Part 3: Boundary value problems for the Laplace equation

Our first goal in part 3 of this course is to solve, on $\mathbb{R}^n$ ($n=2$ or 3), the Poisson equation

$$\Delta u(x) = f(x) \quad , \qquad x \in \mathbb{R}^n,$$

Recall from part 1 that this means that we ask for the equilibrium function $u(x)$, given external sources (heat/diffusion eq.) or external forces (wave equation) $f(x)$.

As in part 2, we apply the Fourier transform on $\mathbb{R}^n$, giving us

$$-|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$$

$$\Longleftrightarrow \quad \hat{u}(\xi) = -\frac{1}{|\xi|^2}\hat{f}(\xi)$$

After inverse Fourier transformation, we get

$$\Delta u = f \quad \Longleftrightarrow \quad u(x) = \Phi * f(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy.$$

<u>Definition 3.1:</u>

The function $\Phi(x)$ such that $F\{\Phi(x)\} = -\frac{1}{|\xi|^2}$, is called <u>the fundamental solution for Δ on $\mathbb{R}^n$</u>.

<u>Theorem 3.2:</u>

$$\Phi(x) = \frac{1}{2\pi} \ln|x| \qquad \text{for } \mathbb{R}^2$$

$$\Phi(x) = -\frac{1}{4\pi|x|} \qquad \text{for } \mathbb{R}^3$$

Proof: Recall that for the heat equation, we had the Fourier tranform, for each fixed $t>0$,

$$F\left\{ \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \right\} = e^{-t|\xi|^2}$$

We find $\Phi(x)$ by integrating both sides for $t \in (0,R)$:

$$\int_0^R \left(\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} e^{-ix\cdot\xi} dx \right) dt = \int_0^R e^{-t|\xi|^2} dt$$

$$\int_{\mathbb{R}^n} \underbrace{\left(\int_0^R \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}} dt \right)} e^{-ix\cdot\xi} dx = \frac{1-e^{-R|\xi|^2}}{|\xi|^2}$$

$$= \Big/ \begin{matrix} t = \frac{|x|^2}{4s} \\ dt = -\frac{|x|^2}{4s^2} ds \end{matrix} \Big/ = \int_{|x|^2/4R}^{\infty} \frac{e^{-s}}{(\pi|x|^2/s)^{n/2}} \frac{|x|^2}{4s^2} ds$$

$$= \frac{1}{4\pi^{n/2}} \frac{1}{|x|^{n-2}} \int_{|x|^2/4R}^{\infty} s^{n/2-2} e^{-s} ds.$$

Write $g(y) := \int_y^\infty s^{n/2-2} e^{-s} ds$. Then we have shown

$$F\left\{ \frac{1}{4\pi^{n/2}} \frac{1}{|x|^{n-2}} g\left(\frac{|x|^2}{4R}\right) \right\} = \frac{1-e^{-R|\xi|^2}}{|\xi|^2}.$$

For $n=3$: Then $g\left(\frac{|x|^2}{4R}\right) \xrightarrow{R\to\infty} \int_0^\infty s^{-1/2} e^{-s} ds$

$$= \Big/ \begin{matrix} s = u^2 \\ ds = 2u\,du \end{matrix} \Big/ = \int_0^\infty \frac{1}{u} e^{-u^2} 2u\,du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi},$$

so $F\left\{ \frac{1}{4\pi|x|} \right\} = \frac{1}{|\xi|^2}$ and $\Phi(x) = -\frac{1}{4\pi|x|}$ follows.

For $n=2$: Then

$$F\left\{ \frac{1}{4\pi} \int_{|x|^2/4R}^{\infty} e^{-s} \frac{ds}{s} \right\} = \frac{1-e^{-R|\xi|^2}}{|\xi|^2}.$$

Now $g\left(\frac{|x|^2}{4R}\right) \to \infty$ as $R \to \infty$. (right?)

One way to solve this convergence problem is differentiate:

$$F\Big\{ \underbrace{\partial_{x_k} \frac{1}{4\pi} g\Big(\frac{|x|^2}{4R}\Big)}_{} \Big\} = i\xi_k \, \frac{1-e^{-R|\xi|^2}}{|\xi|^2}$$

$$= \frac{1}{4\pi} \frac{x_k}{2R} \underbrace{g'\Big(\frac{|x|^2}{4R}\Big)}_{= -\frac{4R}{|x|^2} e^{-\frac{|x|^2}{4R}}}$$

$$\Rightarrow F\Big\{ \frac{1}{2\pi} \frac{x_k}{|x|^2} e^{-\frac{|x|^2}{4R}} \Big\} = i\xi_k \Big(-\frac{1}{|\xi|^2}\Big)\big(1-e^{-R|\xi|^2}\big), \quad k=1,2.$$

Letting $R \to \infty$, we get

$$F\Big\{ \frac{1}{2\pi} \frac{x_k}{|x|^2} \Big\} = i\xi_k \Big(-\frac{1}{|\xi|^2}\Big), \text{ so}$$

$$\nabla \Phi(x) = \frac{1}{2\pi} \frac{x_k}{|x|^2} \quad \text{and after integration}$$

$$\Phi(x) = \frac{1}{2\pi} \ln |x|.$$

A final remark: if you know about distribution theory, the above limits of functions as $R \to \infty$ hold in the sense of distributions. ∎

The fundamental solution has following properties.

- It is locally integrable: The singularity at $x=0$ is weak enough so that $\int_{|x|<1} |\Phi(x)|\,dx < \infty$.
- It decays slowly at $x=\infty$, in fact if $n=2$ then $\Phi(x) \to \infty$ as $|x| \to \infty$. In any dimension $$\int_{|x|>1} |\Phi(x)|\,dx = \infty.$$
- For $x \neq 0$, we have $\Delta \Phi(x) = 0$. (See next page.)
- In any dimension n: $\nabla \Phi = \frac{1}{\text{area of unit sphere}} \cdot \frac{x}{|x|^n}$.

Since Φ is a radial function, you can easily verify that $\Delta\Phi = 0$ away from the origin by using the following expressions for Δ in polar coordinates (do this!).

