Part 2: Evolution problems for heat- and wave equation on $\mathbb{R}^n$

Our first goal in part 2 of this course is to solve the basic evolution problems for the heat- and wave equations on $\mathbb{R}^n$, for $n=1, 2$ and 3.

$$\begin{cases} \partial_t u = \Delta u & , x \in \mathbb{R}^n, t>0, \\ u(x,0) = \varphi(x) & , x \in \mathbb{R}^n, t=0. \end{cases}$$

$$\begin{cases} \partial_t^2 u = \Delta u , & x \in \mathbb{R}^n, t>0, \\ u(x,0) = \varphi(x), & \\ \partial_t u(x,0) = \psi(x), & x \in \mathbb{R}^n, t=0. \end{cases}$$

We assume that initial data $\varphi(x)$ (and $\psi(x)$) are given, and we seek an explicit formula for the solution $u(t,x)$.

To this end, we want to apply the Fourier transform on $\mathbb{R}^n$ to the function $u(\cdot, t)$, for each fixed $t \geq 0$.

Let us recall the Fourier transform of a function $v(x)$ on $\mathbb{R}^n$. It is the function

$$\hat{v}(\xi) = \mathcal{F}\{v(x)\} := \int_{\mathbb{R}^n} v(x)\, e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n,$$

where $x\cdot\xi = x_1\xi_1 + \dots + x_n\xi_n$ is the inner product.

We have the inversion formula

$$v(x) = \mathcal{F}^{-1}\{\hat{v}(\xi)\} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{v}(\xi)\, e^{+ix\cdot\xi}\, dx, \quad x \in \mathbb{R}^n.$$

To use the Fourier tranform, we need to know what operation on $v(x)$ corresponds to what operation on $\hat{v}(\xi)$. The most important operations on $v(x)$ to us are derivations $v(x) \mapsto \partial_{x_j} v(x)$ and convolutions

$$v(x), w(x) \mapsto (v * w)(x) := \int_{\mathbb{R}^n} v(x-y)\, w(y)\, dy.$$

We recall the following correspondences.

$v(x)$	$v(x)$
$\partial_{x_k} v(x)$	$i\xi_k \hat{v}(\xi)$, $k=1,\dots,n$
$(v * w)(x)$	$\hat{v}(\xi)\cdot\hat{w}(\xi)$

<u>Solution of the heat equation on $\mathbb{R}^n$:</u>

Start with the PDE

$$\partial_t u(x,t) = (\partial_{x_1}^2 + \dots + \partial_{x_n}^2)u(x,t).$$

Fix $t>0$ and apply the Fourier transform in the x-variable.

$$\int_{\mathbb{R}^n} \partial_t u(x,t)\, e^{-ix\cdot\xi}\,dx = \int_{\mathbb{R}^n} (\partial_{x_1}^2 + \dots + \partial_{x_n}^2)u(x,t)\, e^{-ix\cdot\xi}\,dx \;, \quad \xi\in\mathbb{R}^n,$$

Use that ∂_{x_k} correspond to $i\xi_k$:

$$\begin{aligned}\partial_t \hat{u}(\xi,t) &= \left((i\xi_1)^2 + \dots + (i\xi_n)^2\right)\hat{u}(\xi,t)\\ &= -(\xi_1^2 + \dots + \xi_n^2)\hat{u}(\xi,t) = -|\xi|^2\hat{u}(\xi,t).\end{aligned}$$

<u>Note</u>: the Laplace operator corresponds to multiplication by $-|\xi|^2$.

Next we vary $t\geq 0$ and fix $\xi\in\mathbb{R}^n$. Instead of the heat PDE in (x,t), we have equivalently the ODE

$$\partial_t \hat{u}(\xi,t) = -|\xi|^2\hat{u}(\xi,t) \quad \text{in } t\geq 0.$$

Solve as usual this linear ODE with integrating factor:

$$\begin{aligned}\partial_t\left(e^{|\xi|^2 t}\,\hat{u}(\xi,t)\right) &= 0\\ e^{|\xi|^2 t}\,\hat{u}(\xi,t) &= C(\xi)\\ \hat{u}(\xi,t) &= C(\xi)\,e^{-|\xi|^2 t}\end{aligned}$$

The function $C(\xi)$ is obtained from initial conditions:

$$\begin{aligned}u(x,0) &= \varphi(x)\\ \hat{u}(\xi,0) &= \hat{\varphi}(\xi)\\ C(\xi) &= \hat{\varphi}(\xi).\end{aligned}$$

