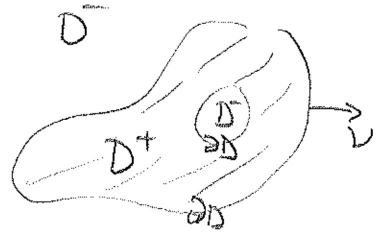
(iv) $h \in L_2(\partial D), (I+K)h \in C(\partial D) \Rightarrow h \in L_2(\partial D)$

hold. In particular, it is unambiguous to write $\alpha(I+K)$ and $\beta(I+K)$.

Single and double layers potentials

The aim in this section is to determine $\alpha(I+K)$ and $\beta(I+K)$. For this we return to the boundary integral method from lecture 1 for solving the Dirichlet problem. We need to consider some additional boundary value problems.

Let D be a bounded open subset of $\mathbb{R}^n$, with $C^{1,\alpha}$-regular boundary ∂D, for some $\alpha > 0$. Write $D^+ := D$ and $D^- := \mathbb{R}^n \setminus (D \cup \partial D)$. Note that we do not make any topological assumptions on D. ν denotes the unit normal vector field on ∂D, pointing into D^-.



Given $h : \partial D \to \mathbb{R}$, we set

$$u(x) := \int_{\partial D} \underbrace{E(y,x)}_{=-E(x,y)} \cdot \nu(y)\, h(y)\, d\sigma(y)$$

$$v(x) := \int_{\partial D} \underbrace{\Gamma(y,x)}_{=\Gamma(x,y)}\, h(y)\, d\sigma(y)$$

for $x \notin \partial D$.

Both the double layer potential $u(x)$ and the single layer potential $v(x)$, of h, is seen to be harmonic functions in $D^+ \cup D^-$. (Apply Δ_x under the integral sign.)

(1) The interior Dirichlet problem,
is to, given φ on ∂D, find u in D^+ such that

$$\begin{cases} \Delta u = 0 & \text{in } D^+ \\ u = \varphi & \text{on } \partial D \end{cases}$$

This we solved by making the ansatz
$u =$ double layer potential of h. The trace calculation leading to Prop. 1.7, showed that

$$\lim_{\substack{x \to x_0 \\ x \in D^+}} u(x) - h(x_0) = \tfrac{1}{2} K h(x_0) - \left(h(x_0) - \tfrac{1}{2} h(x_0)\right)$$

giving the integral equation

$$\tfrac{1}{2}(I + K) h = \varphi \qquad \text{to solve.}$$

(2) The exterior Dirichlet problem,
is to, given φ on ∂D, find u in D^- such that

$$\begin{cases} \Delta u = 0 & \text{in } D^- \\ u = \varphi & \text{on } \partial D \end{cases}$$

(In fact, one should also impose some decay of u at infinity, but we ignore this here as this will be encoded into our integral equation.)
The same trace calculation of the ansatz $u =$ double layer potential of h as in (1) gives

$$\lim_{\substack{x \to x_0 \\ x \in D^-}} u(x) - 0 = \tfrac{1}{2} K h(x_0) - \left(h(x_0) - \tfrac{1}{2} h(x_0)\right).$$

Note that $\int_{\partial D} E(y, x) \cdot \nu(y)\, d\sigma(y) = 0$ if $x \in D^-$.

Thus the integral equation for the exterior Dirichlet problem is

$$\tfrac{1}{2}(-I + K) h = \varphi,$$

Next we turn to the Neumann problem, where we instead make the ansatz

v = single layer potential of h.

We need to compute the trace of

$$\nabla v(x) = -\int_{\partial D} E(y,x)\, h(y)\, d\sigma(y).$$

Recall that the trace calculation for the Dirichlet problem made use of Gauss' theorem. To similarly apply some Stokes' theorem to ∇v, we need the normal vector field in the integrand. For this reason we write

$$E_i h = \sum_{j=1}^n (E_j \nu_j)(\nu_i h) + \sum_{j=1}^n (E_i \nu_j - E_j \nu_i)(\nu_j h),$$

where $E_i(y,x)$ is the i'th coordinate of $E(y,x)$ and $\nu_j(y)$ is the j'th coordinate of $\nu(y)$ = unit normal.

