Part 1: Introduction to PDEs

We start the course by showing how the three main partial differential equations (PDE), namely the diffusion/heat equation, the wave equation, and Laplace's equation appear.

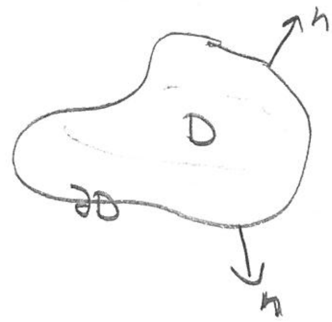
Ex 1.1 We consider diffusion of a substance with density $u(x,y,z,t)$ [mass/volume], depending on spatial variables x,y,z and time t.
We model with Fick's law, which says that the flux J $\left[\frac{\text{mass}}{\text{time}\cdot\text{area}}\right]$ should be

$$J = -k\nabla u = -k(\partial_x u, \partial_y u, \partial_z u),$$

where k is a proportionality constant (large k = fast diffusion).
Conservation of matter gives the equation

$$\underbrace{\partial_t\left(\iiint_D u\, dV\right)}_{\text{increase}} = -\underbrace{\iint_{\partial D} J\cdot n\, dS}_{\text{flow out}} + \underbrace{\iiint_D f\, dV}_{\text{source}},$$

where $f(x,y,z,t)$ $\left[\frac{\text{mass}}{\text{time}\cdot\text{volume}}\right]$ are some given sources of the substance. (If $f<0$ this is a "sink", removing the substance.)



Using Gauss' theorem, we have

$$\iiint_D (\partial_t u - \operatorname{div}(k\nabla u) - f)\, dV = 0.$$

This should hold for any domain D, so we conclude that the diffusion equation

$$\boxed{\partial_t u - k\cdot\Delta u = f}$$ holds for all x,y,z,t.

Here $\Delta u := \operatorname{div}(\nabla u) = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ is the <u>Laplace operator</u>.

Ex 1:2 Consider heat flow in a homogeneous material in three dimensional (3D) space. Proceeding similarly to example 1.1, with $u(x,y,z,t) :=$ temperature and $J :=$ heat flow, we get from Fourier's law $J = -k\nabla u$, and Gauss' law that the heat equation

$$\boxed{\partial_t u - k\Delta u = f}$$

holds for all x, y, z, t.

Defn 1:3: The diffusion equation (or synonymously the heat equation) in n spatial variables is the PDE

$$\partial_t u - k\Delta u = f,$$

for a given source $f(x_1, \dots, x_n, t)$ and unknown function $u(x_1, \dots, x_n, t)$. Here the Laplace operator is $\Delta u = \partial_{x_1}^2 u + \partial_{x_2}^2 u + \dots + \partial_{x_n}^2 u$ and $k > 0$ is a given positive constant.

Some variations:

(1) Diffusion along a one-dimensional (1D) tube, the x-direction is governed by

$$\partial_t u - k\partial_x^2 u = f$$

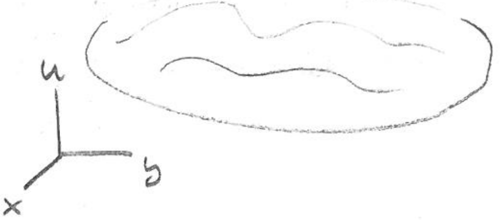
(2) Diffusion along a two-dimensional (2D) plate, the x,y-plane, is governed by $\partial_t u - k(\partial_x^2 u + \partial_y^2 u) = f$.

(3) For heat flow in a non-homogeneous material, $k = k(x,y,z) > 0$ is variable. If the material is anisotropic, then $k = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$ is a positive definite matrix.

The more general form of the heat equation is

$$\partial_t u - \mathrm{div}(k(x,y,z)\nabla u) = f.$$

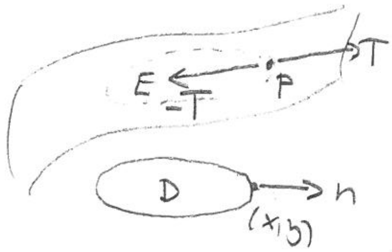
Ex 1.4: Consider a flexible, homogeneous vibrating 2D membrane for example a drum, and its displacement $u(x,y,t)$ [length] at position (x,y) and time t.



We seek a PDE for u.

To this end we consider the stress vector $T=T(n;x,y,t)$ of the membrane, which we now explain.

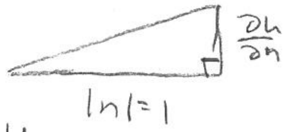
Consider the membrane at some time t and position $p=(x,y,u(x,y,t))$, and an auxiliary domain $D\subset\mathbb{R}^2$ with $(x,y)\in\partial D$, and E = part of membrane above D. The stress vector T is the force density $\left[\frac{\text{force}}{\text{length}}\right]$ with which the rest of the membrane acts on E by at p. (By Newton's third law E acts by $-T$ on the rest of the membrane.).