What remains is to find the function $u(x,t)$, whose Fourier transform is

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi) \cdot e^{-|\xi|^2 t}.$$

If $S(x,t)$ is such that

$$\hat{S}(\xi,t) = e^{-|\xi|^2 t}, \text{ then}$$

$$\hat{u}(\xi,t) = \int_{\mathbb{R}^n} S(x-y,t)\,\varphi(y)\,dy$$

gives the solution. To find $S(x,t)$, consider first the 1D problem $(n=1)$. Here

$$S(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\xi^2 t}\, e^{ix\xi}\,d\xi =$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \cdot \int_{-\infty}^{\infty} e^{-(\sqrt{t}\,\xi - \frac{i}{2\sqrt{t}}x)^2}\,d\xi = \Big/ \begin{array}{l} s = \sqrt{t}\,\xi - \frac{i}{2\sqrt{t}}x \\ ds = \sqrt{t}\,d\xi \end{array} \Big/$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{-s^2}\frac{ds}{\sqrt{t}} = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}.$$

Here we used some contour integration and complex analysis.

<u>Note</u>: For $t=\frac{1}{2}$, the function $e^{-x^2/2}$ is an eigenfunction to the Fourier transform with eigenvalue $\sqrt{2\pi}$:

$$F\{e^{-x^2/2}\} = \sqrt{2\pi}\; e^{-\xi^2/2}.$$

In n dimensions, it follows that

$$S(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-|\xi|^2 t}\, e^{ix\cdot\xi}\,d\xi =$$

$$\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\xi_1^2 t} e^{ix_1\xi_1}\,d\xi_1\right)\cdot\ldots\cdot\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\xi_n^2 t} e^{ix_n\xi_n}\,d\xi_n\right)$$

$$= \left(\frac{1}{\sqrt{4\pi t}}\, e^{-x_1^2/4t}\right)\cdot\ldots\cdot\left(\frac{1}{\sqrt{4\pi t}}\, e^{-x_n^2/4t}\right)$$

$$= \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}.$$

With parameter k, the solution to the heat equation $\partial_t u = k\Delta u$, is the following.

Theorem 2.1: The solution to the heat equation $\partial_t u = k \cdot \Delta u$ on $\mathbb{R}^n$, with initial data $u(x,0) = \varphi(x)$, is

$$u(t,x) = \int_{\mathbb{R}^n} \underbrace{\frac{1}{(4\pi k t)^{n/2}} e^{-\frac{|x-y|^2}{4kt}}}_{S(x-y,t)} \varphi(y)\, dy.$$

The function $S(x,t) = \frac{1}{(4\pi k t)^{n/2}} e^{-|x|^2/4kt}$ is called the heat kernel on $\mathbb{R}^n$.

Solution of the wave equation on $\mathbb{R}^n$.

We repeat the above calculation, now for the wave PDE

$$\partial_t^2 u(x,t) = (\partial_{x_1}^2 + \dots + \partial_{x_n}^2) u(x,t).$$

Fix $t \geq 0$ and apply the Fourier transform in the x-variable, to obtain the ODE

$$\partial_t^2 \hat{u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t)$$

in the t-variable. Now fix $\xi \in \mathbb{R}^n$ and vary $t \geq 0$. The ODE $(\partial_t^2 + |\xi|^2)\hat{u} = 0$ is linear second order and constant coefficients, with roots $\pm i|\xi|$ of the characteristic polynomial $\lambda^2 + |\xi|^2 = 0$. Thus, the general solution of the ODE is

$$\hat{u}(\xi,t) = A(\xi)\cos(|\xi|t) + B(\xi)\sin(|\xi|t).$$

The functions $A(\xi)$ and $B(\xi)$ are found from initial conditions:

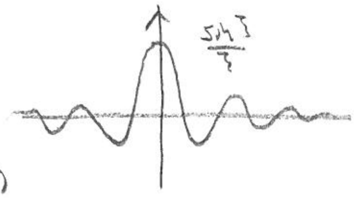
$$\begin{array}{ll} u(x,0) = \varphi(x) & \partial_t u(x,0) = \psi(x) \\ \hat{u}(\xi,0) = \hat{\varphi}(\xi) & \partial_t \hat{u}(\xi,0) = \hat{\psi}(\xi) \\ A(\xi) = \hat{\varphi}(\xi) & |\xi| B(\xi) = \hat{\psi}(\xi). \end{array}$$

