A mean value formula and a maximum principle for the heat equation

We aim to prove the analogue of Prop. 3.11 and Corollary 3.12 for the heat equation.
Let us first simplify the proof of Prop. 3.11 for harmonic functions, using only the fundamental solution Φ (and not the Poisson kernel).
Prop. 3.5 $\Rightarrow$

$$u(\bar{x}_0) = \int_{\partial D} \left(u(\xi) \frac{\partial}{\partial n} \Phi(\xi - \bar{x}_0) - \Phi(\xi - \bar{x}_0) \frac{\partial}{\partial n} u(\xi) \right) dS(\xi)$$

If $D = B(\bar{x}_0, r)$ is a ball, then $\Phi(\xi - \bar{x}_0)$ is constant on ∂D (why?), and

$$\int_{\partial D} \underbrace{\Phi(\xi - \bar{x}_0)}_{=c} \frac{\partial u}{\partial n} \, dS = c \int_{\partial D} \frac{\partial u}{\partial n} dS = c \iint_D \Delta u \, d\bar{x} = 0.$$

Since $\frac{\partial \Phi}{\partial n}(\xi - \bar{x}_0) = \frac{1}{\sigma r^{n-1}}$ as in proof of Prop. 3.5, it follows that

$u(\bar{x}_0) = $ average of u on $\partial B(\bar{x}_0, r)$.

Now consider the heat equation, and for simplicity assume $k = 1$, so that $\partial_t u = \Delta u$.

The mean value formula above says that for solutions to the Laplace equation
$u(\bar{x}_0) = $ average of u over a level curve of $\Phi(\cdot - \bar{x}_0)$ (that is, a circle).

The mean value formula for the heat equation will use a level curve of the heat kernel

$$S(\bar{x},t) = \frac{1}{(4\pi t)^{n/2}} e^{-|\bar{x}|^2/4t}$$

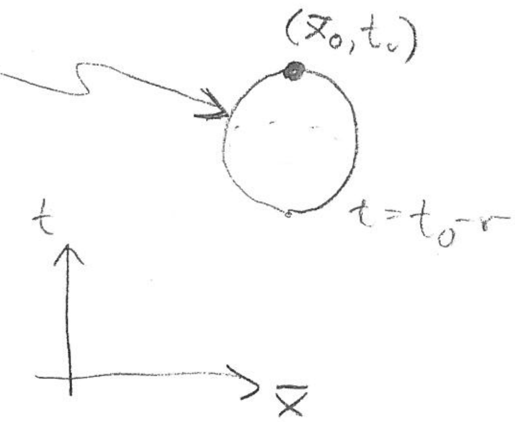
Fix a parameter $r>0$ (a "radius").

$$S(\bar{x},t) = \frac{1}{(4\pi r)^{n/2}} \iff |\bar{x}|^2 = 2nt \ln\frac{r}{t}$$

Mean value formula

Assume that $\partial_t u = \Delta u$ in a neighbourhood of $(\bar{x}_0, t_0)$ in spacetime. Then, for each small enough $r>0$, we have

$$u(\bar{x}_0,t_0) = \frac{1}{(4\pi r)^{n/2}} \iint_{\substack{t_0-r<t<t_0 \\ |\bar{x}-\bar{x}_0| = 2n(t_0-t)\ln\frac{r}{t_0-t}}} \frac{|\bar{x}_0-\bar{x}|^2 \, u(\bar{x},t)\, dS(\bar{x},t)}{\sqrt{4(t_0-t)^2|\bar{x}_0-\bar{x}|^2 + (2n(t_0-t) - |\bar{x}_0-\bar{x}|^2)^2}}$$



Proof:

- We first derive a "parabolic Green's 2nd identity".

Let $u(\bar{x},t)$, $v(\bar{x},t)$ be two functions in a domain D in spacetime.

Define the vector field

$$\bar{f}(\bar{x},t) = u\nabla_x v - v\nabla_x u + e_t u v$$

where $\nabla_x u = (\partial_x u, \partial_y u, \partial_z u)$ is the spatial gradient and e_t is the basis vector along the future time axis.