Proposition 3.3:

$n=2$: In polar coordinates $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$, we have

$$\Delta u = \partial_x^2 u + \partial_y^2 u = \left(\partial_r^2 u + \frac{1}{r}\partial_r u\right) + \frac{1}{r^2}\left(\partial_\varphi^2 u\right).$$

$n=3$: In spherical coordinates $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$,

we have

$$\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u = \left(\partial_r^2 u + \frac{2}{r}\partial_r u\right) +$$
$$+ \frac{1}{r^2}\left(\partial_\theta^2 u + \frac{1}{\tan\theta}\partial_\theta u + \frac{1}{\sin^2\theta}\partial_\varphi^2 u\right).$$

Proof: Use the chain rule as in calculus: compute the Jacobian matrix $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$, invert it to get the matrix $\frac{\partial(r,\theta,\varphi)}{\partial(x,y,z)}$, and use the latter when applying the chain rule. ∎

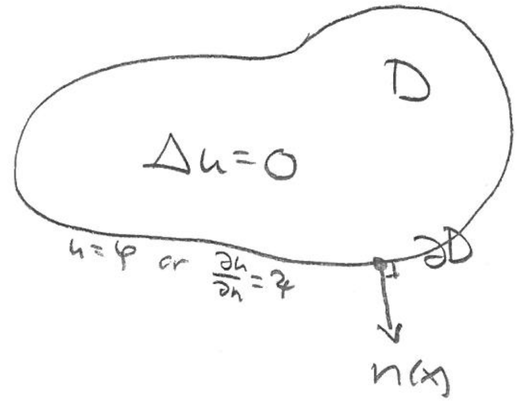
Note that by using the chain rule, we have the following alternative expressions for Δ in polar coordinates:

$n=2$: $\Delta u = \frac{1}{r}\partial_r(r u) + \frac{1}{r^2}\partial_\varphi^2 u,$

$n=3$: $\Delta u = \frac{1}{r^2}\partial_r(r^2 u) + \frac{1}{r^2\sin^2\theta}\left(\sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta u) + \partial_\varphi^2 u\right).$

We now come to the main goal in part 3: to solve the Laplace equation on a given domain $D \subset \mathbb{R}^n$, with given boundary conditions.

Our problem is the following: We are given an open subset $D \subset \mathbb{R}^n$ with boundary ∂D.
At a point $x \in \partial D$, we denote by $n(x)$, the outward pointing (into $\mathbb{R}^n \setminus \bar{D}$), unit ($|n(x)| = 1$) normal ($n(x) \perp \partial D$) vector.
We want to find the/a function $u(x)$ in D which is harmonic, that is $\Delta u(x) = 0$, $x \in D$, and satisfies one of the following two boundary conditions.



Dirichlet: $u(x) = \varphi(x)$ for each $x \in \partial D$, where $\varphi : \partial D \to \mathbb{R}$ is a given function.

Neumann: $\frac{\partial u}{\partial n}(x) = \psi(x)$ for each $x \in \partial D$, where $\psi : \partial D \to \mathbb{R}$ is a given function.

Here the normal derivative

$$\frac{\partial u}{\partial n} = \partial_n u := n(x) \cdot Du(x)$$

denotes the directional derivative of u in the direction $n(x)$.

Our tools will be
- the fundamental solution $\Phi(x)$, and
- Green's (second) identity:

Prop. 3.4: For two functions $u(x)$ and $v(x)$ in D, we have

- Green's first identity:

$$\iint_D (u\,\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial D} u \frac{\partial v}{\partial n}\, dS(y)$$

- Green's second identity:

$$\iint_D (u\,\Delta v - v\,\Delta u)\, dx = \int_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS(y).$$

Proof: Recall from vector calculus: the 1st identity is just Gauss' divergence theorem applied to the vector field $u \nabla v$.
The 2nd identity follows by subtracting the corresponding identity obtained from $v \nabla u$. ∎

We need a Green's 2nd identity for u being some function in D (usually harmonic) and

$$v(x) = \Phi(x - x_0),$$

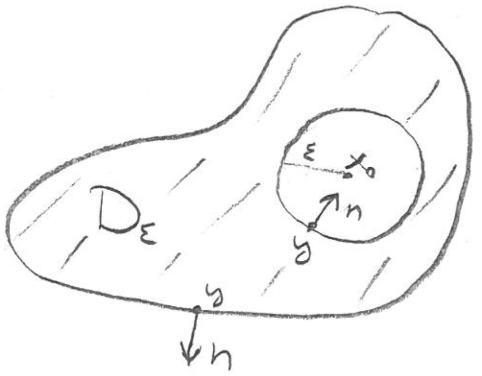
where $x_0 \in D$ is some fixed point. With this choice $\Delta v = 0$ at all points in D except at $x = x_0$.
The following result is sometimes called Green's third identity.

Prop 3.5: For a function u in D, its value at a point $x_0 \in D$ is

$$u(x_0) = \int_{\partial D} \left(u(y) \frac{\partial}{\partial n} \Phi(y - x_0) - \Phi(y - x_0) \frac{\partial}{\partial n} u(y)\right) dS(y)$$
$$+ \iint_D \Phi(x - x_0)\, \Delta u(x)\, dx.$$

Proof: If we did not have singularity of $\Phi(\cdot - x_0)$ at x_0, we could have used Green's 2nd identity. To avoid this problem, we fix $\varepsilon > 0$, and consider the domain

$$D_\varepsilon := D \setminus \underbrace{B(x_0,\varepsilon)}_{\substack{\text{ball of radius } \varepsilon \\ \text{around } x_0.}}$$



Since $\Delta u(x) = \Delta\Phi(x-x_0) = 0$ for $x \in D_\varepsilon$, we get

$$\underbrace{-\iint_{D_\varepsilon} \Phi(x-x_0)\,\Delta u(x)\,dx}_{=:\,\mathrm{I}} = \underbrace{\int_{\partial D}\left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right)ds}_{=:\,\mathrm{II}} + \underbrace{\int_{\partial B(x_0,\varepsilon)}\left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right)ds}_{=:\,\mathrm{III}}$$

since $\partial D_\varepsilon = \partial D \cup \partial B(x_0,\varepsilon)$.

It remains to show $\mathrm{III} \to -u(x_0)$ as $\varepsilon \to 0$.

At $y \in \partial B(x_0,\varepsilon)$, we have

- $|y - x_0| = \varepsilon$
- $n(y) = -\dfrac{y-x_0}{\varepsilon}$
- $\dfrac{\partial \Phi}{\partial n}(y-x_0) = -\dfrac{y-x_0}{\varepsilon} \cdot \dfrac{y-x_0}{\sigma|y-x_0|^n} = -\dfrac{1}{\sigma\varepsilon^{n-1}}$,

  where σ = area of unit sphere / length of unit circle.
- $\Phi(y-x_0) = \begin{cases} \frac{1}{2\pi}\ln\varepsilon, & n=2 \\ -\frac{1}{4\pi}\frac{1}{\varepsilon}, & n=3 \end{cases}$

$$\Rightarrow \mathrm{III} = -\frac{1}{\underbrace{\sigma\varepsilon^{n-1}}_{=|\partial B(x_0,\varepsilon)|}}\int_{\partial B(x_0,\varepsilon)} u\,dS + \underbrace{\Phi(y-x_0)\underbrace{\int_{\partial B(x_0,\varepsilon)}\frac{\partial u}{\partial n}\,dS}_{\le c\cdot\varepsilon^{n-1}}}_{\to 0} \to -u(x_0).$$

We have used the fact that for a continuous function u, the average of u on $\partial B(x_0, \varepsilon)$ converges to $u(x_0)$ when $\varepsilon \to 0$. ∎

Proposition 3.5 gives a very important <u>representation formula</u> for functions u. It shows how u is determined by its

- Dirichlet boundary data $u|_{\partial D}$, its
- Neumann boundary data $\frac{\partial u}{\partial n}|_{\partial D}$, and its
- Poisson source data $\Delta u|_D$.