This gives

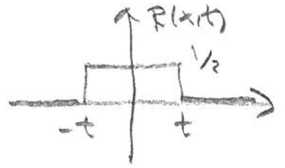
$$\hat{u}(\xi,t) = \hat{\varphi}(\xi)\cos(|\xi|t) + \hat{\psi}(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

We look for a function $R(x,t)$, such that
$$\hat{R}(\xi,t) = \frac{\sin(|\xi| t)}{|\xi|}.$$
Then we get
$$F\{\partial_t R(x,t)\} = \partial_t \frac{\sin(|\xi|t)}{|\xi|} = \cos(|\xi| t),$$
and the solution
$$u(x,t) = \partial_t\left(\int_{\mathbb{R}^n} R(x-y,t)\varphi(y)\,dy\right) + \int_{\mathbb{R}^n} R(x-y,t)\psi(y)\,dy.$$

<u>$n=1$</u>: Consider first the 1D wave equation.
$$\hat{R}(\xi,t) = \frac{\sin(\xi t)}{\xi}.$$

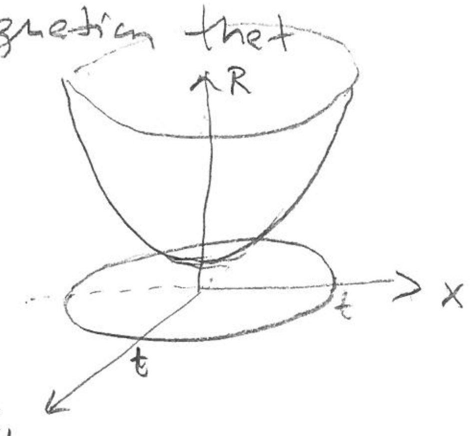


We verify that
$$R(x,t) = \begin{cases} 1/2, & |x|<t, \\ 0, & |x|>t. \end{cases}$$



$$F\{R(x,t)\} = \int_{-t}^{t} \frac{1}{2} e^{-ix\xi}\,dx = \left[\frac{i}{2\xi} e^{-ix\xi}\right]_{-t}^{t} = \frac{e^{it\xi} - e^{-it\xi}}{2i\xi}$$
$$= \frac{\sin(t\xi)}{\xi}.$$

<u>$n=2$</u>: We verify for the 2D wave equation that
$$R(x,t) = \begin{cases} \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}, & |x|<t, \\ 0, & |x|>t. \end{cases}$$



$$F\{R(x,t)\} = \iint_{|x|<t} \frac{e^{-ix\cdot\xi}}{2\pi\sqrt{t^2-|x|^2}}\,dx$$
$= /$ choose basis vectors e_1, e_2 so that $\xi = a\cdot e_1$ for some $a\in\mathbb{R}$ $/$
$$= \iint_{|x|<t} \frac{e^{-ix_1 a}}{2\pi\sqrt{t^2-x_1^2-x_2^2}}\,dx_1\,dx_2 = \int_{-t}^{t}\left(\int_{-\sqrt{t^2-x_1^2}}^{\sqrt{t^2-x_1^2}} \frac{dx_2}{\sqrt{t^2-x_1^2-x_2^2}}\right)\frac{e^{-ix_1 a}}{2\pi}\,dx_1$$
$$= \Big/ \begin{matrix} x_2 = \sqrt{t^2-x_1^2}\, s \\ dx_2 = \sqrt{t^2-x_1^2}\,ds \end{matrix} \Big/ = \int_{-t}^{t}\underbrace{\left(\int_{-1}^{1}\frac{ds}{\sqrt{1-s^2}}\right)}_{=2\arcsin 1 = \pi}\frac{e^{-ix_1 a}}{2\pi}\,dx_1 = \frac{1}{2}\int_{-t}^{t} e^{-ix_1 a}\,dx_1$$
$$= \frac{\sin(at)}{a} = \frac{\sin(|\xi|t)}{|\xi|}.$$

$\underline{n=3}$: For the 3D wave equation, it turns out that $\frac{\sin(|\xi|t)}{|\xi|} \to 0$ too slow as $|\xi| \to \infty$ for there to exist a function $R(x,t)$ with $\hat{R}(\xi,t) = \frac{\sin(|\xi|t)}{|\xi|}$. In this case $R(x,t)$ is a more general distribution. Think of $R(x,t)$ in this case as a function which is $=\infty$ on the sphere $|x|=t$ and $=0$ elsewhere. To avoid these technicalities, we instead look for a function $f(x,t)$ such that

$$\hat{f}(\xi,t) = \hat{\psi}(\xi)\frac{\sin(|\xi|t)}{|\xi|}$$

(assuming $\hat{\psi}(\xi) \to 0$ fast enough as $|\xi| \to \infty$).

