MATEMATISKA VETENSKAPER
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HAND-IN ASSIGNMENTS
Multivectors, spinors and index theorems
HT 2014

Home work 1: exterior algebra

1. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbf{R}^4$, where e_1, e_2 span the z-plane and e_3, e_4 span the w-plane, and consider the 2-surface

$$M = \{(z, w) \in \mathbf{R}^4 \ ; \ w = z^2, |z| < 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\},$$

using standard complex notation. Compute its oriented area $\int_M d\hat{x}$ and its (scalar) area $\int_M |d\hat{x}|$. Discuss the geometric significance of these quantities/coordinates, and relationships between them. Determine if the bivector $\int_M d\hat{x} \in \wedge^2\mathbf{R}^4$ is simple.

2. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis for $\mathbf{R}^4$. Consider an invertible linear map $T : \mathbf{R}^4 \to \mathbf{R}^4$ and its inverse T^{-1}. Express the matrix of 2/2 subdeterminants for T^{-1} in terms of the determinant of T and its 2/2 subdeterminants. Use the basis $\{e_{12}, e_{13}, e_{14}, *e_{14}, *e_{13}, *e_{12}\}$.

3. Proposition 2.28 gives, in particular for $w \in \wedge^2 V$, a criteria for w to be simple in terms of the dimension of a certain subspace of V. State and prove an analogous criteria for tensors in the tensor product $V_1 \otimes V_2$ of two linear spaces V_1 and V_2 to be simple (in terms of the dimension of a certain subspace of either V_1 or V_2). Also briefly explain how to determine the simplicity of a tensor in $V_1 \otimes V_2 \otimes \ldots \otimes V_k$, when $k \geq 3$?

Home work 2: Clifford algebra

1. Let $v_1 = \frac{1}{\sqrt{2}}(e_1+e_2)$ and $v_2 = \frac{1}{3}(2e_1-e_2+2e_3)$ be two unit vectors in a three dimensional euclidean space V with ON-basis $\{e_1, e_2, e_3\}$. Find all bivectors $b \in \wedge^2 V$ such that the rotor/unit quaternion $q = v_1 v_2$ equals $q = e^{b/2}$. Describe the corresponding rotation $v \mapsto qvq^{-1}$ (rotation-axis, angle and sense of rotation).

2. Consider the bivector

$$b = e_{12} + 2e_{13} + 5e_{14} + 5e_{23} - 2e_{24} + e_{34}$$

in a four dimensional euclidean space V with ON-basis $\{e_1, e_2, e_3, e_4\}$. Write b in canonical form $b = b_1+b_2$, where b_1, b_2 are simple and commuting bivectors.

3. Let $\{e_1, e_2\}$ be an ON-basis for the euclidean plane V. Let $\{e_1^+, e_2^+, e_1^-, e_2^-\}$ be the Clifford generators for $\mathcal{L}(\wedge V)$ from Definition 3.33. Do part of Exercise 3.38: Write the left Hodge star operator $w \mapsto *w$ in the Clifford basis $\{e_s^+ e_t^-\}_{s,t\subset\{1,2\}}$.

4. Consider Minkowski spacetime W with two spatial dimensions: We fix ON-basis $\{e_0, e_1, e_2\}$ with $e_0^2 = -1, e_1^2 = e_2^2 = +1$. Find equations for the coordinates of a multivector in the induced basis which describe the non-invertible elements in the Clifford algebra $\triangle V$.

Home work 3: Complex spinor spaces

1. Consider the anti-euclidean plane V, with ON-basis $\{\tilde{e}_1, \tilde{e}_2\}$, $\tilde{e}_1^2 = \tilde{e}_2^2 = -1$. Set $e_{-1} := i\tilde{e}_2$ and $e_1 := i\tilde{e}_1$ and consider the standard representation of the complex spinor space $\not\triangle V$.

   (a) Calculate a sesquilinear spinor duality $\langle \cdot, \cdot \rangle_*$ on $\not\triangle V$. Is it possible to normalize this duality so that it becomes a complex inner product in the sense of Definition 1.31, and if so, what is the signature?

   (b) Calculate a spinor conjugation $\psi^\dagger$ on $\not\triangle V$. Is it possible to normalize this spinor conjugation so that it becomes a real structure on $\not\triangle V$ in the sense discussed after Definition 1.29?

2. Consider the euclidean plane V, with ON-basis $\{\tilde{e}_1, \tilde{e}_2\}$, $\tilde{e}_1^2 = \tilde{e}_2^2 = +1$. Set $e_{-1} := \tilde{e}_2$ and $e_1 := \tilde{e}_1$ and consider the standard representation of the complex spinor space $\not\triangle V$, equipped with the duality and

conjugation from Proposition 5.18. Calculate the two spinor maps $T_S : \not\triangle V \to \not\triangle V$ induced by the linear map $T : V \to V$ with matrix

$$T = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

in the basis $\{\tilde{e}_1, \tilde{e}_2\}$.