An important fact is that there is redundancy in the boundary data: only Dirichlet <u>or</u> only Neumann data suffices (together with knowing Δu). Unlike Prop. 3.5, the resulting solution formula for the Dirichlet problem, or for the Neumann problem depends heavily on the geometry of D. In part 3 we mainly consider D = disk / ball / half-plane / half-space. The more general situation is left for part 4. We will focus on the Dirichlet problem. The Neumann problem is handled analogously (see for example problem 7.4.21).

The Dirichlet boundary value problem

Definition 3.6: The Poisson kernel for a given domain $D \subset \mathbb{R}^n$, is the function $P_D(x,y)$, defined for $x \in D$, $y \in \partial D$, such that

$$u(x) = \int_{\partial D} P_D(x,y)\, u(y)\, dS(y), \quad x \in D,$$

for any harmonic function u in D ($\Delta u = 0$).

Note that if we find a Poisson kernel for a domain D, and show that has good estimates, then one can show that the Dirichlet boundary problem is well posed: from $u|_{\partial D}$ we can calculate $u|_D$.

We now describe two methods for finding the Poisson kernel, both building on Prop. 3.5.

Ⓐ Green function method.

Defn. 3.7: The Green function for a given domain $D \subset \mathbb{R}^n$, is the function $G_D(x,y)$, defined for $x \in D$, $y \in D$, $x \neq y$, such that

$$G_D(x,y) = \Phi(x-y) + v_y(x),$$

where for each fixed $y \in D$

- v_y is harmonic in all D ($\Delta v_y = 0$), and
- $G_D(x,y) = \Phi(x-y) + v_y(x) = 0$ for all $x \in \partial D$.

The method Ⓐ is as follows:

- first solve the Dirichlet problem for the particular boundary data $u(x) = -\Phi(x-y)$:

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u(x) = -\Phi(x-y) & \text{for } x \in \partial D, \end{cases}$$

where $y \in D$ is fixed. Write $v_y(x)$ for the solution u.

- This gives the Green function

  $$G_D(x,y) := \Phi(x-y) + v_y(x).$$

  Now apply the following result to obtain the Poisson kernel, and therefore solving the Dirichlet problem for <u>general</u> boundary data.

<u>Prop 3.8</u>: Given the Green function $G_D(x,y)$, the Poisson kernel is

$$P_D(x,y) = \frac{\partial G_D(x,y)}{\partial n(y)} = n(y) \cdot \nabla_y G_D(y,x).$$

<u>Proof</u>: Write $x = x_0$ for this fixed point in D.

By Props. 3.4 and 3.5, we have

$$0 = \int_{\partial D} \left(u(y) \frac{\partial v_{x_0}(y)}{\partial n(y)} - v_{x_0}(y) \frac{\partial u(y)}{\partial n(y)} \right) dS(y),$$

$$u(x_0) = \int_{\partial D} \left(u(y) \frac{\partial \Phi(y-x_0)}{\partial n(y)} - \Phi(y-x_0) \frac{\partial u(y)}{\partial n(y)} \right) dS(y),$$

for any harmonic function u in D.

Addition gives

$$u(x_0) = \int_{\partial D} u(y)\, n(y) \cdot \nabla_y \left(\Phi(y-x_0) + v_{x_0}(y) \right) dS(y),$$

since $\Phi(y-x_0) + v_{x_0}(y) = 0$ on ∂D.

By definition 3.6, this means $P_D(x_0, y) = \dfrac{\partial G_D(y, x_0)}{\partial n(y)}$. ∎

(B) The integral equation method.

It is not true in general that
$$u(x) = \int_{\partial D} u(y) \frac{\partial}{\partial n} \Phi(y-x)\, dS(y), \qquad x \in D,$$
for all harmonic functions u (that is $v_x \neq 0$).
However, it is reasonable that there exists another function $v(y)$ on ∂D such that
$$u(x) = \int_{\partial D} v(y) \frac{\partial}{\partial n} \Phi(y-x)\, dS(y), \quad x \in D,$$
when u is harmonic in D, for the reason that
$$\Delta_x \left(\int_{\partial D} v(y)\, n(y) \cdot \nabla_y \Phi(y-x)\, dS(y) \right)$$
$$= \int_{\partial D} \Delta_x \left(v(y)\, n(y) \cdot \nabla_y \Phi(y-x) \right) dS(y)$$
$$= \int_{\partial D} v(y)\, n(y) \cdot \nabla_y \left(\Delta_x \Phi(y-x) \right) dS(y) = 0, \quad x \in D.$$
Therefore, for any function $v(y)$ on ∂D, the right hand side defines a harmonic function in D.

The method (B) for solving the Dirichlet problem / computing the Poisson kernel is as follows.

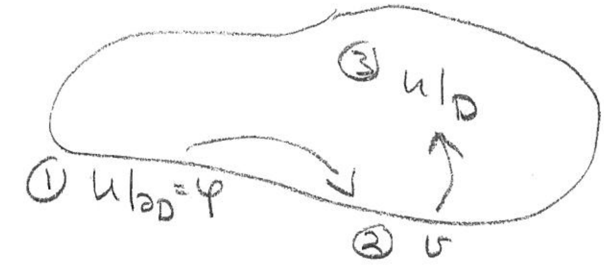
- We are given the boundary values $\varphi := u|_{\partial D}$ of an harmonic function u in D, and want to compute u in D.
- Find an auxiliary function v on ∂D such that
$$\varphi(x_0) = \lim_{\substack{x \to x_0 \\ x \in D}} \int_{\partial D} v(y) \frac{\partial}{\partial n} \Phi(y-x)\, dS(y), \quad \text{for all } x_0 \in \partial D.$$

This is an integral equation on ∂D which called the double layer potential equation (due to the electrostatic interpretation of $\frac{\partial \Phi}{\partial n}$, see part 5).

- Given $u|_{\partial D} = \varphi$, and having solved the double layer potential equation to obtain v on ∂D, the harmonic function which we seek is

$$u(x) = \int_{\partial D} v(y) \frac{\partial \Phi}{\partial n}(y-x)\, dS(y), \quad x \in D.$$

The main result needed for method Ⓑ is the following trace theorem.



<u>Prop. 3.9</u>: Assuming that $v : \partial D \to \mathbb{R}$ and the boundary ∂D is sufficiently regular, we have the limit

$$\lim_{\substack{x \to x_0 \\ x \in D}} \int_{\partial D} v(y) \frac{\partial \Phi}{\partial n}(y-x)\, dS(y) =$$

$$= \frac{1}{2} v(x_0) + \int_{\partial D} v(y) \frac{\partial \Phi}{\partial n}(y - x_0)\, dS(y),$$

for each $x_0 \in \partial D$.