Gauss' theorem $\Rightarrow$

$$\iint_{\partial D} \bar{n}\cdot\bar{f}\, dS(\bar{x},t) = \iiint_D \operatorname{div}_{\bar{x},t} \bar{f}\, d\bar{x}\,dt$$

If $\bar{n} = (\bar{n}_x, n_t)$, then

$$\bar{n}\cdot\bar{f} = u\frac{\partial v}{\partial \bar{n}_x} - v\frac{\partial u}{\partial \bar{n}_x} + n_t u v, \quad \text{where } \frac{\partial u}{\partial \bar{n}_x} = n\cdot\nabla_x u$$

If $H := \Delta_x - \partial_t$ and $H^* = \Delta_x + \partial_t$, then

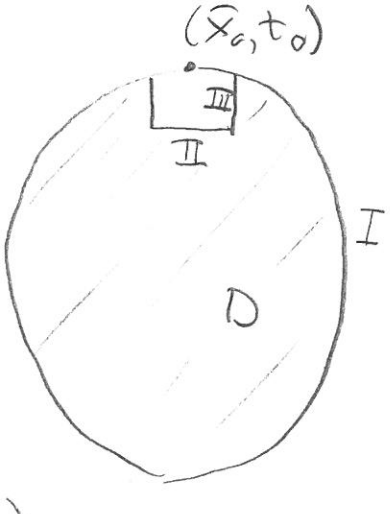
$$\mathrm{div}_{\bar x, t}\, \bar f = \mathrm{div}_{\bar x}(u D_{\bar x} v) - \mathrm{div}_{\bar x}(v D_{\bar x} u) + \partial_t (uv)$$

$$= u\Delta v - v\Delta u + (\partial_t u) v + u\,\partial_t v = u(H^* v) - v(Hu)$$

Therefore

$$(*) \quad \iint_{\partial D} \left(u \frac{\partial v}{\partial \bar n_x} - v \frac{\partial u}{\partial \bar n_x} + n_t u v \right) dS(\bar x, t) = \iiint_D \left(u(H^* v) - v(Hu) \right) d\bar x\, dt$$

- Now we apply this parabolic Green's formula with
  - $\partial_t u = \Delta u$ around $(\bar x_0, t_0)$ $\Rightarrow$ $Hu = 0$
  - $v(\bar x, t) := S(\bar x_0 - \bar x, t_0 - t)$
    $\Rightarrow$ $H^* v = 0$ for $t < t_0$
  - D being the domain



$$\begin{cases} |\bar x - \bar x_0| < 2n(t_0 - t) \ln \frac{r}{t_0 - t} \\ t_0 - r < t < t_0 \\ t < t_0 - \varepsilon \text{ or } |\bar x - \bar x_0| > \varepsilon \end{cases}$$

(Of course, the last condition is in order to avoid the singularity of S at $(\bar x_0, t_0)$.)

- On the boundary I, defined by $|\bar x - \bar x_0|^2 - 2n(t_0 - t) \ln \frac{r}{t_0 - t} = 0$, we have

$$\bar n \parallel \left(2(\bar x - \bar x_0),\ +2n \ln \frac{r}{t_0 - t} - 2n \right)$$

$$\Rightarrow \bar n_x = \frac{\bar x - \bar x_0}{\sqrt{|\bar x - \bar x_0|^2 + \left(\frac{|\bar x - \bar x_0|^2}{2(t_0 - t)} - n \right)^2}}$$

With

$$\nabla_{\bar{x}} v = -\nabla_{\bar{x}} S = +\frac{\bar{x}-\bar{x}_0}{2(t_0-t)} S = / \bar{x} \in \mathrm{I} / = \frac{\bar{x}-\bar{x}_0}{2(t_0-t)} \frac{1}{(4\pi r)^{n/2}},$$

we get

$$\frac{\partial \bar{v}}{\partial \bar{n}_x} = \frac{1}{(4\pi r)^{n/2}} \frac{1}{2(t_0-t)} \frac{|\bar{x}-\bar{x}_0|^2}{\sqrt{|\bar{x}-\bar{x}_0|^2 + \left(n - \frac{|\bar{x}-\bar{x}_0|^2}{2(t_0-t)}\right)}}$$

On the other hand, since v is constant on I, we have

$$\iint_{\partial D} v\left(n_t u - \frac{\partial u}{\partial \bar{n}_x}\right) dS = v \iint_{\partial D} \left(n_t u - \frac{\partial u}{\partial \bar{n}_x}\right) dS$$

$$= v \iiint_D (\partial_t u - \Delta u)\, d\bar{x}\, dt = 0 \quad \text{by Gauss' theorem.}$$