Under suitable assumptions, in particular that the displacement u and the slope Du are small, we have

horizontal component of T = $k\cdot n$ (outward unit normal to D at (x,y)),

vertical component of T = $k\cdot\frac{\partial u}{\partial n}$,



where $k=k(x,y,t)>0$ and $\frac{\partial u}{\partial n}=n\cdot Du$ is the directional derivative.

We now apply Newton's second law ($F=ma$), assuming the membrane to have density ρ $\left[\frac{\text{mass}}{\text{area}}\right]$.

$$\Rightarrow\begin{cases}\displaystyle\int_{\partial D} k\cdot n\,dl=0 \quad \text{(no horizontal motion)} & (1)\\[2ex] \displaystyle\underbrace{\int_{\partial D} k\cdot\frac{\partial u}{\partial n}\,dl}_{\text{stress force}}+\underbrace{\iint_D f\,dS}_{\text{external force}}=\underbrace{\iint_D \rho\,\partial_t^2 u\,dS}_{\text{mass}\cdot\text{acceleration}} & (2)\end{cases}$$

where $f(x,y,t)$ $\left[\frac{\text{force}}{\text{area}}\right]$ is some given vertical force, i.e. gravitation $f=-\rho\cdot g$.

With integral formulas from vector calculus, we get

$$\begin{cases} \iint_D Dk \, dS = 0, \\ \iint_D \left(\operatorname{div}(k \nabla u) + f - \rho \partial_t^2 u\right) dS = 0. \end{cases}$$

Since D is arbitrary, $Dk = 0 = \operatorname{div}(k \nabla u) + f - \rho \partial_t^2 u$.
Thus k can depend only on t.
Assume for simplicity that the stress k of the membrane is constant in time t.
We see that the displacement u satisfies the wave equation

$$\boxed{\rho \partial_t^2 u - k \Delta u = f} \quad \text{for all } x, y \text{ and } t.$$

Dividing by ρ and rescaling f, we make the following defn.

<u>Defn 1.5</u>: The wave equation in n spatial variables is the PDE

$$\partial_t^2 u - c^2 \Delta u = f,$$

for given constant c and source $f(x_1, \dots, x_n, t)$ and unknown function $u(x_1, \dots, x_n, t)$. Here the Laplace operator is $\Delta u = \partial_{x_1}^2 u + \partial_{x_2}^2 u + \dots + \partial_{x_n}^2 u$.

<u>Some variations</u>:

(1) The displacement of a vibrating 1D string, like a violin string, is governed by the 1D wave equation
$$\partial_t^2 u - c^2 \partial_x^2 u = f.$$

(2) The evolution of acoustic waves (sound) is given by the 3D wave equation
$$\partial_t^2 u - c^2 (\partial_x^2 u + \partial_y^2 u + \partial_z^2 u) = f. \qquad (u = \text{pressure})$$

(3) Vector valued generalizations of the wave equation are important, where u is a vector valued function. For example Maxwell's equations for electromagnetic waves (light etc.) and the elastic wave equation for seismic waves. See part 5.

We now consider stationary waves or diffusion, i.e. $\partial_t u = 0$. It follows that at equilibrium we have the PDE $\Delta u = f$ for $u(x_1, \ldots, x_n)$ time-independent.

<u>Defn 1.6</u>: The Laplace equation in n spatial variables is the PDE

$$\Delta u = 0,$$

for an unknown function $u(x_1, \ldots, x_n)$. The inhomogeneo Laplace equation $\Delta u = f$, for given source $f(x_1, \ldots, x_n)$, is called Poisson's equation.

<u>Some variations</u>:

(1) Laplace equation in 1D is trivial: $\Delta u = \partial_x^2 u = 0$ $\Leftrightarrow$ $u(x) = a + bx$ is linear.

(2) Consider a time-harmonic solution $u(x_1, \ldots, x_n, t) = \bar{u}(x_1, \ldots, x_n) e^{i\omega t}$ to the wave equation. Then

$$\Delta \bar{u} + k^2 \bar{u} = \bar{f},$$

where $k := \frac{\omega}{c}$ is the wave number and $\bar{f} := -\frac{1}{c^2} f$. This is called the Helmholtz equation.

<u>Well posedness (in the sense of Hadamard)</u>

Our main task is to obtain a solution to the PDE, for given data.

