Part 4.1: Eigenfunctions to Δ in one dimension

Definition 4.1: Let $D \subset \mathbb{R}^n$ ($n=1,2$ or 3) be a domain. If

$$\Delta u(x) = \lambda \cdot u(x) \quad , \quad \text{for all } x \in D, \ u \not\equiv 0,$$

holds, then u is an eigenfunction to Δ, with eigenvalue λ. If u satisfies boundary condition X (X = Dirichlet, Neumann, ...), then we say that u is an X-eigenfunction and λ is an X eigenvalue.

We first consider dimension $n=1$, where

$$\Delta = \frac{d^2}{dx^2}$$

and the domain is an interval $D=(0,\ell)$, of some length $\ell > 0$. We compute below the eigenfunctions/values for 4 different boundary conditions, and show in each case that the set of eigenfunctions is complete, meaning that any function f can be written

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

as a sum of eigenfunctions, with convergence in appropriate sense.

Periodic boundary conditions (P)

We need to solve

$$\begin{cases} u''(x) = \lambda u(x), & x \in (0,\ell) \\ u(\ell) = u(0), \\ u'(\ell) = u'(0). \end{cases}$$

The ODE has the general solution

$$u(x) = A e^{\mu x} + B e^{-\mu x},$$

where $\pm\mu$ are the roots of $\mu^2 = \lambda$.

The boundary conditions give

$$\begin{cases} A+B = A e^{\mu l} + B e^{-\mu l} \\ \mu A - \mu B = A\mu e^{\mu l} - B\mu e^{-\mu l}, \end{cases}$$

$$\Leftrightarrow \begin{cases} A(1-e^{\mu l}) = B(e^{-\mu l}-1) \\ \mu A(1-e^{\mu l}) = -\mu B(e^{-\mu l}-1) \end{cases}$$

$$\Rightarrow 2\mu(1-e^{\mu l})A = 0$$

Two cases:

- $\mu(1-e^{\mu l}) \neq 0 \Rightarrow A = 0 = B \Rightarrow u(x) = 0.$
- $\mu(1-e^{\mu l}) = 0 \Rightarrow \mu l = i2\pi n$ for some integer $n \in \mathbb{Z}$.

We have shown:

<u>Prop. 4.2</u>: For periodic boundary conditions, the eigenvalues are $\lambda_n = -\left(\frac{2\pi n}{l}\right)^2$, $\underline{n = 0, 1, 2, \ldots}$

The corresponding eigenfunctions are

$n = 0$: $u_0(x) = A = \text{constant}$

$n \in \mathbb{Z}_+$: $u_n(x) = A e^{i\frac{2\pi n}{l}x} + B e^{-i\frac{2\pi n}{l}x}$

$$= C\cos\left(\frac{2\pi n}{l}x\right) + D\sin\left(\frac{2\pi n}{l}x\right).$$

<u>Prop. 4.3</u>: Any function $f: (0, l) \to \mathbb{C}$ can be written as a convergent series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{l}nx} = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{2\pi}{l}nx\right) + b_n\sin\left(\frac{2\pi}{l}nx\right)\right).$$

Convergence is good if the periodic extension of f is smooth.

This is standard result of Fourier analys.
The reader should recall the exact meaning
of "convergent" (L_2-convergence...),
"good" (uniform convergence ...) and
"smooth" (C^1-regular, ...).

<u>Dirichlet boundary conditions (D):</u>

We need to solve

$$\begin{cases} u''(x) = \lambda u(x), & x \in (0, l), \\ u(0) = 0 = u(l). \end{cases}$$

Proceed as for (P). The boundary conditions now mean

$$A + B = 0 = A e^{\mu l} + B e^{-\mu l}.$$

Two cases:

- $e^{\mu l} \neq \pm 1 \Rightarrow A = B = 0$
- $e^{\mu l} = \pm 1 \Rightarrow \mu l = \pi n \cdot i$ & $B = -A$.