Before the proof, we need to discuss where the strange term $\frac{1}{2} v(x_0)$ comes from. The answer is that it is due to the singularity of

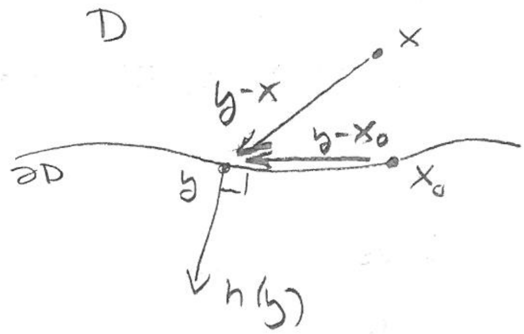
$$\frac{\partial \Phi}{\partial n}(y-x) = \frac{(y-x) \cdot n(y)}{\sigma |y-x|^n}$$

normal vector

dimension 2 or 3

area of unit sphere / length of unit circle

If ∂D is smooth, then $y-x_0$ and $n(y)$ are almost orthogonal, so $(y-x_0)\cdot n(y)\approx 0$.



More precisely, one can show that

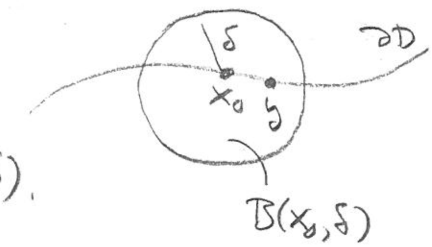
$$\left|\frac{\partial \Phi}{\partial n}(y-x_0)\right| \le \underbrace{c}_{\text{some constant.}} \frac{1}{|y-x_0|^{n-2}}.$$

However, this orthogonality is not true when $x\in D$, and $\left|\frac{\partial \Phi}{\partial n}(y-x)\right|$ is much bigger that $\left|\frac{\partial \Phi}{\partial n}(y-x_0)\right|$. This "missing part" is $\frac{1}{2}v(x_0)$.

Let us do the proof now.

"Proof": Let $x_0\in\partial D$ be given.

Choose $\delta>0$ so that $v(y)\approx v(x_0)$ for $y\in\partial D\cap B(x_0,\delta)$.

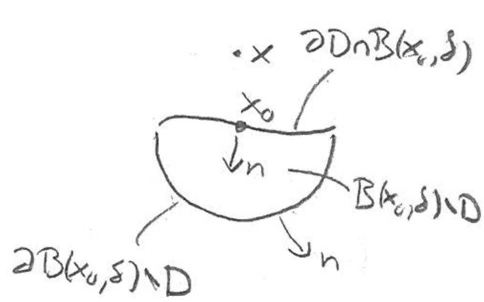


Then

$$\int_{\partial D\cap B(x_0,\delta)} v(y)\frac{\partial \Phi}{\partial n}(y-x)\,dS(y) \approx v(x_0)\int_{\partial D\cap B(x_0,\delta)} \frac{\partial \Phi}{\partial n}(y-x)$$

By Gauss' theorem

$$-\int_{\partial D\cap B(x_0,\delta)} \frac{\partial \Phi}{\partial n}(y-x)\,dS(y) + \int_{\partial B(x_0,\delta)\setminus D} \frac{\partial \Phi}{\partial n}(y-x)\,dS(y)$$

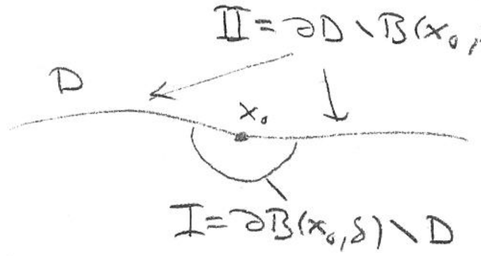


$$= \iint_{B(x_0,\delta)\setminus D} \Delta\Phi(y-x)\,dS(y) = 0$$

Therefore, when $x\approx x_0$, we have

$$\int_{\partial D} v(y)\frac{\partial \Phi}{\partial n}(y-x)\, dS(y) \approx$$

$$v(x_0)\int_{\mathrm{I}} \frac{\partial \Phi}{\partial n}(y-x_0)\, dS(y) + \int_{\mathrm{II}} v(y)\frac{\partial \Phi}{\partial n}(y-x_0)\, dS(y).$$



Here

$$\int_{\mathrm{I}} \frac{\partial \Phi}{\partial n}(y-x)\, dS(y) = \int_{\mathrm{I}} \frac{1}{\sigma\, \delta^{n-2}}\, dS(y)$$

$$\approx \frac{\text{area of half-sphere}}{\text{area of sphere}} = \frac{1}{2}$$

On the other hand

$$\int_{\mathrm{I}} v(y)\frac{\partial \Phi}{\partial n}(y-x_0)\, dS(y) \le c\int_{\mathrm{I}} \frac{1}{|y-x_0|^{n-2}}\, dS(y) \approx 0$$

if δ is small.

This shows that

$$\int_{\partial D} v(y)\frac{\partial \Phi}{\partial n}(y-x)\, dS(y) \approx \frac{1}{2}v(x_0) + \int_{\partial D} v(y)\frac{\partial \Phi}{\partial n}(y-x_0)\, dS(y),$$

when $x \approx x_0$. ■

A first observation about the Poisson kernel is that we always have

$$\int_{\partial D} P(x,y)\, dS(y) = 1.$$

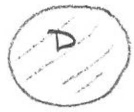
This follows directly from Definition 3.6, by taking the harmonic function 1.

We shall soon learn more properties, but first some concrete examples.

Computations of Poisson kernels

We now compute $P_D(\bar{x},\bar{y})$ for D being

1. the upper half-plane, 

2. the unit disk, 

3. the upper half-space 

4. the unit ball 

Notation: Before we wrote x, y for vectors. If we want to emphasize that x, y are vectors, we write $\bar{x}, \bar{y}$. Below we use coordinates $\bar{x} = (x, y, z)$ or (x, y) and $\bar{y} = (x_0, y_0, z_0)$ or (x_0, y_0) $= \bar{x}_0$

1. Use for example method Ⓐ.

$$\Phi(\bar{x}-\bar{x}_0) = \frac{1}{2\pi}\ln|\bar{x}-\bar{x}_0|$$

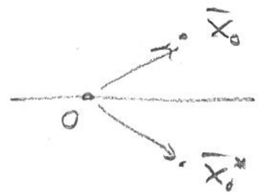
We need to solve

$$\begin{cases} \Delta v_{\bar{x}_0} = 0, & \text{in } D, \\ v_{\bar{x}_0}(\bar{x}) = -\frac{1}{2\pi}\ln|\bar{x}-\bar{x}_0|, & \text{for } \bar{x}=(x,0)\in\partial D. \end{cases}$$

A solution is

$$v_{\bar{x}_0}(\bar{x}) = -\frac{1}{4\pi}\ln|\bar{x}-\bar{x}_0^*|$$

where $\bar{x}_0^* = (x_0, -y_0)$ is the conjugate.