<u>Defn 1.7</u> A PDE problem is said to well posed (in the sense of Hadamard) with set of data X and set of solutions Y, if for each data in X there (1) exists a solution in Y, (2) the solution in Y is unique, and (3) the solution in Y depends in a stable way on the data in X.

Note that this means that the map

$Y \ni$ solution $\longmapsto$ data $\in X$

should have a continuous inverse.

We now consider what kind of data which are natural for PDEs. The data set X may in general consist of triples (f, g, h), where

(A) f is a <u>source term</u>, as we already have seen in the heat-, wave- and Poisson equation.

(B) g is some <u>initial data</u>.

Heat equation: $u(x,0) = g(x)$ is some prescribed distribution of heat at time $t=0$.

Wave equation: $\begin{cases} u(x,0) = g_1(x) \\ \partial_t u(x,0) = g_2(x) \end{cases}$ are some prescribed position and velocity of the string (1D) or membrane (2D).

Initial conditions are not relevant to Laplace's equation.

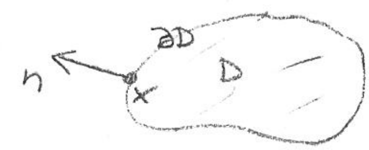
(C) h is some <u>boundary data</u>.

Often the PDE is only assumed to hold in some domain $D \subsetneq \mathbb{R}^n$. In this case, boundary conditions at ∂D need to be specified. The two main conditions are the following.

(D) Dirichlet condition: $u(x) = h(x)$ for $x \in \partial D$.

(N) Neumann condition: $\frac{\partial u}{\partial n}(x) = h(x)$ for $x \in \partial D$.

outward normal derivative.



Note that for time-evolution equations like heat- and wave equation, the boundary condition should hold for each t, and $h = h(x,t)$ may depend on time.

<u>Some variations</u>:

(R) Robin condition: $\frac{\partial u}{\partial n}(x) + a\,u(x) = h(x)$ for $x \in \partial D$ and some positive constant $a > 0$ (or more generally a function $a(x)$)

The Robin condition is also called "impedance boundary condition".

(M) Mixed boundary condition: the boundary is split in two parts $\partial D = E_1 \cup E_2$, and we prescribe

$$u(x) = h_1(x), \quad \text{for } x \in E_1,$$

$$\frac{\partial u}{\partial n}(x) = h_2(x), \quad \text{for } x \in E_2.$$

(T) Transmission condition:

here we look for a pair of solutions to the PDE, u_1 in D and u_2 in $\mathbb{R}^n \setminus \overline{D}$. At ∂D, we have the solution jumps from u_1 to u_2, e.g.

$$u_1(x) - u_2(x) = h_1(x) \quad \text{for } x \in \partial D,$$

$$a \frac{\partial u_1}{\partial n}(x) - b \frac{\partial u_2}{\partial n}(x) = h_2(x) \quad \text{for } x \in \partial D,$$

where a and b are given constants.

<u>Ex. 1.8</u>: Consider heat conduction in a domain D, and assume that there is a thin insulating layer at ∂D and that the exterior domain $\mathbb{R}^n \setminus \overline{D}$ is at a constant temperature u_0.

A good model is that the heat flow across ∂D at $x \in \partial D$ out from D is

$$= \alpha\,(u(x) - u_0).$$

(Newton's law of cooling. $\alpha > 0$ large $\Leftrightarrow$ thin insulation)

By Fourier's law, we get

$$n \cdot J = -k \frac{\partial u}{\partial n} = \alpha (u - u_0)$$

$$\Leftrightarrow \quad k \frac{\partial u}{\partial n} + \alpha u = \alpha u_0.$$

Dividing by k, this is the Robin boundary condition. As special cases:

- No insulation $\Leftrightarrow$ $\alpha = \infty$ $\Leftrightarrow$ $u - u_0 = \frac{k}{\alpha}\frac{\partial u}{\partial n} = 0$ $\Leftrightarrow$ Dirichlet boundary condition $u = u_0$ at ∂D.
- Total insulation $\Leftrightarrow$ $\alpha = 0$ $\Leftrightarrow$ $\frac{\partial u}{\partial n} = 0$ $\Leftrightarrow$ Neumann boundary condition. ∎

Finally a comment on the solution set Y.

In this course we shall require that solutions to the second order PDEs are C^2-regular, so that the PDE makes sense pointwise. (More generally, one can consider socalled weak solution, where the PDE is satisfied in a distributional sense.)

In general, the solution set Y may consist of all functions satisfying three conditions.

(A) u is sufficiently smooth inside D as above.

(B) u is assumed to be sufficiently regular towards the boundary ∂D, so that boundary values like $u|_{\partial D}$ and $\frac{\partial u}{\partial n}\big|_{\partial D}$ make sense.