We verify that

$$f(x,t) = \frac{1}{4\pi t}\iint_{|y-x|=t} \psi(y)\, dS(y)$$

(surface integral over the sphere with centre x and radius t).

$$F\{f(x,t)\} = \iiint_{\mathbb{R}^3} \left(\frac{1}{4\pi t}\iint_{|y-x|=t} \psi(y)\,dS(y)\right) e^{-ix\cdot\xi}\,dx$$

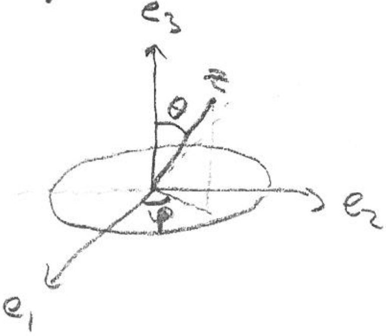
$$= \iiint_{\mathbb{R}^3} \frac{1}{4\pi t}\iint_{|z|=t} \psi(x-z)\,dS(z)\; e^{-ix\cdot\xi}\,dx$$

$= /$change order of integration$/$

$$= \frac{1}{4\pi t}\iint_{|z|=t} \underbrace{\left(\iiint_{\mathbb{R}^3} \psi(x-z)e^{-i(x-z)\cdot\xi}dx\right)}_{=\hat{\psi}(\xi)} e^{-iz\cdot\xi}\,dS(z)$$

$$= \frac{\hat{\psi}(\xi)}{4\pi t}\iint_{|z|=t} e^{-iz\cdot\xi}\,dS(z) =$$ /choose basis vectors e_1, e_2, e_3 so that $\xi = ae_3$ for some $a \in \mathbb{R}$/

$$= \frac{\hat{\psi}(\xi)}{4\pi t}\int_0^{2\pi}\int_0^{\pi} e^{-iat\cos\theta}\, t^2\sin\theta\, d\theta\, d\varphi$$



$$= \frac{t}{2}\hat{\psi}(\xi)\int_0^{\pi} \sin\theta\, e^{-iat\cos\theta}\,d\theta$$

$$= \hat{\psi}(\xi)\left[\frac{-i}{2a}e^{-iat\cos\theta}\right]_0^{\pi} =$$

$$= \hat{\psi}(\xi) \frac{e^{iat} - e^{-iat}}{2ia} = \hat{\psi}(\xi) \frac{\sin(at)}{a} = \hat{\psi}(\xi) \frac{\sin(|\xi|t)}{|\xi|}.$$

With parameter c, the solution to the wave equation $\partial_t^2 u = c^2 \Delta u$, is the following.

<u>Theorem 2.2</u> The solution to the wave equation $\partial_t^2 u = c^2 \Delta u$, with initial data $u(x,0) = \varphi(x)$ and $\partial_t u(x,0) = \psi(x)$, is

<u>$n=1$</u>: $$u(x,t) = \frac{1}{2}\left(\varphi(x+ct) + \varphi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

(d'Alembert's formula)

<u>$n=2$</u>: $$u(x,t) = \partial_t\left(\iint_{|y-x|<ct} \frac{\varphi(y)\,dy}{2\pi c\sqrt{c^2t^2 - |x-y|^2}} \right) + \iint_{|y-x|<ct} \frac{\psi(y)\,dy}{2\pi c\sqrt{c^2t^2 - |x-y|^2}}$$

<u>$n=3$</u>: $$u(x,t) = \partial_t\left(\frac{1}{4\pi c^2 t} \iint_{|y-x|=ct} \varphi(y)\,dS(y) \right) + \frac{1}{4\pi c^2 t} \iint_{|y-x|=ct} \psi(y)\,dS(y).$$

(Kirchhoff's formula)

<u>Note</u>:

- For $n=1$, we used $\partial_t \frac{1}{c}\int_{x-ct}^{x+ct} \varphi(s)\,ds = \varphi(x+ct) + \varphi(x-ct)$.
- Both the formulas for $n=2$ and $n=3$ use 2D integrals!
- For even dimensions $n=4,6,8,\ldots$, solution formulas are similar to $n=2$, but with higher t-derivatives in front of the integrals.
- For odd dimensions $n=5,7,9,\ldots$, solution formulas are similar to $n=3$, but with higher t-derivatives in front of the integrals.
- The central function/distribution $R(x,t)$ is called "the Riemann function".