From (*), we get

$$\frac{1}{(4\pi r)^{n/2}} \iint_{\mathrm{I}} \frac{|\bar{x}-\bar{x}_0|^2 u(\bar{x},t) \quad dS(\bar{x},t)}{\sqrt{4|\bar{x}-\bar{x}_0|^2(t_0-t) + (2n(t_0-t) - |\bar{x}-\bar{x}_0|^2)^2}}$$

$$+ \iint_{\mathrm{II}} \left(u \frac{\partial v}{\partial \bar{n}_x} - v \frac{\partial u}{\partial \bar{n}_x} + n_t u v\right) dS(\bar{x},t) + \iint_{\mathrm{III}} \left(u \frac{\partial v}{\partial \bar{n}_x} - v \frac{\partial u}{\partial \bar{n}_x} + n_t u v\right) dS(\bar{x},t)$$

$$= 0.$$

- For II and III, we have

$$\iint_{\mathrm{II}} dS + \iint_{\mathrm{III}} dS \approx \varepsilon^n \quad \text{as } \varepsilon \to 0.$$

u and $\frac{\partial u}{\partial \bar{n}_x}$ are bounded functions

$$v: \quad v = \frac{1}{(4\pi\varepsilon)^{n/2}} e^{-\frac{|\bar{x}-\bar{x}_0|^2}{4\varepsilon}} \le \frac{1}{\varepsilon^{n/2}} \quad \text{on II}$$

$$v = \frac{1}{(4\pi(t_0-t))^{n/2}} e^{-\frac{\varepsilon^2}{4(t_0-t)}} \quad \text{on III}$$

$$\Rightarrow \iint_{\mathrm{II}} v\, dS \le c\, \varepsilon^n \cdot \frac{1}{\varepsilon^{n/2}} \to 0, \quad \text{as } \varepsilon \to 0.$$

$$\iiint_{\mathrm{III}} v\, dS \le c\, \varepsilon^{n-1} \int_0^{\varepsilon} \frac{1}{s^{n/2}} e^{-\varepsilon^2/4s}\, ds = / \frac{\varepsilon^2}{4s} = x, \; dx = -\frac{\varepsilon^2}{4s^2} ds /$$

$$= C\varepsilon^{n-1}\int_{\varepsilon/4}^{\infty}\left(\frac{4x}{\varepsilon^2}\right)^{n/2} e^{-x}\,\frac{\varepsilon^2}{4x}\,dx$$

$$= C\varepsilon\cdot\frac{2^n}{4}\int_{\varepsilon/4}^{\infty} x^{n/2-1}e^{-x}\,dx \to 0 \text{ as } \varepsilon\to 0.$$

- It remeins to consider the terms $u\frac{\partial v}{\partial \bar n_x}$ on II and III

On II: $\bar n_x = \bar 0$, so $\frac{\partial v}{\partial \bar n_x} = 0$.

On III: $\bar n_x = \frac{\bar x_0 - \bar x}{\varepsilon}$, so

$$\frac{\partial v}{\partial \bar n_x} = \bar n_x \cdot \nabla_x v = \frac{\bar x_0-\bar x}{\varepsilon}\cdot\frac{\bar x - \bar x_0}{2(t_0-t)}\,\frac{1}{(4\pi(t_0-t))^{n/2}}e^{-\frac{\varepsilon^2}{4(t_0-t)}}$$

$$= -\frac{2\pi\varepsilon}{(4\pi(t_0-t))^{n/2+1}}e^{-\frac{\varepsilon^2}{4(t_0-t)}}$$

