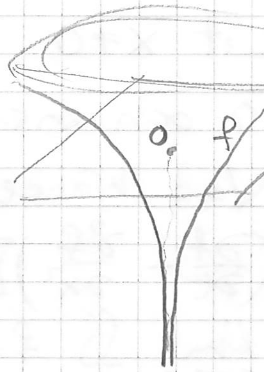
Distributioner och fundamentallösningar

① Laplaces ekvation i planet:

Betrakta $f(x,y) = \ln(x^2+y^2)$.

f är lokalt integrerbar och definierar en distribution på $\mathbb{R}^2$.



Utanför origo:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{2}{x^2+y^2} - \frac{(2x)^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{(2y)^2}{x^2+y^2} = 0.$$

Vad händer i origo?

$$\Delta f[\varphi] = /\text{definition av distributionsderivata}/$$

$$= (-1)^2 f[\Delta\varphi] = \iint f(x,y)\,\Delta\varphi(x,y)\,dx\,dy$$

$$= \lim_{\varepsilon\to 0} \iint_{x^2+y^2>\varepsilon^2} f\,\Delta\varphi\,dx\,dy = /\text{Greens andra formel}/$$

$$\lim_{\varepsilon\to 0}\left(\int_{x^2+y^2=\varepsilon^2} \Big(\underbrace{f}_{=\ln\varepsilon^2} \frac{\partial\varphi}{\partial n} - \underbrace{\frac{\partial f}{\partial n}}_{=\nabla f\cdot\frac{(-x,-y)}{\sqrt{x^2+y^2}} = -\frac{2}{\varepsilon}} \varphi\Big)\,ds + \iint_{x^2+y^2>\varepsilon^2} \underbrace{(\Delta f)}_{=0}\cdot\varphi\,dx\,dy \right)$$

$$= \lim_{\varepsilon\to 0}\left(\underbrace{\ln\varepsilon^2 \int_{x^2+y^2=\varepsilon^2} \frac{\partial\varphi}{\partial n}\,ds}_{=\mathcal{O}(\varepsilon) \text{ då } \nabla\varphi \text{ är begränsad.}} + 4\pi\,\underbrace{\frac{1}{2\pi\varepsilon}\int_{x^2+y^2=\varepsilon^2} \varphi\,ds}_{\text{medelvärden på cirkeln } \to \varphi(0,0)} \right)$$

$$= 4\pi\,\varphi(0,0)$$

$$\therefore\quad \Delta\Big(\underbrace{\tfrac{1}{4\pi}\ln(x^2+y^2)}_{=\text{fundamentallösning till } \Delta \text{ i } \mathbb{R}^2.}\Big) = \delta$$

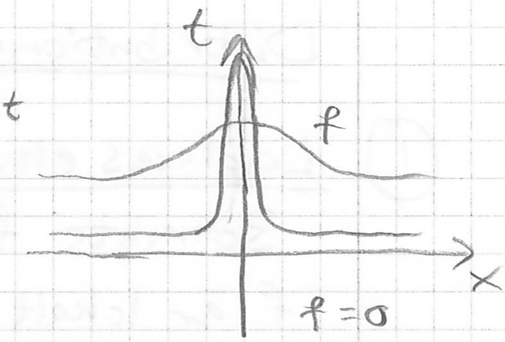
② Värmeledningsekvationen på linjen:

Betrakta

$$f(x,t) = \begin{cases} \frac{1}{\sqrt{t}} e^{-x^2/4t}, & t>0 \\ 0, & t\le 0. \end{cases}$$

f är lokalt integrerbar och definierar en distribution på $\mathbb{R}^2$.

För $t>0$:



$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = -\frac{1/2}{t^{3/2}} e^{-x^2/4t} + \frac{x^2}{4t^{5/2}} e^{-x^2/4t} + \frac{1}{\sqrt{t}} \underbrace{\frac{\partial}{\partial x}\left(\frac{x}{2t} e^{-x^2/4t}\right)}_{= \frac{1}{2t} e^{-x^2/4t} - \frac{x^2}{4t^2} e^{-x^2/4t}} = 0$$

Låt $x \neq 0$. Då $t \to 0^+$ avtar f mycket snabbt mot 0 och man kan visa att f är C^∞ kring $(x,0)$

Vad händer i origo?

$$\left(\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2}\right)[\varphi] = f\left[-\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2}\right] =$$

$$\lim_{\varepsilon \to 0} \iint_{t>\varepsilon} f\left(-\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2}\right) dx\, dt =$$

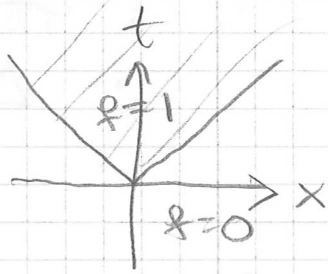
$$= \lim_{\varepsilon \to 0}\left(\int f(x,\varepsilon)\varphi(x,\varepsilon)\, dx + \iint_{t>\varepsilon} \underbrace{\left(\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2}\right)}_{=0} \varphi \, dx\, dt \right)$$

$$= /\, x = 2\sqrt{\varepsilon}\, s \,/$$

$$= \lim_{\varepsilon \to 0} \int \frac{1}{\sqrt{\varepsilon}} e^{-s^2} \underbrace{\varphi(2\sqrt{\varepsilon}\, s, \varepsilon)}_{\to \varphi(0,0)} 2\sqrt{\varepsilon}\, ds = 2 \underbrace{\int_{-\infty}^{\infty} e^{-s^2} ds}_{=\sqrt{\pi}} \cdot \varphi(0,0)$$

$$\therefore \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\Bigg(\underbrace{\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} H(t)}_{\text{fundamental lösning till värmelednings-ekvationen med 1 rumsvariabel.}} \Bigg) = \delta$$

③ Vågekvationen på linjen



Betrakta $f(x,t) = \begin{cases} 1 & , t > |x| \\ 0 & , \text{för övrigt.} \end{cases}$

$$\frac{\partial^2 f}{\partial t^2} = 0 = \frac{\partial^2 f}{\partial x^2} \quad \text{då} \quad t \neq |x|$$

Vad händer vid språnget över $t = |x|$?

$$\left(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2}\right)[\varphi] = f\left[\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2}\right]$$

$$= \iint_{t>|x|} \left(\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2}\right) dt\, dx = \Big/ \begin{matrix} x = u - v \\ t = u + v \end{matrix} \Big/$$

kedjeregeln

$$= \iint_{\substack{u>0 \\ v>0}} \frac{\partial^2 \varphi}{\partial u \partial v} \cdot (2\, du\, dv)$$

$$= 2\int_0^\infty \left(0 - \frac{\partial \varphi}{\partial v}(u,0) \right) dv = 2\varphi(0,0)$$

$$\therefore \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \Big(\underbrace{\tfrac{1}{2} H(t-|x|)} \Big) = \delta$$

fundamentallösning till vågekvation
med 1 rumsvariabel.

Här:

$$H(s) = \begin{cases} 1, & s>0 \\ 0, & s\le 0 \end{cases}$$ Heavisidefunktionen.

$$\delta[\varphi] = \varphi(0)$$ Diracs delta-distribution

$$\varphi \in C_0^\infty$$ testfunktioner.