Let $h_i(y) := \nu_i(y) h(y)$ and

$$f_i^{(1)}(x) := \int_{\partial D} E(y,x)\cdot \nu(y)\, h_i(y)\, d\sigma(y),$$

$$f_i^{(2)}(x) := \sum_{j=1}^n \int_{\partial D} \big(E_i(y,x)\nu_j(y) - E_j(y,x)\nu_i(y)\big)\, h_j(y)\, d\sigma(y), \quad x \in D^+ \cup D^-,$$

and

$$K_{ij} g(x) := 2\,\text{p.v.} \int_{\partial D} \big(E_i(y,x)\nu_j(y) - E_j(y,x)\nu_i(y)\big) g(y)\, d\sigma(y), \quad x \in \partial D.$$

Thus $\partial_i v(x) = -f_i^{(1)}(x) - f_i^{(2)}(x)$.

Note that

$$\int_{\partial D} E(y,x)\cdot \nu(y)\, d\sigma(y) = \begin{cases} 1, & x \in D^+ \\ 0, & x \in D^- \end{cases}$$

$$\int_{\partial D} \big(E_i(y,x)\nu_j(y) - E_j(y,x)\nu_i(y)\big)\, d\sigma(y) = 0, \quad x \in D^+ \cup D^-,$$

by Gauss' and Stokes' theorems.

Finally, note that

$$2\,\text{p.v.} \int_{\partial D} E_i(y,x)\, h(y)\, d\sigma(y) = K h_i(x) + \sum_{j=1}^n K_{ij} h_j(x), \quad x \in \partial D.$$

Now we are in position to formulate boundary integral equations for the Neumann problem:

③ The interior Neumann problem,

is to, given ψ on ∂D, find v in D^+ such that

$$\begin{cases} \Delta v = 0 & \text{in } D^+ \\ \frac{\partial v}{\partial \nu} = \psi & \text{on } \partial D. \end{cases}$$

Here $\frac{\partial v}{\partial \nu} := \nu \cdot \nabla v$ is the (outward) normal derivative of v. We solve this by making the ansatz v = single layer potential of h. As above, we have

$$-\partial_i v = f_i^{(1)} + f_i^{(2)}.$$

As in ①,

$$\lim_{\substack{x \to x_0 \\ x \in D^+}} f_i^{(1)}(x) = \frac{1}{2} h_i(x_0) + \frac{1}{2} K h_i(x_0).$$

Similar arguments also yield

$$\lim_{\substack{x \to x_0 \\ x \in D^+}} f_i^{(2)}(x) - 0 = \frac{1}{2} \sum_{j=1}^{n} K_{ij} h_j(x_0) - (0-0)$$

Thus $$-\lim_{\substack{x \to x_0 \\ x \in D^+}} \partial_i v(x) = \frac{1}{2} \nu_i(x_0) h(x_0) + \text{p.v.} \int_{\partial D} E_i(y,x) h(y) \, d\sigma(y).$$

Multiplying with $\nu_i(x)$ and summing, yields the integral equation

$$\frac{1}{2}(-I + K^*) h = \psi.$$

K^* denotes the $L_2(\partial D)$-adjoint of K: $K^* h(x) = 2 \,\text{p.v.} \int_{\partial D} \nu(x) \cdot E(x,y) h(y) \, d\sigma(y)$

④ The exterior Neumann problem,

is to, given ψ on ∂D, find v in D^- such that

$$\begin{cases} \Delta v = 0 & \text{in } D^- \\ \frac{\partial v}{\partial \nu} = \psi & \text{on } D^+, \end{cases}$$

Similar to ③, for the single layer potential ansatz, we have $-\partial_i v = f_i^{(1)} + f_i^{(2)}$ in D^-, where

$$\lim_{\substack{x \to x_0 \\ x \in D^-}} f_i^{(1)}(x) = -\frac{1}{2} h_i(x_0) + \frac{1}{2} K h_i(x_0) \qquad (\text{see } ②)$$

and

$$\lim_{\substack{x \to x_0 \\ x \in D^-}} f_i^{(2)}(x) = \frac{1}{2} \sum_{j=1}^{n} K_{ij} h_j(x_0)$$

Thus $$-\lim_{\substack{x \to x_0 \\ x \in D^-}} \partial_i v(x) = -\frac{1}{2} \nu_i(x_0) h(x_0) + \text{p.v.} \int_{\partial D} E_i(y,x) h(y) \, d\sigma(y),$$

giving the integral equation

$$\frac{1}{2}(I + K^*) h = \psi.$$

We now solve the problem posed at the beginning of this section.