Then $u(x) = A\left(e^{i\frac{\pi}{l}nx} - e^{-i\frac{\pi}{l}\cdot nx}\right)$
$= 2iA \sin\left(\frac{\pi}{l} nx\right)$

<u>Prop. 4.4:</u> For Dirichlet boundary conditions, the eigenvalues are $\lambda_n = -\left(\frac{\pi n}{l}\right)^2$, $n = 1, 2, \ldots$
The corresponding eigenfunctions are

$$u_n(x) = A \sin\left(\frac{\pi}{l} nx\right), \quad n = 1, 2, \ldots$$

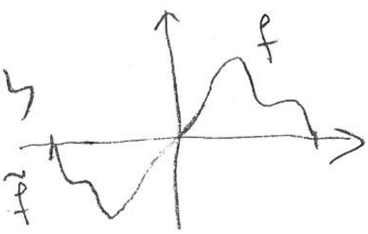
<u>Prop. 4.5</u> Any function $f:(0,\ell)\to\mathbb{C}$ can be written as a convergent series

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{\ell}nx\right), \quad x\in(0,\ell).$$

Convergence is good if f is smooth and $f(0)=0$
$f(\ell)=0$.

<u>Proof</u>: We derive this result from Prop. 4.3. Define $\tilde f:(-\ell,\ell)\to\mathbb{C}$ by

$$\tilde f(x)=\begin{cases} f(x), & 0<x<\ell,\\ -f(-x), & -\ell<x<0.\end{cases}$$



Apply the analogue of Prop. 4.3 for $(-\ell,\ell)$:

$$\tilde f(x) = 0+\sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{2\ell}nx\right), \quad x\in(-\ell,\ell),$$

/ $a_n=0$ since $\tilde f$ is odd /

$$\Rightarrow f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{\ell}nx\right), \quad x\in(0,\ell). \qquad \blacksquare$$

<u>Neumann boundary conditions (N):</u>

We need to solve

$$\begin{cases} u''(x)=\lambda u(x), & x\in(0,\ell),\\ u'(0)=0=u'(\ell). \end{cases}$$

Proceed as for (P). Boundary conditions yield

$$A\mu - B\mu = 0 = A\mu e^{+\mu\ell} - B\mu e^{-\mu\ell}$$

Two cases:

- $e^{+\mu\ell}\neq\pm 1 \Rightarrow A=0=B$
- $e^{\mu\ell}=\pm 1 \Rightarrow B_\mu = A_\mu$
  $\mu\ell = 2\pi n\cdot i$

Then $u(x) = A(e^{\mu x} + e^{-\mu x}) = 2A\cos(\frac{\pi}{l}nx)$.
We get:

<u>Prop. 4.6</u>: For Neumann boundary conditions, the eigenvalues are $\lambda_n = -(\frac{\pi n}{l})^2$, $n = \underline{\underline{0}}, 1, 2, \dots$
The corresponding eigenfunctions are

$$u_n(x) = B\cos(\frac{\pi}{l}nx), \quad n = 0, 1, 2, \dots$$

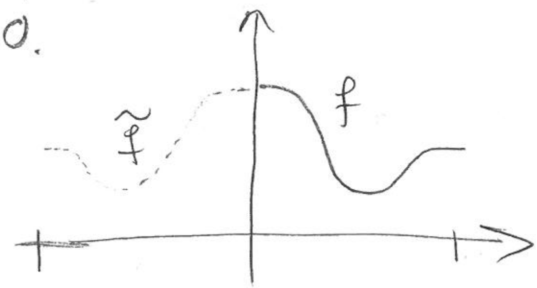
<u>Prop. 4.7</u>: Any function $f: (0, l) \to \mathbb{C}$ can be written as a convergent series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi}{l}nx), \quad x \in (0, l).$$

Convergence is good if f is smooth and
$f'(0) = 0$, $f'(l) = 0$.