The inhomogeneous equations:

Before going further, let us see how to use the Fourier transform to solve the PDE with sources $f(x,t)$.

The inhomogeneous heat equation:

We want to solve $\begin{cases} \partial_t u - k\Delta u = f & , \quad t>0, \\ u = \varphi & , \quad t=0. \end{cases}$

Proceeding as before, we get

$$\partial_t \hat{u}(\xi,t) + k|\xi|^2 \hat{u}(\xi,t) = \hat{f}(\xi,t)$$

$$\partial_t\left(e^{kt|\xi|^2}\hat{u}(\xi,t)\right) = e^{kt|\xi|^2}\hat{f}(\xi,t)$$

$$e^{kt|\xi|^2}\hat{u}(\xi,t) = C(\xi) + \int_0^t e^{ks|\xi|^2}\hat{f}(\xi,s)\,ds$$

$$\hat{u}(\xi,t) = \underbrace{C(\xi)}_{=\hat{\varphi}(\xi)} e^{-kt|\xi|^2} + \int_0^t e^{-(t-s)k|\xi|^2}\hat{f}(\xi,s)\,ds$$

$$\therefore u(x,t) = \int_{\mathbb{R}^n} S(x-y,t)\varphi(y)\,dy + \int_0^t \int_{\mathbb{R}^n} S(x-y,t-s) f(y,s)\,dy\,ds$$

The inhomogeneous wave equation:

We want to solve $\begin{cases} \partial_t^2 u - c^2\Delta u = f, & t>0 \\ u = \varphi & , \quad t=0 \\ \partial_t u = \psi & , \quad t=0 \end{cases}$

Proceeding as before, we get

$$\partial_t^2 \hat{u}(\xi,t) + c^2|\xi|^2\hat{u}(\xi,t) = \hat{f}(\xi,t)$$

$$\hat{u}(\xi,t) = A(\xi)\cos(c|\xi|t) + B(\xi)\sin(c|\xi|t) + \hat{u}_p(\xi,t),$$

where $\hat{u}_p$ is some particular solution to the ODE.

One checks that

$$\hat{u}_p(\xi,t) = \int_0^t \frac{\sin((t-s)c|\xi|)}{c|\xi|}\hat{f}(\xi,s)\,ds \quad \text{is a solution.}$$

As before $A(\xi) = \hat{\varphi}(\xi)$ and $B(\xi) = \dfrac{\hat{\psi}(\xi)}{c|\xi|}$.

(Another way: Laplace transformation in t.)

We get the solution

$$u(x,t) = \partial_t \int_{\mathbb{R}^n} R(x-y,t)\varphi(y)\,dy + \int_{\mathbb{R}^n} R(x-y,t)\psi(y)\,dy$$

$$+ \int_0^t \int_{\mathbb{R}^n} R(x-y,t-s) f(y,s)\,dy\,ds.$$

In dimension $n=1,2$, $R(x,t)$ is the function $R = \frac{1}{2c}$ and $R = \frac{1}{2\pi c\sqrt{c^2t^2 - |x|^2}}$ for $|x| < ct$ and $R = 0$ for $|x| > ct$ respectively. For $n=3$, we have the solution

$$u(x,t) = \partial_t \left(\frac{1}{4\pi c^2 t} \iint_{|x-y|=ct} \varphi(y)\,dS(y) \right) + \frac{1}{4\pi c^2 t} \iint_{|x-y|=ct} \psi(y)\,dS(y)$$

$$+ \int_0^t \left(\frac{1}{4\pi c^2 (t-s)} \iint_{|x-y|=c(t-s)} f(y,s)\,dS(y) \right) ds.$$

Note the pattern: knowing a solution formula $\int_{\mathbb{R}^n} h(x-y,t)\varphi(y)\,dy$ for initial data gives the solution formula $\int_0^t \int_{\mathbb{R}^n} h(x-y,t-s) f(y,s)\,dy\,ds$ for sources.
This fact is called Duhamel's principle.

We next study solutions to the PDEs based on obtained solution formulas.