Prop. 6.3: Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set with $C^{1,\alpha}$-regular boundary for some $\alpha > 0$. Let

$$Kh(x) := 2\,\text{p.v.} \int_{\partial D} E(y,x) \cdot \nu(y)\, h(y)\, d\sigma(y), \quad x \in \partial D,$$

denote the double layer potential operator on ∂D. Then in both spaces $C(\partial D)$ and $L_2(\partial D)$, $I+K$ and $I-K$ are Fredholm operators with index 0, and

$$\alpha(I+K) = \text{number of bounded connected components of } D^- := \mathbb{R}^n \setminus \overline{D},$$

$$\alpha(I-K) = \text{number of connected components of } D.$$

Proof: Everything except the statements for $\alpha(I+K)$ and $\alpha(I-K)$ has already been proved. Consider $\alpha(I+K)$. (The proof for $\alpha(I-K)$ is similar and is left as an exercise.)

Let $N :=$ number of bounded components of D^-, and let $D_1^-, \dots, D_N^-$ be these components and D_∞^- being the unbounded component of D^-.

Gauss theorem on D_k^- shows

$$(I+K)(\chi_{\partial D_k^-})(x_0) = \lim_{\substack{x \to x_0 \\ x \in D}} \int_{\partial D_k^-} E(y,x) \cdot \nu(y)\, d\sigma(y) = 0, \quad x_0 \in \partial D$$

Thus $\text{span}\{\chi_{\partial D_1^-}, \dots, \chi_{\partial D_N^-}\} \subset N(I+K)$, so $\alpha(I+K) \geq N$.

- For the converse, we note

$$\alpha(I+K; C(\partial D)) = \alpha(I+K; L_2(\partial D)) = \beta(I+K^*; L_2(\partial D)) =$$
$$= \alpha(I+K^*; L_2(\partial D)) = \alpha(I+K^*; C(\partial D)),$$

using Prop. 6.1, Cor. 4.2, Cor. 5.20.

We show $\alpha(I+K^*) \leq N$.

Assume $h \in C(\partial D)$ and $(I+K^*)h = 0$.

If $u(x) := \int_{\partial D} \Gamma(y,x)\, h(y)\, d\sigma(y)$ denotes the single layer potential of h, we have that

$\frac{\partial v^-}{\partial \nu} = 0$ on ∂D, where $v^\pm := v|_{D^\pm}$, according to (4).

Apply Green's first identity:

$$\iint_{D^-} |Dv^-|^2 dx = -\int_{\partial D} v^-(y) \underbrace{\frac{\partial v^-}{\partial \nu}(y)}_{=0} d\sigma(y) = 0 \qquad (*)$$

At the moment, assume $n \geq 3$. Then the formula is seen to hold for the unbounded domain D^- as v^- decays sufficiently fast at infinity.

Thus $v^-|_{\partial D_k^-} =$ constant for each $k=1,\dots,N$, and $v^-|_{\partial D_\infty^-} = 0$.

Due to the weak singularity of $\Gamma(y,x)$ at $y=x$, v is seen to be continuous across ∂D, i.e. $v^+|_{\partial D} = v^-|_{\partial D}$.

Let $V := \{u : \bar{D} \to \mathbb{R} \,;\, \Delta u = 0 \text{ in } D^+,\ u|_{\partial D_k^-} = c_k,\ k=1,\dots,N \text{ and } u|_{\partial D_\infty^-} = 0\}$

Then $v^+ \in V$. The Green's formula

$$\iint_{D^+} |Du|^2 dx = \int_{\partial D} u(y) \frac{\partial u}{\partial \nu}(y) d\sigma(y)$$ shows that

$u \in V$ is uniquely determined by $c_1, \dots, c_N$. Thus $\dim V \leq N$. By (3) and (4)

$$h = \frac{\partial v^-}{\partial \nu} - \frac{\partial v^+}{\partial \nu} = -\frac{\partial v^+}{\partial \nu} \text{ on } \partial D, \text{ where } v^+ \in V.$$

This shows $\alpha(I+K^*) \leq N$.

- It remains to modify the above argument when $n=2$. Then $\Gamma(y,x) = \frac{1}{2\pi}\ln|y-x|$ does not decay at ∞, and it is not clear if $(*)$ holds.

We note that $(I+K)1 = 1$. Thus

$$0 = \langle \underbrace{(I+K^*)h}_{=0}, 1 \rangle = \langle h, 1 \rangle = \int_{\partial D} h(y) d\sigma(y), \text{ and}$$

$$v^-(x) = \int_{\partial D} (\Gamma(y,x) - \Gamma(y_0,x)) h(y) d\sigma(y), \quad x \in D^-,$$

for any fixed $y_0 \in D^+$. From this it follows that indeed $\lim_{x\to\infty} v^-(x) = 0$, as a consequence of $h \in N(I+K^*)$, and that $(*)$ holds. ∎

Note that with general Fredholm theory, we can prove $\alpha(T) < \infty$. However, to find the exact value of $\alpha(T)$, a concrete investigation of the specific operator T is needed.