<u>Proof</u>: Similar to proof of Prop. 4.5.
Now do even extension of f to $(-l, l)$:

$$\tilde{f}(x) := \begin{cases} f(x), & 0 < x < l, \\ f(-x), & -l < x < 0. \end{cases}$$



<u>Mixed boundary conditions (M)</u>

We need to solve

$$\begin{cases} u''(x) = \lambda u(x), & x \in (0, l) \\ u(0) = 0 \\ u'(l) = 0 \end{cases}$$

Proceed as for (P). Boundary conditions yield

$$A+B=0=A\mu e^{\mu l}-B\mu e^{-\mu l}$$

$$\Rightarrow A\mu\underbrace{(e^{\mu l}+e^{-\mu l})}_{2\cos(\mu l/i)}=0$$

Two cases:

- $\mu\cos(\mu l/i)\neq 0 \Rightarrow A=B=0$
- $\mu\cos(\mu l/i)=0 \Rightarrow B=-A$

  $\mu l=(\frac{\pi}{2}+n\pi)i$, $n\in\mathbb{Z}$,

  or $\mu=0$.

Then $u(x)=A\left(e^{i\frac{\pi}{l}(n+\frac{1}{2})x}-e^{-i\frac{\pi}{l}(n+\frac{1}{2})x}\right)$

$$=2iA\sin\left(\tfrac{\pi}{l}(n+\tfrac{1}{2})x\right)$$

<u>Prop. 4.8</u>: For mixed boundary conditions, the eigenvalues are $\lambda_n=\left(\frac{\pi}{l}(n+\frac{1}{2})\right)^2$, $n=0,1,2,\dots$

The corresponding eigenfunctions are

$$u_n(x)=A\sin\left(\tfrac{\pi}{l}(n+\tfrac{1}{2})x\right),\quad n=0,1,2,\dots$$

<u>Prop. 4.9</u>: Any function $f:(0,l)\to\mathbb{C}$ can be written as a convergent series

$$f(x)=\sum_{n=0}^{\infty}b_n\sin\left(\tfrac{\pi}{l}(n+\tfrac{1}{2})x\right),\quad x\in(0,l).$$

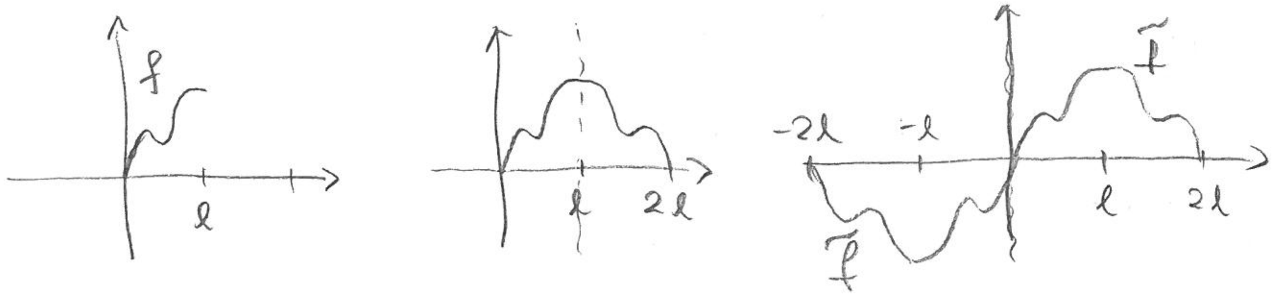
Convergence is good if f is smooth and

$$f(0)=0,\quad f'(l)=0.$$

<u>Proof</u>: We now extend f on $(0,l)$ to the function

$$\tilde{f}(x):=\begin{cases} f(x), & 0<x<l\\ f(2l-x), & l<x<2l\\ -f(-x), & -l<x<0\\ -f(2l+x), & -2l<x<-l\end{cases}$$

which is even with respect to $x=l$ and odd with respect to $x=0$.



Apply the analogue of Prop. 4.3 for $(-2l, 2l)$

$$\Rightarrow \tilde{f}(x) = 0 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{4l} nx\right),$$

since $\tilde{f}$ odd $\Rightarrow a_n = 0$.

Similarly one shows from the formulas for b_n, that $b_n = 0$ when n is even, due to that $\tilde{f}$ is even with respect to $x=l$.