Ⓐ Propagation speed:

Consider initial data φ such that $\varphi(z) = 0$ if $|z-y| > \varepsilon$ and $\int_{\mathbb{R}^n} \varphi(z)\,dz = 1$, where $\varepsilon \approx 0$.
(The Dirac delta distribution.)
Question: for which (x,t), $t > 0$, is the solution $u(x,t) \neq 0$?
The set of such (x,t) is called the domain of influence of $(y,0)$. Conversely, the domain of dependence of (x,t) is the set of all $(y,0)$ for

which (x,t) belong to the domain of influence of $(y,0)$.

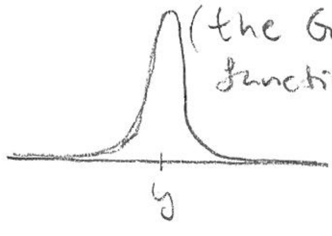
<u>Proposition 2.3</u> The domain of influence of $(y,0)$ is

(1) the future cone $\{(x,t);\ |x-y|\le ct,\ t>0\}$ for the wave equation

(2) the full future $\{(x,t);\ x\in\mathbb{R}^n,\ t>0\}$ for the heat equation.

This is clear from the solution formulas, and means that the propagation speed for solutions to the wave equation is $\le c$, so the parameter c has the meaning of propagation speed. On the other hand, the propagation speed for solutions to the heat equation is infinite (which clearly indicates that the heat equation is only an ideal model of real diffusion / heat conduction) since

$e^{-|x-y|^2/4kt} > 0$ for all x (although <u>very</u> little heat propagates fast).



(the Gauss function.)

Actually, the above result for the wave equation can be improved when dimension $n=3$.

<u>Prop. 2.3.</u> (cont') The domain of influence of $(y,0)$ is the surface of the cone:

$\{(x,t);\ |x-y|=ct,\ t>0\}$

for the <u>3D</u> wave equation.

This is also true in odd dimension $n=5,7,9,\ldots$ and half-true in dimension $n=1$ (true for φ, but not ψ), but <u>not</u> true in even dimension $n=2,4,6,\ldots$.

This phenomenon for 3D waves is called Huygens' principle, which says that 3D waves travels at exactly speed $= c$. A 2D world is much noisier than our 3D world!

Ⓑ Well posedness

Existence of solutions to the heat- and wave equations, with given initial data and sources follows from the solution formulas. Note that the formulas give a solution for any data and sources which are bounded ($|\psi(x)| \leq C$ for all x, etc.) and sufficiently regular.

Also the uniqueness of solutions and continuous dependence in appropriate sense follows from our computations. However, we shall now give a more direct proof of this.

Prop. 2.4: Let $u(x,t)$ be a solution to the wave equation $\partial_t^2 u = c^2 \Delta u$ for $t>0$, $x \in \mathbb{R}^n$. Define the (total) energy at time t to be

$$E(t) := \int_{\mathbb{R}^n} \left((\partial_t u)^2 + c^2 |\nabla u|^2 \right) dx.$$

Then $E(t)$ is a constant independent of t, i.e. $\frac{dE}{dt} = 0$.

Proof:

$$\frac{d}{dt} \int_{\mathbb{R}^n} (\partial_t u)^2 = \int_{\mathbb{R}^n} \partial_t u \cdot \partial_t^2 u + \partial_t^2 u \cdot \partial_t u = \int_{\mathbb{R}^n} 2\, \partial_t u\, \partial_t^2 u$$

$$= \int_{\mathbb{R}^n} 2\, \partial_t u \cdot c^2 \Delta u$$

By Gauss' theorem $0 = \int_{\mathbb{R}^n} \operatorname{div}(u \nabla u) = \int_{\mathbb{R}^n} |\nabla u|^2 + u \Delta u$,

so

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 = - \int_{\mathbb{R}^n} \partial_t u \cdot \Delta u + u \cdot \underbrace{(\partial_t \Delta u)}_{= \Delta(\partial_t u)} = \text{/integrate by parts twice/}$$

$$= -2 \int_{\mathbb{R}^n} \partial_t u \cdot \Delta u. \quad \boxed{}$$

From Prop. 2.4 follows uniqueness and stability of solutions in the following sense.

Assume that $\begin{cases} \partial_t^2 u_i = c^2 \Delta u_i & , t>0, \\ u_i = \varphi_i & , t=0, \\ \partial_t u_i = \psi_i & , t=0, \end{cases}$ for $i=1,2$.