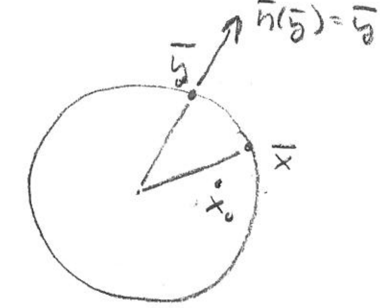


$$\Rightarrow G_D(\bar{x},\bar{x}_0) = \frac{1}{2\pi}\ln\left(\frac{|\bar{x}-\bar{x}_0|}{|\bar{x}-\bar{x}_0^*|}\right)$$

$$\Rightarrow P_D(\bar{x}_0,\bar{x}) = -\frac{\partial}{\partial y}\frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\Big|_{y=0}$$

$$= -\frac{1}{4\pi}\left(\frac{2(0-y_0)}{(x-x_0)^2+y_0^2} - \frac{2(0+y_0)}{(x-x_0)^2+y_0^2}\right) = \boxed{\frac{1}{\pi}\frac{y_0}{|\bar{x}-\bar{x}_0|^2}}$$

② Use for example method ⑬



Adopting the notation, we want to find $P_D(\bar x_0, \bar x)$ by solving

$$u(\bar x) = \frac{1}{2} v(\bar x) + \int_{\partial D} v(\bar y) \frac{\partial \Phi}{\partial n}(\bar y - \bar x)\, dS(\bar y).$$

The very special geometry of the disk gives

$$\frac{\partial \Phi}{\partial n}(\bar y - \bar x) = \frac{(\bar y - \bar x)\cdot \bar y}{2\pi |\bar y - \bar x|^2} = \frac{1 - \bar x \cdot \bar y}{2\pi (2 - 2\bar y \cdot \bar x)} = \frac{1}{4\pi}$$

Denote by $[v] := \frac{1}{2\pi} \int_{\partial D} v\, dS$, the average of v on ∂D. Then

$$u(\bar x) = \frac{1}{2} v(\bar x) + \frac{1}{2}[v].$$

Taking the average on both sides gives $[u] = [v]$, so $v(\bar x) = 2u(\bar x) - [u]$ is the solution.

This yields

$$u(\bar x_0) = \int_{\partial D} \left(2u(\bar x) - [u]\right) \frac{(\bar x - \bar x_0)\cdot \bar x}{2\pi |\bar x - \bar x_0|^2}\, dS(\bar x),$$

$$= 2\int_{\partial D} u(\bar x) \frac{1 - \bar x_0 \cdot \bar x}{2\pi (1 + |\bar x_0|^2 - 2\bar x \cdot \bar x_0)}\, dS(\bar x) - [u] \underbrace{\int_{\partial D} \frac{\partial \Phi}{\partial n}(\bar x - \bar x_0)\, dS(\bar x)}_{\substack{= 1 \text{ by Prop 3.5} \\ (\text{set } u = 1)}}$$

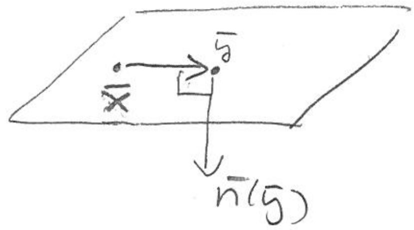
$$= \int_{\partial D} u(\bar x) \underbrace{\left(\frac{1}{\pi} \frac{1 - \bar x \cdot \bar x_0}{1 + |\bar x_0|^2 - 2\bar x \cdot \bar x_0} - \frac{1}{2\pi} \right)}_{= \frac{1}{2\pi} \frac{2 - \cancel{2\bar x \cdot \bar x_0} - (1 + |\bar x_0|^2 - \cancel{2\bar x \cdot \bar x_0})}{1 + |\bar x_0|^2 - 2\bar x \cdot \bar x_0}} dS(\bar x)$$

$$\Rightarrow P(\bar x_0, \bar x) = \boxed{\frac{1}{2\pi} \frac{1 - |\bar x_0|^2}{|\bar x - \bar x_0|^2}}$$

③ Use for example method Ⓑ.

In this case $(\bar{y}-\bar{x})\cdot\bar{n}(\bar{y})=0$,

so $u(\bar{x})=\frac{1}{2}v(\bar{x})$.



Therefore

$$u(\bar{x}_0)=\int_{\partial D} 2u(\bar{x})\,\frac{(\bar{x}-\bar{x}_0)\cdot\bar{n}(\bar{x})}{4\pi|\bar{x}-\bar{x}_0|^3}\,dS(\bar{x})$$

$$=\int_{\partial D} u(\bar{x})\,\frac{1}{2\pi}\,\frac{z_0}{|\bar{x}-\bar{x}_0|^3}$$

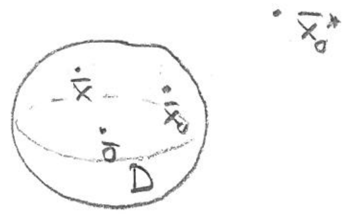
$$\Rightarrow P(\underset{\substack{\parallel \\ (x_0,y_0,z_0)}}{\bar{x}_0},\bar{x})=\boxed{\frac{1}{2\pi}\,\frac{z_0}{|\bar{x}-\bar{x}_0|^3}}$$

④ Use for example method Ⓐ.

We use a "reflection technique" similar to that in ①, to find a Green's function. Our guess is

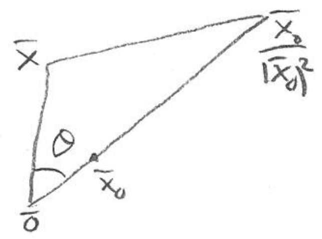
$$G_D(\bar{x},\bar{x}_0)=\Phi(\bar{x}-\bar{x}_0)-a\,\Phi(\bar{x}-\bar{x}_0^*),$$

where we need to find suitable constant a and point $\bar{x}_0^*$ outside D.



Claim: The choices $\bar{x}_0^*=\dfrac{\bar{x}_0}{|\bar{x}_0|^2}$ and $a=\dfrac{1}{|\bar{x}_0|}$ works.

To see this, calculate for $\bar{x}\in\partial D$



$$0=G_D(\bar{x},\bar{x}_0)=-\frac{1}{4\pi}\left(\frac{1}{|\bar{x}-\bar{x}_0|}-a\,\frac{1}{|\bar{x}-\bar{x}_0^*|}\right)$$

$$\Leftrightarrow\quad a^2=\frac{|\bar{x}-\bar{x}_0^*|^2}{|\bar{x}-\bar{x}_0|^2}=\frac{1+|\bar{x}_0^*|^2-2|\bar{x}_0^*|\cos\theta}{1+|\bar{x}_0|^2-2|\bar{x}_0|\cos\theta}$$

$$=\frac{1+\frac{1}{|\bar{x}_0|^2}-2\frac{1}{|\bar{x}_0|}\cos\theta}{1+|\bar{x}_0|^2-2|\bar{x}_0|\cos\theta}=\frac{1}{|\bar{x}_0|^2}$$

$$\Rightarrow G_D(\bar{x},\bar{x}_0) = -\frac{1}{4\pi}\left(\frac{1}{|\bar{x}-\bar{x}_0|} - \frac{1}{|\bar{x}_0|}\,\frac{1}{\left|\bar{x}-\frac{\bar{x}_0}{|\bar{x}_0|^2}\right|}\right)$$

$$= -\frac{1}{4\pi}\left(\frac{1}{\left(|\bar{x}|^2+|\bar{x}_0|^2-2\,\bar{x}\cdot\bar{x}_0\right)^{1/2}} - \frac{1}{\left(|\bar{x}_0|^2|\bar{x}|^2+1-2\,\bar{x}\cdot\bar{x}_0\right)^{1/2}}\right)$$