Using that u is continuous at $(\bar x_0, t_0)$, we celculete

$$\iiint_{III} u\frac{\partial v}{\partial \bar n_x}\,dS \approx u(\bar x_0,t_0)\iiint_{III}\frac{\partial v}{\partial \bar n_x}\,dS$$

$$\approx u(\bar x_0,t_0)\underbrace{\sigma\varepsilon^{n-1}}_{\substack{\text{meesure of}\\ \text{sphere in}\\ \text{dimension } n-1}}\underbrace{\int_0^{\varepsilon}\left(-\frac{2\pi\varepsilon}{(4\pi u)^{n/2+1}}e^{-\frac{\varepsilon^2}{4u}}\right)du}_{\text{verticel part of integrel}}$$

$$= \Big/\ \frac{\varepsilon^2}{4v} = u\ \Big/ = -u\cdot\sigma\varepsilon^{n-1}\,2\pi\varepsilon\int_{\varepsilon/4}^{\infty}\left(\frac{v}{\pi\varepsilon^2}\right)^{n/2+1}e^{-v}\,\frac{\varepsilon^2}{4v^2}\,dv$$

$$= -u\,\underbrace{\frac{\sigma}{2\pi^{n/2}}\int_{\varepsilon/4}^{\infty} v^{n/2-1}e^{-v}\,dv}_{\to 1 \text{ can be shown as } \varepsilon\to 0.}$$

(For exemple, if $n=1$:

$$\frac{2}{2\pi^{1/2}}\int_0^{\infty} e^{-v}\frac{dv}{\sqrt v} = 1.)$$

In total, letting $\varepsilon \to 0$, we obtain the mean value formula from the parabolic Green's formula. $\boxtimes$

Let

$$k(\bar{x},t) = \frac{|\bar{x}|^2}{\sqrt{4t^2|\bar{x}|^2 + (2nt - |\bar{x}|^2)^2}}.$$

Then the mean value formula is

$$u(\bar{x}_0,t_0) = \frac{1}{(4\pi r)^{n/2}} \iint_{\partial D_r} k(\bar{x}_0-\bar{x}, t_0-t)\, u(\bar{x},t)\, dS(\bar{x},t).$$

In particular, this applies with $u=1$, so

$$\frac{1}{(4\pi r)^{n/2}} \iint_{\partial D_r} k(\bar{x}_0-\bar{x}, t_0-t)\, dS(\bar{x},t) = 1.$$

Thus the right hand side in the formula is a weighted average of u for some <u>past</u> times $t<t_0$ and some $\bar{x}$.

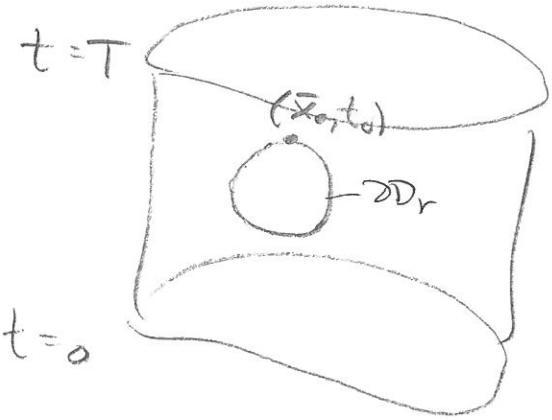
As we did for harmonic functions, we can now deduce the maximum principle for solutions to the heat equation from the mean value formula.

<u>Maximum principle</u>

Let D be an open connected set in $\mathbb{R}^n$ and let $T>0$. Assume that $\partial_t u = \Delta u$ in the spacetime region $D\times(0,T)$ and is continuous on $D\times(0,T]$. Then

- u does not attain a max or min on $D\times(0,T]$, unless
- $u=$ constant.

Proof:



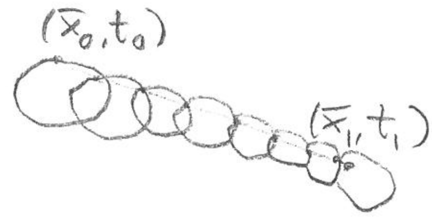
Let
$$M := \sup_{D\times(0,T]} u(\bar{x},t).$$

Assume $u(\bar{x}_0,t_0)=M$ at some $(\bar{x}_0,t_0)\in D\times(0,T]$.

Consider a part domain D_r as in the mean value formula, for any $r>0$ such that $D_r \subset D\times(0,T)$.

We get
$$M = u(\bar{x}_0,t_0) = \underbrace{\text{weighted average of } u \text{ on } \partial D_r}_{\leq M},$$
and a contradiction $M<M$ unless $u=M$ on all ∂D_r.

Repeating this argument shows that $u=M$ in all $D\times(0,T]$.



(It is important to note that the plane $t=t_0$ is tangent to ∂D_r. Why?)

$\blacksquare$