More on operators I + compacts:

Unlike previous (and later) sections, we here require the B-spaces to be over the complex field.

We have seen that for a Fredholm operator $T: X \to Y$, we have splittings $X = N(T) \oplus X_1$ and $Y = R(T) \oplus Y_1$, where $T: X_1 \to R(T)$ is an isomorphism. When $X = Y$, it is natural to ask if we can take $X_1 = R(T)$. The following result shows that this almost can be done for operators $T = I + \text{compact}$.

Thm 6.4 (F. Riesz' theorem)
Let X be a B-space, and let $S \in C(X,X)$. Then there is a unique splitting $X = N \oplus M$, where $\dim N < \infty$ and
$I - S: N \to N$ is nilpotent, i.e. $((I-S)|_N)^m = 0$ for some $m < \infty$,
and $I - S: M \to M$ is an isomorphism.
Moreover, there exists $\varepsilon > 0$ such that $\lambda I - S$ is an isomorphism for each $0 < |\lambda - 1| < 2\varepsilon$, and
$$P := \frac{1}{2\pi i} \int_{|\lambda - 1| = \varepsilon} (\lambda I - S)^{-1} d\lambda$$
is the projection onto N along M.

Proof: Consider operators $(I-S)^k$, $k = 1, 2, \ldots$

Claim: there exists $m < \infty$ such that $N((I-S)^k) = N((I-S)^m)$ for all $k > m$.

If not, then
$$N((I-S)) \subsetneqq N((I-S)^2) \subsetneqq N((I-S)^3) \subsetneqq N((I-S)^4) \subsetneqq \ldots$$
By F. Riesz' lemma 2.2, there are $x_k \in N((I-S)^k)$ such that $\|x_k\| = 1 = \operatorname{dist}(x_k, N((I-S)^{k-1}))$, $k = 2, 3, \ldots$
For $k > j$, write
$$S(x_k - x_j) = x_k - (Sx_j + (I-S)x_k)$$
Since $Sx_j \in N((I-S)^{k-1})$ and $(I-S)x_k \in N((I-S)^{k-1})$, we get
$$\|S(x_k - x_j)\| \geq \operatorname{dist}(x_k, N((I-S)^{k-1})) = 1 \quad \text{for all } k > j.$$

This contradicts the compactness of S ($S(B_X)$ cannot be totally bounded). Thus the claim holds.

- Define
$$N := N((I-S)^m) \quad , \quad M := R((I-S)^m)$$
By thm 5.9 and cor. 5.20, $(I-S)^m$ is a Fredholm operator with index 0. Thus N and M are closed subspaces and $\dim X/M = \dim N$. To show $X = N \oplus M$ it remains to prove $N \cap M = \{0\}$.
But if $x = (I-S)^m y$ satisfies $(I-S)^m x = 0$, then $(I-S)^{2m} y = 0$. Thus $x = (I-S)^m y = 0$ since $N((I-S)^{2m}) = N((I-S)^m)$.
To show that $(I-S)|_N$ is nilpotent and $(I-S)|_M$ is an isomorphism is straightforward.

- Next consider the operator $\lambda I - S$ in the splitting $X = N \oplus M$ given by $I-S$. Clearly both subspaces N and M are invariant under $\lambda I - S$, so it suffices to show that $\lambda I - S : N \to N$ and $\lambda I - S : M \to M$ are invertible for $0 < |\lambda - 1| < 2\varepsilon$. But this is true since
$$\lambda I - S = (\lambda - 1) I + (I - S),$$
where $\lambda - 1$ is small and $I-S$ is invertible on M, and $\lambda - 1 \neq 0$ and $I - S$ is nilpotent on N.
(From geometric series, $(I-T)^{-1} = \sum_{k=1}^{m-1} T^k$ if $T^m = 0$.)
Finally, consider the Dunford integral
$$P = \frac{1}{2\pi i} \int_{|\lambda-1|=\varepsilon} (\lambda I - S)^{-1} d\lambda$$ around the circle $|\lambda - 1| = \varepsilon$ counterclockwise.
We need to show that $P|_N = I$ and $P|_M = 0$.
$\forall x \in M, f \in M^* : \langle f, Px \rangle = \frac{1}{2\pi i} \int_{|\lambda-1|=\varepsilon} h(\lambda)\, d\lambda = 0$ since
$h(\lambda) := \langle f, (\lambda I - S)^{-1} x \rangle$ is analytic in $|\lambda - 1| < 2\varepsilon$ (see below.)
On N, we write $(\lambda I - S)^{-1} = \sum_{k=0}^{m-1} \frac{(S-I)^k}{(\lambda-1)^{k+1}}$. Then
$$\forall x \in N, f \in N^* : \langle f, Px \rangle = \sum_{k=0}^{m-1} \langle f, (S-I)^k x \rangle \frac{1}{2\pi i} \int_{|\lambda-1|=\varepsilon} \frac{d\lambda}{(\lambda-1)^{k+1}} = \langle f, x \rangle,$$