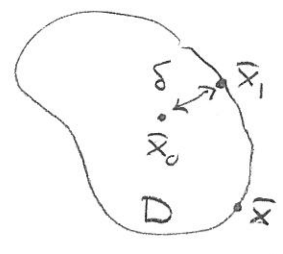
$$\Rightarrow P_D(\bar{x}_0,\bar{x}) = -\frac{1}{4\pi}\,\frac{\partial}{\partial r}\Bigg|_{r=1}\left(\frac{1}{\left(r^2+|\bar{x}_0|^2-2r\,\bar{x}\cdot\bar{x}_0\right)^{1/2}} - \frac{1}{\left(|\bar{x}_0|^2r^2+1-2r\,\bar{x}\cdot\bar{x}_0\right)^{1/2}}\right)$$

$$= -\frac{1}{4\pi}\left(\frac{2-2\cancel{\bar{x}\cdot\bar{x}_0}}{|\bar{x}-\bar{x}_0|^3} - \frac{2|\bar{x}_0|^2-2\cancel{\bar{x}\cdot\bar{x}_0}}{|\bar{x}-\bar{x}_0|^3}\right)\left(-\frac{1}{2}\right)$$

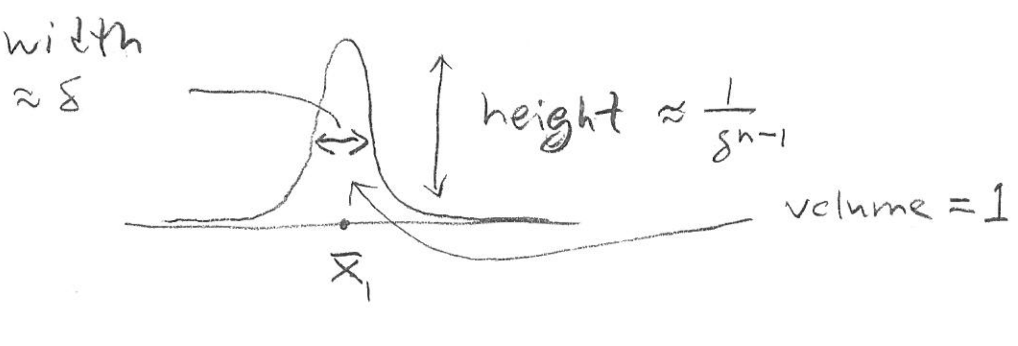
$$= \boxed{\frac{1-|\bar{x}_0|^2}{4\pi\,|\bar{x}-\bar{x}_0|^3}}$$

Summary of properties of the Poisson kernel:

The exact form of the Poisson kernel $P_D(\bar{x}_0,\bar{x})$ depends on the geometry of D, but the following general properties always hold. (You should verify these in the four examples above.)



- $\int_{\partial D} P_D(\bar{x}_0,\bar{x})\,dS(\bar{x}) = 1$, for all $\bar{x}_0 \in D$.
- $P_D(\bar{x}_0,\bar{x}) \geq 0$ for all $\bar{x}_0 \in D$, $\bar{x} \in \partial D$.
- $\Delta_{\bar{y}} P_D(\bar{y},\bar{x}) = 0$ for each $\bar{x} \in \partial D$
- For fixed $\bar{x}_0 \in D$, δ denoting the distance to ∂D, $P_D(\bar{x}_0,x)$ has a graph looking like this:



$P \geq 0$ follows from the maximum principle, and we will return to this. For the 3rd property, we calculate

$$\Delta_{\bar y} P_D(\bar y, \bar x) = \Delta_{\bar y}\left(\bar n(\bar x)\cdot \nabla_{\bar x} G_D(\bar x, \bar y)\right)$$
$$= \bar n(\bar x)\cdot \nabla_{\bar x}\left(\Delta_{\bar y} G_D(\bar x, \bar y)\right).$$

It is clear from Defn. 3.7 that $\Delta_{\bar x} G_D(\bar x, \bar y) = 0$. That $\Delta_{\bar y} G_D(\bar x, \bar y) = 0$ follows from the following fundamental symmetry property of G_D.

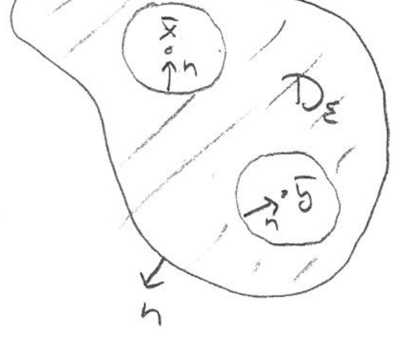
<u>Prop. 3.10</u>: For $\bar x, \bar y \in D$, $\bar x \neq \bar y$, we have

$$G_D(\bar x, \bar y) = G_D(\bar y, \bar x).$$

<u>Proof</u>: Fix $\bar x, \bar y \in D$ and consider the two functions $u(\bar z) := G_D(\bar z, \bar y)$ and
$v(\bar z) := G_D(\bar z, \bar x)$.

We need to show $u(\bar x) = v(\bar y)$.
We have $u = v = 0$ on ∂D and $\Delta u = \Delta v = 0$ in D except at $\bar x$ and $\bar y$. The following is a 2 point singularity version of Prop 3.5.
Let $D_\varepsilon := D \setminus (B(\bar x, \varepsilon) \cup B(\bar y, \varepsilon))$ and apply Green's second identity in D_ε.

$$0 = \int_{\partial D}\Big(\underbrace{u}_{=0}\frac{\partial v}{\partial n} - \underbrace{v}_{=0}\frac{\partial u}{\partial n}\Big)dS + \underbrace{\int_{\partial B(\bar x, \varepsilon)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS}_{= I}$$
$$+ \underbrace{\int_{\partial B(\bar y, \varepsilon)}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS}_{= II}$$



Consider first the term I. Note

that around $\bar{x}$, $u(\bar{z})$ is a nice function where

$v(\bar{z}) = \Phi(\bar{z}-\bar{x}) + v_{\bar{x}}(\bar{z})$, where $v_{\bar{x}}$ is nice and

$\Phi(\cdot - \bar{x})$ has a singularity at $\bar{x}$.