Home work 4: Affine multivector calculus

1. Projecting with pullbacks: Do Exercise 7.14.

2. Let $\{e_1, e_2, e_3\}$ be a basis and consider the constant vector field $v(x) = e_1$. Compute the pushed forward vector field $\rho_*(v)$ under

   $$\rho : (x_1, x_2, x_3) \mapsto (x_1, e^{x_1} x_2, e^{x_1} x_3).$$

   Compute the divergence of $\rho_*(v)$ as well as the divergence of the normalized pushed forward field $\tilde{\rho}_*(e_1)$ defined as in Definition 7.17. Make sure that your result agrees with the fundamental commutation theorem!

3. Let $F : D \to \triangle V$ be a multivector field in an oriented three-dimensional euclidean space V. Write $F(x) = \alpha(x) + v(x) + *u(x) + *\beta(x)$, where α, β are scalar functions and v, u are vector fields. Rewrite the Hodge–Dirac equation $\mathbf{D}F = 0$ for monogenic fields in classical vector calculus notation, i.e. as a number of equations involving α, v, u, β and the gradient, curl (the classical vector field!) and divergence operators.

Home work 5: Multivectors and manifolds

1. Do Exercise 6.33: Compute the Christoffel symbols for the Levi-Civita covariant derivative for the tangent bundle TM over a Riemannian manifold M, in a general frame $\{\mathbf{e}_i\}$. In particular, write down the formula for the vector fields ω_{ij}.

2. Outline the details of the proof of Proposition 11.8: The Hodge and L_2 dualities between d_M and δ_M on a Riemannian manifold. We assume that the manifold is compact (without boundary). Your job is to identify all the key steps in the proof.

3. *Poincaré's inequality* for a compact Riemannian manifold M, states there exists a constant $C < \infty$ such that

   $$\int_M |u(p) - u_M|^2 dp \leq C \int_M |\nabla_M u(p)|^2 dp,$$

   for all scalar functions $u \in C^1(M; \mathbf{R})$. Here $u_M := \int_M u dp / \int_M dp$ denotes the average of u over M. Show how such inequality directly follows from the Hodge decomposition in Section 11.5.

Home work 6: Chern–Gauss–Bonnet

1. Let V be euclidean n-dimensional space. Consider the real algebra isomorphism between $\mathcal{L}(\wedge V)$ and $\triangle(V^2)$ from Theorem 3.32, where $V^2 = V \oplus V$ has signature zero. Show that under this isomorphism, we have

   $$\text{Tr}(T) = 2^n T|_{\wedge^0 V^2}$$

   for all $T \in \mathcal{L}(\wedge V) \approx \triangle(V^2)$.

2. Do Exercise 12.12. The integrand should be expressed in terms of $|R|^2 := \sum_{ijkl} R_{ijkl}^2$, $|\text{Ric}|^2 := \sum_{ij} \text{Ric}_{ij}^2$ and scalar curvature S, following the notation in Section 11.3.

3. Compute the three Betti numbers $\beta_0(M)$, $\beta_1(M)$ and $\beta_2(M)$, for the two dimension sphere $M = S^2$ as well as the two dimensional torus $M = S^1 \times S^1$, using Hodge star maps and Gauss–Bonnet's theorem.

Home work 7: Atiyah–Singer

1. Prove that $\not{D}_M$ maps $L_2(M;\not{\Delta}^+M)$ into $L_2(M;\not{\Delta}^-M)$, and vice versa. That is, prove that $\not{D}$ swaps the sub bundles $\not{\Delta}^{\pm}M$.

2. Following the lecture notes (20), and not Definition 12.28 in the book, let $p(t_1, t_2, \ldots, t_{m/2})$ be the polynomial for which
$$(a_1, \ldots, a_m) \mapsto p(\mathrm{Tr}(A^2), \mathrm{Tr}(A^4), \ldots, \mathrm{Tr}(A^m))$$
is the m-homogeneous part in the Taylor expansion of
$$f(a_1, \ldots, a_m) = \frac{a_1/2}{\sin(a_1/2)} \cdot \ldots \cdot \frac{a_m/2}{\sin(a_m/2)}.$$
Here A is the block diagonal $2m/2m$ matrix with diagonal blocks $\begin{bmatrix} 0 & a_i \\ -a_1 & 0 \end{bmatrix}$. Compute p, at least for $m = 2$ and 4 (that is for 4 and 8 dimensional manifolds).

3. Fix $\omega > 0$. Consider the rescaled Mehler kernel
$$K_\omega(t, x, y) := \sqrt{\frac{\omega}{2\pi \sinh(2\omega t)}} \exp\left(\frac{\omega}{\sinh(2\omega t)}\Big(- \cosh(2\omega t)(x^2 + y^2)/2 + xy\Big)\right).$$
Show that, for fixed $y \in \mathbf{R}$, the PDE $\partial_t K_\omega = \partial_x^2 K_\omega - \omega^2 x^2 K_\omega$ holds for $x \in \mathbf{R}$, $t > 0$. Compute also $\lim_{t \to 0^+} \int_{\mathbf{R}} K_\omega(t, x, y) f(y) dy$ for $f \in C_0^\infty(\mathbf{R})$.