by residue calculus. This proves the theorem. ∎

Analyticity of operator-valued functions is meant in the following sense.

<u>Defn 6.5</u>: Let X, Y be B-spaces, and let $D \subset \mathbb{C}$ be an open set. Then an operator-valued function
$$D \longrightarrow L(X,Y) : z \longmapsto T_z$$
is said to be <u>analytic</u> (or holomorphic) if
$$\forall z \in D \; \exists T_z' \in L(X,Y) : \lim_{h \to 0} \left\| \frac{T_{z+h} - T_z}{h} - T_z' \right\|_{X \to Y} = 0.$$

<u>Problem 6.6</u>:

(i) Let $S \in L(X,Y)$. Show that $\lambda \mapsto \lambda I - S$ is analytic on $\rho(S)$, the resolvent set of S.

(ii) Let $D \to L(X,Y) : z \mapsto S_z$ be an operator-valued analytic function. Assume that $\lambda I - S_z$ is invertible for all $z \in D$ and $|\lambda - 1| = \varepsilon$ ($\varepsilon > 0$ given). Show that
$P_z := \int_{|\lambda - 1| = \varepsilon} (\lambda I - S_z)^{-1} d\lambda$ is an operator-valued analytic function.

A useful tool is the following lemma, that roughly says that an operator-valued function is analytic iff all (almost all) its matrix elements are scalar analytic functions.

<u>Lemma 6.7</u>: Let $D \to L(X,Y) : z \mapsto T_z$ be an operator-valued function. Then it is analytic iff $h(z) := \langle f, T_z x \rangle$ is a scalar analytic function for all $f \in \tilde{Y}^*$ and $x \in \tilde{X}$ (where $\tilde{X} \subset X$ and $\tilde{Y}^* \subseteq Y^*$ are dense subspaces), and $\sup_{z \in K} \|T_z\| < \infty$ for each compact set $K \subset D$.

<u>Proof</u>: ($\Longrightarrow$) is straightforward.

($\Longleftarrow$) Given $f \in \tilde{Y}^*$ and $x \in \tilde{X}$, we apply Cauchy's formula for $\gamma \subset D$ around $z \in D$.

$$\Rightarrow \langle f, \frac{T_{z+h}-T_z}{h}x\rangle = \frac{1}{2\pi i}\int_\gamma \langle f, T_w x\rangle \frac{dw}{(w-z)(w-z-h)}.$$

For small h_1, h_2, we set for $U_h := \frac{T_{z+h}-T_z}{h}$ that

$$|\langle f, (U_{h_1}-U_{h_2})x\rangle| = \left|\frac{1}{2\pi i}\int_\gamma \frac{\langle f, T_w x\rangle}{w-z} \frac{h_1-h_2}{(w-z-h_1)(w-z-h_2)}dw\right|$$

$$\leq C\cdot|h_1-h_2|\cdot\|f\|\cdot\|w\|, \quad f\in\tilde{Y}^*, x\in\tilde{X}$$

Density shows $\|U_{h_1}-U_{h_2}\| \leq C|h_1-h_2|$.
Completeness shows $U_h = \frac{1}{h}(T_{z+h}-T_z) \to T_z'$ for some $T_z'\in L(X,Y)$. ∎

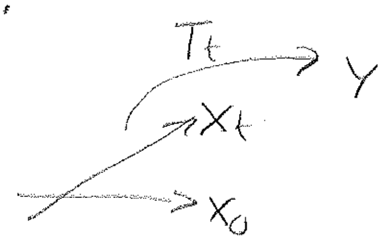
Our main result in this section is the following analytic perturbation results.