Then the difference $v := u_1 - u_2$ satisfies

$$\begin{cases} \partial_t^2 v = c^2 \Delta v & , t>0, \\ v = \varphi_1 - \varphi_2 & , t=0, \\ \partial_t v = \psi_1 - \psi_2 & , t=0, \end{cases}$$

and Prop. 2.4 gives

$$\int_{\mathbb{R}^n} \left((\partial_t (u_1 - u_2))^2 + c^2 |\nabla (u_1 - u_2)|^2 \right) dx$$
$$= \int_{\mathbb{R}^n} \left((\psi_1 - \psi_2)^2 + c^2 |\nabla (\varphi_1 - \varphi_2)|^2 \right) dx$$

for all $t>0$. Uniqueness and stability follow.

The analogous energy method for the heat equation is the following.

<u>Prop. 2.5</u>: Let $u(x,t)$ be a solution to the heat equation $\partial_t u = k \Delta u$. Define the energy at time t to be $E(t) := \int_{\mathbb{R}^n} u^2 \, dx$.

Then E is a decreasing function of t, i.e. $\frac{dE}{dt} \leq 0$.

<u>Proof</u>: $\frac{d}{dt} \int_{\mathbb{R}^n} u^2 = \int_{\mathbb{R}^n} 2u \underbrace{\partial_t u}_{= k\Delta u} = 2k \int_{\mathbb{R}^n} u \Delta u =$

$= /\text{Gauss' thm}/ = -2k \int_{\mathbb{R}^n} |\nabla u|^2 \leq 0.$ ∎

Similar to above, uniqueness and stability follows.

Assume $\begin{cases} \partial_t u_i = k \Delta u_i & , t>0, \\ u_i = \varphi_i & , t=0, \end{cases}$ $i=1,2$.

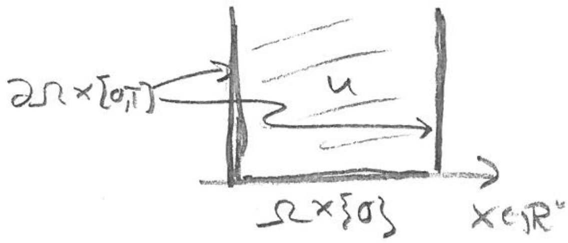
Then $v := u_1 - u_2$ satisfies $\begin{cases} \partial_t v = k \Delta v & , t>0, \\ v = \varphi_1 - \varphi_2 & , t=0, \end{cases}$

and Prop. 2.5 gives

$$0 \le \int_{\mathbb{R}^n} (u_1 - u_2)^2 dx \le \int_{\mathbb{R}^n} (\varphi_1 - \varphi_2)^2 dx \quad \text{for all } t > 0.$$

Another basic result for the heat equation is the following maximum principle.

<u>Prop. 2.6</u>: Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $T > 0$. If $u(x,t)$ is a solution to the heat equation $\partial_t u = h \Delta u$ for $x \in \Omega$, $0 < t < T$, and is continuous on $\overline{\Omega} \times [0,T]$, then u attains its max and min on $\Omega \times \{0\}$ or on $\partial\Omega \times [0,T]$.



<u>Proof</u>: Consider first the maximum. Since u is continuous on the compact set $\overline{\Omega} \times [0,T]$, there exist $p \in \overline{\Omega} \times [0,T]$ such that
$M = u(p) \ge u(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,T]$.
We must show that such p can be found on $\Omega \times \{0\} \cup \partial\Omega \times [0,T]$.
To this end, let $\varepsilon > 0$ and define $v(x,t) := u(x,t) - \varepsilon t$.
Then $\partial_t v - h \Delta v = \partial_t u - h \Delta u - \varepsilon = -\varepsilon < 0$.
Let p_ε be the maximum of v on the compact set $\overline{\Omega} \times [0, T-\varepsilon]$.
<u>Claim</u>: p_ε must lie on $\Omega \times \{0\}$ or $\partial\Omega \times [0, T-\varepsilon]$.
To show this, assume that $p_\varepsilon \in \Omega \times (0,T)$.
Then $\partial_t v = 0$ and $\Delta v = \partial_{x_1}^2 v + \dots + \partial_{x_n}^2 v \le 0$ since p_ε is a max. But then the heat eq. gives the contradiction $0 \le \partial_t v - h \Delta v < 0$. Similar contradiction comes if $p_\varepsilon \in \Omega \times \{T - \varepsilon\}$, since then $\partial_t v \ge 0$ and $\Delta v \le 0$.