$$\Rightarrow I = \int_{\partial B(\bar{x},\varepsilon)} \Big(u(\bar{x}) \underbrace{\frac{\partial}{\partial n}\Phi(\bar{z}-\bar{x})}_{= -\frac{1}{\sigma \varepsilon^{n-1}}} \Big) dS(\bar{z}) + \int_{\partial B(\bar{x},\varepsilon)} \underbrace{(u(\bar{z}) - u(\bar{x}))}_{\to 0} \underbrace{\frac{\partial}{\partial n}\Phi(\bar{z}-\bar{x})}_{= -\frac{1}{\sigma \varepsilon^{n-1}}} \Big) dS(\bar{z})$$

$$+ \int_{\partial B(\bar{x},\varepsilon)} \underbrace{\Big(u(\bar{z}) \frac{\partial v_{\bar{x}}(\bar{z})}{\partial n} - v_{\bar{x}}(\bar{z}) \frac{\partial u(\bar{z})}{\partial n} \Big)}_{\text{bounded}} dS(\bar{z}) - \int_{\partial B(\bar{x},\varepsilon)} \Phi(\bar{z}-\bar{x}) \underbrace{\frac{\partial u(\bar{z})}{\partial n}}_{\text{bounded}} dS(\bar{z})$$

$$\Phi = \begin{cases} \frac{1}{2\pi} \ln \varepsilon, & n=2 \\ -\frac{1}{4\pi}\frac{1}{\varepsilon}, & n=3 \end{cases}$$

$\to -u(\bar{x})$.

Similarly, for term II we show

$$II \approx -\int_{\partial B(\bar{y},\varepsilon)} v(\bar{y}) \frac{\partial \Phi(\bar{z}-\bar{y})}{\partial n} dS(\bar{z}) = v(\bar{y}).$$

Thus $0 - u(\bar{x}) + v(\bar{y}) = 0$, that is $G_D(\bar{x},\bar{y}) = G_D(\bar{y},\bar{x})$. ∎

Properties of harmonic functions:

We now derive some important consequences of the Poisson kernel for the unit disk / ball.

$$P_D(\bar{x}_0, \bar{x}) = \frac{1}{2\pi} \frac{1-|\bar{x}_0|^2}{|\bar{x}-\bar{x}_0|^2} \quad \text{for the disk } (n=2).$$

$$P_D(\bar{x}_0, \bar{x}) = \frac{1}{4\pi} \frac{1-|\bar{x}_0|^2}{|\bar{x}-\bar{x}_0|^3} \quad \text{for the ball } (n=3).$$

Seeing the pattern, we write

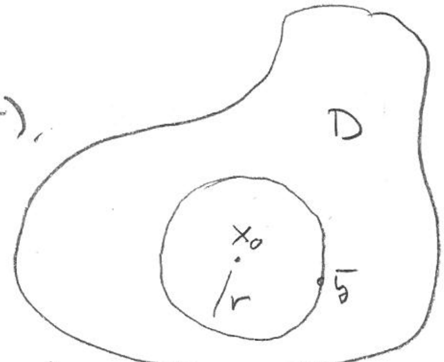
$$P_D(\bar{x}_0, \bar{x}) = \frac{1}{\sigma} \frac{1-|\bar{x}_0|^2}{|\bar{x}-\bar{x}_0|^n}$$

, where n = dimension and σ = length of unit circle / area of unit sphere.

Prop. 3.11: We have the following mean value property for harmonic functions. If $\Delta u = 0$ in D and $B(\bar{x}_0, r) \subset D$, then

$$u(\bar{x}_0) = \frac{1}{\sigma r^{n-1}} \int_{\partial B(\bar{x}_0, r)} u(\bar{y})\, dS(\bar{y})$$

= mean value of u on $\partial B(\bar{x}_0, r)$.



Proof: Let

$v(\bar{x}) := u(\bar{x}_0 + r\bar{x})$.

Then $\Delta v = 0$ in the unit disk/ball.

$$\Rightarrow v(\bar{0}) = \int_{|\bar{x}|=1} \underbrace{P_D(\bar{0}, \bar{x})}_{= \frac{1}{\sigma}} v(\bar{x})\, dS(\bar{x}) = \text{average of } v \text{ on the unit circle/sphere.}$$

The corresponding result for u follows. ∎

Corollary 3.12: We have the following maximum principle for harmonic functions.

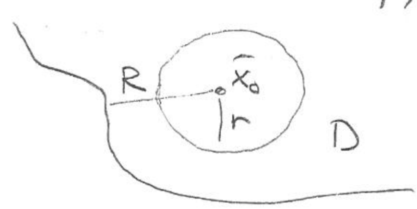
Assume that $\Delta u = 0$ in D and u is continuous on $\bar{D}$, where D is a bounded, open and connected set. Then

- u does not attain maximum or minimum inside D (and therefore attains them on ∂D), unless
- u = constant.

Proof: Assume for example that $u(\bar{x}_0) = M$ = maximum of u on $\bar{D}$, at an interior point $\bar{x}_0 \in D$.

Define the distance $R := \text{dist}(\bar{x}_0, \partial D)$ to the boundary, and let $0 < r < R$.

By Prop. 3.11

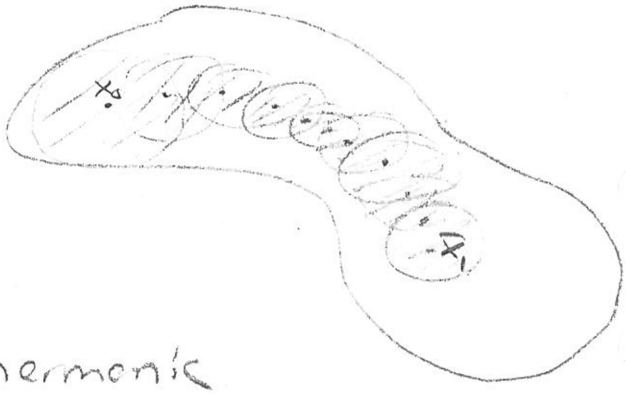


we have

$$\underbrace{u(\bar{x}_0)}_{M} = \underbrace{\text{average of } u \text{ on } \partial B(\bar{x}_0, r)}_{\leq M}.$$

Since we get a contradiction $M < M$ unless $u = M$ on all $\partial B(\bar{x}_0, r)$, this latter must hold. Since this holds for all $0 < r < R$, we have shown that $u = M$ on all $B(\bar{x}_0, R)$.

Repeating the argument shows that $u = M$ in all D.



∎

<u>Prop. 3.13</u>: Let u be a harmonic function in an open set D. Then all partial derivatives of all orders of u exist in all D. (u is C^∞ <u>smooth</u>)

<u>Proof</u>: Consider a ball $B(\bar{x}_0, r) \subset D$ and change variables as in the proof of Prop. 3.11. We have

$$v(\bar{x}) = \int_{|\bar{y}|=1} \underbrace{\frac{1 - |\bar{x}|^2}{\sigma |\bar{y} - \bar{x}|^n}}_{\text{smooth function of } \bar{x} \text{ for } |\bar{x}|<1} v(\bar{y}) \, dS(\bar{y})$$

Differentiate with respect to $\bar{x}$ under the integral sign. Thus v is smooth, hence u. ∎

Well posedness of the Dirichlet problem

Consider the Dirichlet boundary value problem for Δ on a domain D:

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = \varphi, & \text{on } \partial D, \end{cases}$$

where $\varphi : \partial D \to \mathbb{R}$ is a given function.