<u>Thm 6.8</u> (Analytic Fredholm theorem.)
Let X, Y be B-spaces, and let $D\subset\mathbb{C}$ be open. Assume that $D\to L(X,Y) : z\mapsto T_z$ is an operator-valued analytic function and that $T_{z_1}-T_{z_2}$ is compact for all $z_1, z_2\in D$.
If T_{z_0} is invertible for some $z_0\in D$, then there exists a discrete set $D_0\subset D$ such that T_z is invertible for all $z\in D\setminus D_0$.

To prove this we need, besides Thm 6.4, the following lemma. We state this in a slightly more general form than needed, in order to illustrate that it is a variant of thm 5.14 for variable spaces (rather than variable operators).

<u>Lemma 6.9</u>: Let $T\in L(X,Y)$, with X, Y B-spaces.
Let X_t, $t\in(-\delta,\delta)$, $\delta>0$, be a family of closed subspaces of X, and assume P_t are projections onto X_t with $(-\delta,\delta)\to L(X,X) : t\mapsto P_t$ being continuous.
Define $T_t := T|_{X_t} : X_t\to Y$.
If $T_0\in\mathcal{F}(X_0,Y)$, then $\exists\, 0<\varepsilon<\delta$ such that for all $|t|<\varepsilon$,

$T_t \in SF(X_t, Y)$, $i(T_t) = i(T_0)$,
$\alpha(T_t) \leq \alpha(T_0)$ and $\beta(T_t) \leq \beta(T_0)$.



<u>Proof</u>:

Consider the restricted projection $P_t : X_0 \to X_t$. We wish to show that this is invertible, and to find an inverse $Q_t : X_t \to X_0$. For a left inverse, note that

$\forall f \in X_0$: $P_0 P_t f = (P_0(P_t - P_0) + I) f$

When $|t|$ is small, $I + P_0(P_t - P_0) : X \to X$ is invertible and leaves X_0 invariant. Thus

$\tilde{Q}_t := (I + P_0(P_t - P_0))^{-1} P_0 |_{X_t}$ is a left inverse.

For a right inverse, note

$\forall g \in X_t$: $P_t P_0 g = (P_t(P_0 - P_t) + I) g$

When $|t|$ is small, $I + P_t(P_0 - P_t) : X \to X$ is invertible and leaves X_t invariant.

$\tilde{\tilde{Q}}_t := P_0 (I + P_t(P_0 - P_t))^{-1}|_{X_t}$ is a right inverse.

Thus $P_t : X_0 \to X_t$ is invertible for $|t| < \varepsilon$, with inverse $Q_t := \tilde{Q}_t = \tilde{\tilde{Q}}_t$.

- Now consider

  $S_t := T_t \circ P_t : X_0 \to Y$, which are operators between fixed spaces. Thm 5.14 applied to S_t now gives the stated results for T_t, since $P_t : X_0 \to X_t$ are isomorphisms. ∎

<u>Proof of thm 6.8</u>:

Define $S_z := T_{z_0}^{-1}(T_{z_0} - T_z) = I - T_{z_0}^{-1} T_z \in C(X, X)$, so that $T_z = T_{z_0}(I - S_z)$. We need to show that $I - S_z$ is invertible for all $z \in D$ except a discrete set.

For this, it suffices to show, for each $z_1 \in D$, that $I - S_z$ is invertible for $0 < |z - z_1| < \delta = \delta(z_1)$.

Fix $z_1 \in D$, and apply Thm 6.4 to S_{z_1}.

We have $I-S_{z_1}$ nilpotent on the finite-dimensional space N_{z_1}, $I-S_{z_1}$ an isomorphism on M_{z_1}, $\lambda I-S_{z_1}$ invertible for $0<|\lambda-1|<2\varepsilon=2\varepsilon(z_1)$, and

$$P_{z_1}=\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon(z_1)}(\lambda I-S_{z_1})^{-1}d\lambda = \text{projection onto } N_{z_1} \text{ along } M_{z_1}.$$

Similarly, apply Thm 6.4 to S_z to obtain projection

$$P_z=\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon(z)}(\lambda I-S_z)^{-1}d\lambda$$ onto N_z in the splitting

$$X=N_z\oplus M_z.$$

It is seen (how?) that $\lambda I-S_z$ is invertible when $\frac{\varepsilon(z_1)}{2}<|\lambda-1|<\frac{3\varepsilon(z_1)}{2}$ and $|z-z_1|<\delta$, for some $\delta>0$.