We get a sequence p_ε in the compact set $\Omega \times \{0\} \cup \partial\Omega \times [0,T]$. Let p be a limit point of some subsequence.

For any $\varepsilon > 0$ and $x \in \bar{\Omega}$, $0 \le t \le T-\varepsilon$,
we had

$u(x,t) \le v(x,t) + \varepsilon T \le v(p_\varepsilon) + \varepsilon T \le u(p_\varepsilon) + \varepsilon T$.

Letting $\varepsilon \to 0$, we get

$u(x,t) \le u(p)$ for all $(x,t) \in \bar{\Omega} \times [0,T]$.

The result for the minimum is obtained by applying the above result to $-u$. ∎

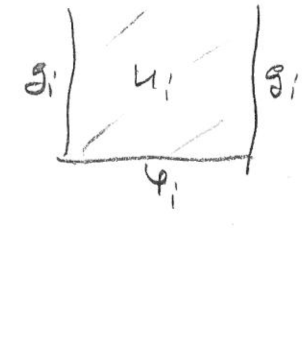
We shall return to this max. principle in part 3 and prove a stronger form of it, after having studied the (simpler) max. principle for the Laplace equation.

The maximum principle implies uniqueness and stability of the initial value problem for the heat equation on a domain Ω with Dirichlet boundary conditions, in the following sense.

Assume
$$\begin{cases} \partial_t u_i - k\Delta u_i = 0 & , t > 0, x \in \Omega, \\ u_i = \varphi_i & , t = 0, x \in \Omega, \\ u_i = g_i & , t > 0, x \in \partial\Omega, \quad i = 1,2. \end{cases}$$

Then $v := u_1 - u_2$ solves $\partial_t v - k\Delta v = 0$
and Prop. 2.6 shows

$$\sup_{\substack{x\in\Omega \\ t>0}} |u_1(x,t) - u_2(x,t)| \le \max\Big(\sup_{x\in\Omega} |\varphi_1(x) - \varphi_2(x)|, \sup_{\substack{x\in\partial\Omega \\ t>0}} |g_1(x,t) - g_2(x,t)| \Big)$$



(c) Smoothing and reversibility:

Given initial data at $t=0$, consider now the PDE for $t<0$.

Question: Can we solve the PDE backward in time?

Making the change of variable $t \to s=-t$, the PDEs transform

$$\partial_t u - k\Delta u = 0 \quad \longrightarrow \quad \partial_s u + k\Delta u = 0,$$

$$\partial_t^2 u - c^2\Delta u = 0 \quad \longrightarrow \quad \partial_s^2 u - c^2\Delta u = 0,$$

so we expect different answers to the question for the heat- and wave equations.

(The heat equation $\partial_t u - k\Delta u = 0$ with $k<0$ is called the backward heat equation.)

Prop 2.7: The initial value problem for the heat equation $\partial_t u - k\Delta u = 0$, $k>0$, is

(1) well-posed for $t>0$, and the solution u is C^∞-smooth for $t>0$.

(2) not well-posed for $t<0$. Solutions may not exist even for very nice initial data.

Proof (1) Smoothness is clear from Thm. 2.1 (differentiate under the integral sign.)

(2) Consider for example the initial data

$$\varphi(x) := S(x, \varepsilon) \quad \text{for some } \varepsilon>0 \text{ fixed.}$$

The solution $u(x,t) = S(x, t+\varepsilon)$ blows up at $(0, -\varepsilon)$. ∎

Prop. 2.8: The initial value problem for the wave equation $\partial_t^2 u - c^2 \Delta u = 0$ is well posed for both $t > 0$ and $t < 0$, but solutions are no smoother than the initial data in either case.

Proof: By inspection of the proof of Thm 2.2, wellposedness of the backward wave equation is clear. Indeed

$$\hat{u}(\xi,t) = \hat{\varphi}(\xi)\cos(ct|\xi|) + \hat{\psi}(\xi)\frac{\sin(ct|\xi|)}{c|\xi|}$$

gives a solution also for $t<0$.

Note that the dependence on φ should be even in t, and the dependence on ψ should be odd in t.

To see that solution are in general not smoother than initial data, consider for example the d'Alembert solution

$$u(x,t) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct))$$

with $n=1$, $\psi=0$. This solution u is a superposition of two waves with same shape as φ, one travelling to the left and one to the right

∎