Uniqueness:

① Using the maximum principle:

If u_1 and u_2 are solutions, with boundary data φ_1, φ_2, let $v := u_1 - u_2$.

Then $\begin{cases} \Delta v = 0 & \text{in } D, \\ v = \varphi_1 - \varphi_2 & \text{on } \partial D. \end{cases}$

We get

$$\max_{\bar{x} \in D} |u_1(\bar{x}) - u_2(\bar{x})| \le \max_{\bar{y} \in \partial D} |\varphi_1(\bar{y}) - \varphi_2(\bar{y})|.$$

This shows uniqueness and stability for the Dirichlet BVP.

② Using Green's 1st identity:

If u_1 and u_2 are solutions, both with boundary data φ, let $v := u_1 - u_2$.

Then $\begin{cases} \Delta v = 0 & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases}$

Prop. 3.4 (with $u = v$), gives

$$\iint_D |\nabla v|^2 = \int_{\partial D} \underbrace{v}_{=0} \frac{\partial v}{\partial n} \, dS = 0.$$

Thus $\nabla v = 0$, so $v = $ constant. Since $v|_{\partial D} = 0$, we get $v|_D = 0$, so $u_1 = u_2$.

(Note that this "energy method" applies also to the Neumann problem.)

Existence:

① For a domain D for which we know a Poisson kernel $P_D(\bar{x},\bar{\xi})$: This means that we know that

$$u(\bar{x}) = \int_{\partial D} P_D(\bar{x},\bar{\xi})\, u(\bar{\xi})\, dS(\bar{\xi}), \quad \bar{x}\in D, \text{ whenever}$$

u is harmonic in D.

Assume now that φ is some given function on ∂D (not necessarily $= u|_{\partial D}$), and define

$$v(\bar{x}) := \int_{\partial D} P_D(\bar{x},\bar{\xi})\, \varphi(\bar{\xi})\, dS(\bar{\xi}), \quad \text{for } \bar{x}\in D.$$

Differentiating, using that $\Delta_{\bar{x}} P_D(\bar{x},\bar{\xi}) = 0$, we see that v is harmonic in D.

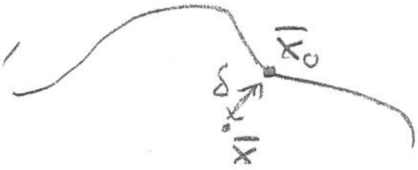
To show $v|_{\partial D} = \varphi$ (so that v is a solution to the Dirichlet BVP), one needs further estimates of D, which hold if D is "nice". We shall not discuss this further here. The basic idea is that

$$\int_{\partial D} P_D(\bar{x},\bar{\xi})\,\varphi(\bar{\xi})\, dS(\bar{\xi}) \approx \text{average of } \varphi \text{ on } \partial D \cap B(\bar{x}_0,\delta) \longrightarrow \varphi(\bar{x}_0),$$

as $\bar{x}\to\bar{x}_0\in\partial D$,

where $\delta = |\bar{x}-\bar{x}_0|$.

Besides the property $\int_{\partial D} P_D(\bar{x},\bar{\xi})\,dS(\bar{\xi}) = 1$, which we saw,



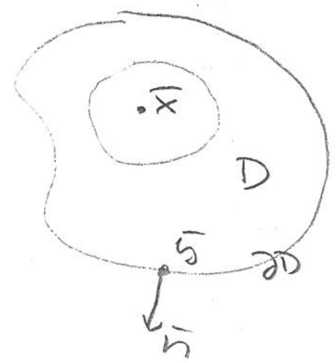
let us limit ourselves to proving the positivity property $P_D(\bar{x},\bar{\xi}) \geq 0$.

Prop. 3.14: Let $P_D(\bar{x},\bar{y})$ be the Poisson kernel for a domain D. Then

$$P_D(\bar{x},\bar{y}) \geq 0 \text{ for } \bar{x}\in D,\ \bar{y}\in \partial D.$$

Proof: By Prop. 3.8

$$P_D(\bar{x},\bar{y}) = \bar{n}(\bar{y})\cdot D_{\bar{y}} G_D(\bar{y},\bar{x}).$$



Consider the harmonic function $u(\bar{y}) := G_D(\bar{y},\bar{x})$ on a domain $D_\varepsilon := D \setminus B(\bar{x},\varepsilon)$.

Since $G_D(\bar{y},\bar{x}) \to -\infty$ as $\bar{y}\to\bar{x}$, we can choose $\varepsilon>0$ small so that $u<0$ on $\partial B(\bar{x},\varepsilon)$.

By Defn 3.7, $u=0$ on ∂D, and $\Delta u = 0$ on D_ε. Hence by the maximum principle 3.12, $u \leq 0$ in D_ε. It follows that $\frac{\partial u}{\partial n} \geq 0$ on ∂D, that is $P_D(\bar{x},\bar{y}) \geq 0$. ∎

(2) For a general domain D.

The socalled Dirichlet's principle is a very general method for obtaining a solution to the Dirichlet problem.

Let $M := \{ v : \bar{D} \to \mathbb{R} ;\ v = \varphi \text{ on } \partial D \}$, and minimize the Dirichlet integral

$$E(v) := \iint_D |\nabla v(\bar{x})|^2 d\bar{x}$$

over all possible v in M.

Prop. 3.15: Dirichlet's principle holds:
The following are equivalent.

- u is a minimizer to E, that is $E(u) \le E(v)$ for all $v \in M$.
- u solves the Dirichlet BVP $\begin{cases} \Delta u = 0 \text{ in } D \\ u = \varphi \text{ on } \partial D \end{cases}$

Proof: Let $v = u + \varepsilon w$, with $w|_{\partial D} = 0$.

$$\Rightarrow E(v) - E(u) = \iint_D \left(|\nabla(u+\varepsilon w)|^2 - |\nabla u|^2 \right) d\bar{x}$$

$$= \underbrace{2\varepsilon \iint_D \nabla u \cdot \nabla w \, d\bar{x}}_{= \iint_D \Delta u \cdot w \, d\bar{x} \text{ by Green's first identity.}} + \varepsilon^2 \iint_D |\nabla w|^2 d\bar{x}$$

If $\Delta u = 0$, then $E(v) - E(u) \ge 0$.
Conversely, if $E(v) \ge E(u)$, then

$$0 \le \lim_{\varepsilon \to 0} \frac{E(v) - E(u)}{2\varepsilon} = \iint_D \Delta u \cdot w \, d\bar{x}$$

If this holds for all w, then we must have $\Delta u = 0$ in all D. $\blacksquare$

There is a <u>very important subtlety</u> in Dirichlet's principle: a minimizer need not exist (if D is irregular).
This was in fact a key problem in the development of the important concept of <u>compactness</u> during the 1800's.