Define the projection

$$P_z^1:=\frac{1}{2\pi i}\int_{|\lambda-1|=\varepsilon(z_1)}(\lambda I-S_z)^{-1}d\lambda$$

As in the proof thm 6.4, it is shown that $P_z^1x=x$ for all $x\in N_z$, so $N(I-S_z)\subset N_z\subset R(P_z^1)$. In the finite-dim. space $X_z:=R(P_z^1)$, we have

$I-S_z: X_z\to X_z$ is invertible iff $\det(I-S_z|_{X_z})=0$.

(Note that X_z is invariant under $I-S_z$, since P_z^1 commutes with $(I-S_z)$.)

Now consider the restricted projection

$P_z^1: X_{z_1}=N_{z_1}\to X_z$. By lem. 6.9 this is invertible, with inverse $Q_z^1: X_z\to N_{z_1}$ when $|z-z_1|$ is small enough. Inspection of the proof of lem. 6.9. shows moreover that $z\mapsto Q_z^1$ is the restriction to X_z of an analytic operator-valued function.

$h(z):=\det(Q_z^1(I-S_z)P_z^1|_{N_{z_1}})$ is seen to be analytic in a neighborhood of z_1.

We have shown that

$I-S_z$ is injective iff $h(z)=0$. Since zero's of analytic functions are isolated, the proof is complete. ∎

Note that in the above proof, P_z^1 may be projection onto a strictly larger subspace than P_z ($\varepsilon(z)$ may need to be much smaller than $\varepsilon(z_1)$). This can happen if 1 is a multiple eigenvalue of P_{z_1}, which splits into several distinct eigenvalues for P_z, $z \approx z_1$.

A simple application of thm 6.8 gives the basic spectral properties of compact operators.

Defn 6.10: Let X be a B-space and $T \in L(X, X)$. Its spectrum is the set

$\sigma(T) := \{\lambda \in \mathbb{C};\ \lambda I - T \text{ is not invertible}\}$.

Its essential spectrum is the set

$\sigma_{ess}(T) := \{\lambda \in \mathbb{C};\ \lambda I - T \text{ is not a Fredholm operator}\}$.

The corresponding resolvent sets are

$\rho(T) := \mathbb{C} \setminus \sigma(T)$, $\rho_{ess}(T) := \mathbb{C} \setminus \sigma_{ess}(T)$.

Note that $\sigma_{ess}(T) \subset \sigma(T)$, both being compact sets, and that $\sigma_{ess}(T)$ can be seen as the spectrum of T as an element in the Calkin algebra (Defn. 5.4.).

Cor. 6.11: Let $S \in C(X, X)$ be a compact operator on a B-space X. Then $\sigma_{ess}(S) = \{0\}$ and the only possible limit point for $\sigma(S)$ is 0.

Proof: $\sigma_{ess}(S) = \{0\}$ since $[S] = 0$ in the Calkin algebra. Next let $Y = X$ and $T_\lambda := I - \frac{1}{\lambda} S$, with $D := \mathbb{C} \setminus \{0\}$, in Thm 6.8. The result for $\sigma(S)$ is immediate. ∎

A less elementary application of thm 6.8 is the following.

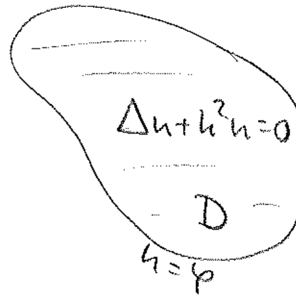
Ex 6.12:

The method of solving the Dirichlet and Neumann problem for the Laplace equation, by inverting the double layer potential operator or its adjoint, can be generalized to different equations and boundary value problems. Here we consider the Helmholtz equation $(\Delta+k^2)u=0$, for given wave number $k\in\mathbb{C}$. This equation appears when seeking time-harmonic solutions $u(t,x)=u(x)\cdot e^{ikt}$ to the wave-equation $(\Delta-\frac{\partial^2}{\partial t^2})u=0$. Laplace's equation, $k=0$, is the special case of time-independent solutions.

For simplicity, assume

$$\mathbb{R}^n=\mathbb{R}^3,$$

which is the main interest in applications.