For evolution equations, the initial values should make sense similarly.

(c) If the domain D is unbounded (like $\mathbb{R}^n \setminus D$ in a transmission problem), one typically need to assume that $u(x) \to 0$ as $x \to \infty$, or satisfies some "radiation condition".

Classification of PDEs

The general form of a PDE is an equation

$$F(x, u, \nabla u, \nabla^2 u, \nabla^3 u, \dots, \nabla^m u) = 0,$$

where $x = (x_1, \dots, x_n)$ and the unknown function is $u = u(x_1, \dots, x_n)$. Here $\nabla^k u = (\partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_k}} u)_{1 \le i_j \le n}$ stands for all k'th order partial derivatives of u.

<u>Defn 1.9</u>: A linear partial differential operator (PDO) with coefficients $a = (a_{i_1, \dots, i_k})_{\substack{1 \le i_j \le n \\ 0 \le k \le m}}$, is a map of the form

$$u \mapsto L_a u = \sum a_{i_1, \dots, i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} u,$$

i.e. a linear combination of partial derivatives of various orders, of u.

We make the following definitions for PDEs:

(1) The order of the PDE is m. ($m \ge 3$ is often referred to as "higher order PDE".)

(2) If $n = 1$, then we speak of an ODE (ordinary differential equation).

(3) If F and u are assumed to be vector-valued, then we speak of a system of PDEs.

(4) If the PDE is of the form

$L_a u = f$, for some linear PDO with coefficients a and f depending only on $x = (x_1, \dots, x_n)$,

then we speak of an (inhomogeneous) linear PDE. If $f = 0$, we say homogeneous.

(5) If in (4), the operator L_a involves only derivates of order m, while the coefficents and f may depend on $x, u, Du, \dots, D^{m-1}u$, but not $D^m u$, then the PDE is called quasilinear. If the coefficents a only depend on x, then the PDE is called semilinear. If the PDE is not quasilinear, it is called fully non-linear.

Ex. 1.10

- The Laplace, heat- and wave equations are linear second order PDEs.
- The 1D Poisson equation $u'' = f$ is an ODE.
- Maxwell's equations are first order linear systems of PDEs.

In this course, we shall only study linear PDEs, except in part 5. Non-linear equations are much harder to understand. It should be noted that when modelling with PDEs, simple models produce a linear PDE. The "real" equation is typically non-linear. For example, in Ex. 1.4, assuming small oscillations lead to the linear wave equation. Nevertheless, this is often a good model.

The focus in the course is on the Laplace equation, the wave equation and the heat equation. We next show that these essentially are all second order linear PDEs.

Consider the PDE

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \partial_{x_i} \partial_{x_j} u = L_b u ,$$ where L_b is a first order linear PDO.

(a_{ij} may depend on x.)

Make the change of variables $y_k = \sum_{j=1}^{n} b_{kj} x_j$.

Chain rule $\Rightarrow$ $\partial_{x_j} u = \sum_{k=1}^{n} b_{kj} \partial_{y_k} u$

$$\therefore \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \sum_{l=1}^{n} b_{li} \partial_{y_l} \Big(\sum_{k=1}^{n} b_{kj} \partial_{y_k} u \Big) = \tilde{L}_b u$$

$$\Leftrightarrow \sum_{kl} \Big(\sum_{ij} b_{li} a_{ij} b_{kj} \Big) \partial_{y_l} \partial_{y_k} u = \tilde{\tilde{L}}_b u$$

some first order PDOs

We may assume that the matrix $A = (a_{ij})$ is symmetric (why?). Thus, by the spectral theorem in linear algebra, we may find an invertible matrix $B = (b_{ij})$ at each point x, so that

$$BAB^T = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$ is a diagonal matrix with elements being all $+1$, -1 or 0.

Defn 1.11:

(1) If A (or $-A$) is positive definite, then the PDE is called elliptic. In this case, after a change of variables, the PDE is the Laplace equation (with lower order terms).

(2) If A (or $-A$) has $n-1$ positive eigenvalues and one negative eigenvalue, then the PDE is called hyperbolic. In this case, after a change of variables, the PDE is the wave equation (with lower order terms).

(3) If A (or $-A$) has $n-1$ positive eigenvalues and one eigenvalue $= 0$, then the PDE is called parabolic.

Ex 1, 12:

(1) The Laplace equation, Poisson equation and Helmholtz equation are elliptic

(2) The wave equation is hyperbolic

(3) The heat equation $\partial_t u = \Delta u$, the backward heat equation $\partial_t u = -\Delta u$ and the n-Laplace equation $\partial_{x_1}^2 u + \dots + \partial_{x_n}^2 u = 0$ in $\mathbb{R}^{1+n}$ are parabolic.