One finds the fundamental solution

$$\Gamma_k(y,x)=-\frac{1}{4\pi|y-x|}\cdot e^{ik|y-x|}$$

Proceeding as in the case $k=0$, to solve the Dirichlet problem for the Helmholtz equation

$$\begin{cases}\Delta u+k^2u=0 & \text{in } D\\ u=\varphi & \text{on } \partial D,\end{cases}$$

we need to solve

$$\tfrac{1}{2}(I+K_k)h=\varphi,$$

where the modified double layer potential is

$$K_kh(x)=2\,\text{p.v.}\int_{\partial D}E_k(y,x)\cdot\nu(y)\,h(y)\,d\sigma(y),$$

$$E_k(y,x)=\nabla_y\Gamma_k(y,x)=\left(\frac{y-x}{|y-x|^3}-ik\frac{y-x}{|y-x|^2}\right)\frac{e^{ik|y-x|}}{4\pi}.$$

One check that Prop. 6.3 applies to K_k, for any $k\in\mathbb{C}$, except that the formulae for $\alpha(I\pm K)$ do not hold for $I\pm K_k$.

However, since $k\mapsto E_k(y,x)$ is analytic for fixed $x\neq y$, one can verify from this that

$\mathbb{C}\to\mathcal{L}(X,X): k\mapsto K_k$ is analytic in the sense of Defn 6.5, both for $X=C(\partial D)$ and $X=L_2(\partial D)$.

Thus Thm 6.8 applies and shows that

$I\pm K_k$ is an isomorphism for all $k\in\mathbb{C}$ except for

a discrete set of exceptional k, where $I \pm K_k$ is "only" a Fredholm operator of index 0.
(It can be shown that $I \pm K_k$ is invertible when $\operatorname{Im} k > 0$.)

Problem 6.13:

Apply Thm 6.4 to a finite-dimensional B-space X. Use this theorem to develop an algorithm for calculating the Jordan normal form of a matrix representing an operator $S: X \to X$.

In preparation for Problem 6.14, consider the unit circle $\mathbb{T} := \{z \in \mathbb{C} ; |z| = 1\}$ in the complex plane.
Given any function $f: \mathbb{T} \to \mathbb{C}$, apply the Cauchy integral:

$$F(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w)}{w - z} dw, \quad |z| < 1.$$

With calculations similar to those for the double layer potential, giving Prop. 1.7, we have trace

$$P^+ f(z_0) := \lim_{\substack{z \to z_0 \\ |z|<1}} F(z) = \frac{1}{2} f(z_0) + \frac{1}{2}\left(\frac{1}{\pi i} \text{p.v.} \int_{\mathbb{T}} \frac{f(w)}{w - z_0} dw\right), \quad z_0 \in \mathbb{T}$$

under appropriate regularity assumptions.
Next consider the action of P^+ on a finite Laurent-series $f(z) = \sum_{-N}^{N} a_k z^k$.
Complex analysis shows that

$$P^+ f(z) = \sum_{0}^{N} a_k z^k.$$

Interpreting this in terms of Fourier series

$$\mathcal{F}: L_2(\mathbb{T}) \longrightarrow l_2(\mathbb{Z}) : f \longmapsto \widehat{\mathcal{F}f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) e^{-in\theta} d\theta,$$

we have

$$\mathcal{F} P^+ \mathcal{F}^{-1} : (a_k)_{k=-\infty}^{\infty} \longmapsto (\chi_+(k) a_k)_k, \quad \text{where}$$

$$\chi_+(k) := \begin{cases} 1, & k \geq 0 \\ 0, & k < 0. \end{cases}$$

From this it is clear that $P^+ : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is an orthogonal projection.

Problem 6.14: With notation as above, let

$H^+ := R(P^+) \subset L_2(\mathbb{T})$ be the interior Hardy subspace. For a continuous function $\varphi \in C(\mathbb{T})$, define the Toeplitz operator

$$T_\varphi : H^+ \to H^+ : f \mapsto P^+(\varphi f).$$

It is clear that $\|T_\varphi\|_{L_2 \to L_2} \leq \|\varphi\|_{L_\infty(\mathbb{T})}$.

(i) Let $e_n(\theta) := e^{in\theta}$. Show that

$$\forall n, l \in \mathbb{Z} : T_{e_n} T_{e_l} - T_{e_n e_l} : H^+ \to H^+ \text{ is compact.}$$

(ii) Show that $T_\varphi : H^+ \to H^+$ is a Fredholm operator for each non-vanishing $\varphi \in C(\mathbb{T})$.

(iii) Calculate the index $i(T_\varphi)$ if $\varphi \in C(\mathbb{T})$ is a non-vanishing function which is homotopic to e_k through continuous maps $\mathbb{T} \to \mathbb{C} \setminus \{0\}$ (i.e. φ has winding-number k around 0).

We end Part I of the course with the remark that the index of an operator usually is topological in its nature. The reason is the method of continuity Thm 5.19, by which continuous perturbations of the operator do not change the index.