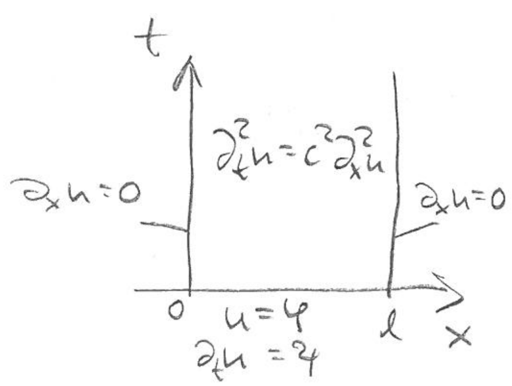
Writing $n = 2k+1$, gives

$$\tilde{f}(x) = \sum_{k=0}^{\infty} b_{2k+1} \sin\left(\frac{\pi}{l}\left(k+\tfrac{1}{2}\right)x\right). \quad \blacksquare$$

We now show by three examples how to use these eigenfunction expansions to solve initial/boundary value problems in simple bounded domains.

<u>Example 4.10:</u>

We want to solve the wave equation



$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u, & x \in (0,l),\ t>0 \\ \partial_x u = 0, & x=0 \text{ or } x=l \\ u = \varphi(x), & t=0 \\ \partial_t u = \psi(x), & t=0 \end{cases}$$

$\leftarrow$ boundary conditions

$\leftarrow$ initial data.

Consider, for each fixed t, the one variable function

$$u_t(x) := u(x,t).$$

Apply Prop. 4.7 with $f = u_t$:

$$u_t(x) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos\left(\frac{\pi}{\ell} n x\right).$$

Insert into the wave equation:

$$a_0''(t) + \sum_{n=1}^{\infty} a_n''(t) \cos\left(\frac{\pi}{\ell} n x\right) = c^2 \sum_{n=1}^{\infty} a_n(t) \cdot \left(-\left(\frac{\pi n}{\ell}\right)^2\right) \cos\left(\frac{\pi}{\ell} n x\right).$$

Identify coefficients:

$$\begin{cases} a_0''(t) = 0 & \Rightarrow a_0(t) = A_0 + B_0 t \\ a_n''(t) = -\left(\frac{\pi c}{\ell} n\right)^2 a_n(t) & \end{cases}$$

$$\Rightarrow a_n(t) = A_n \cos\left(\frac{\pi c}{\ell} n t\right) + B_n \sin\left(\frac{\pi c}{\ell} n t\right).$$

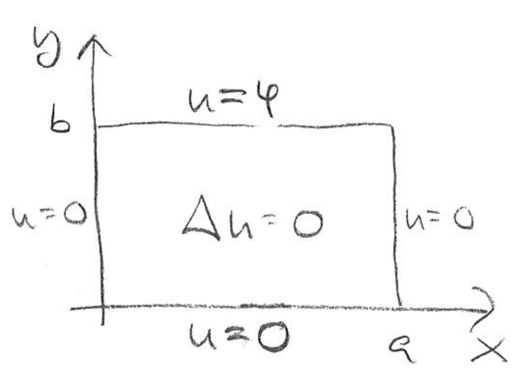
Solution:

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi c}{\ell} n t\right) + B_n \sin\left(\frac{\pi c}{\ell} n t\right)\right) \cos\left(\frac{\pi}{\ell} n x\right).$$

The coefficients A_0, B_0, A_n, B_n are determined by the initial data φ and ψ, using the integral formulas from Fourier analysis.

Ex 4.11:

We want to solve the following Dirichlet problem for Laplace on the rectangle:



$$\begin{cases} \Delta u = 0, & 0 < x < a,\ 0 < y < b, \\ u = \varphi, & y = b \\ u = 0 & \text{on the other three sides.} \end{cases}$$

Consider for each fixed y, the one variable function

$$u_y(x) := u(x,y).$$

Apply Prop. 4.5 with $f = u_y$:

$$u_y(x) = \sum_{n=1}^{\infty} b_n(y) \cdot \sin\left(\frac{\pi}{\ell} n x\right).$$

Insert into Laplace equation:

$$\sum_{n=1}^{\infty} \left(-\left(\frac{\pi n}{\ell}\right)^2 b_n(y) + b_n''(y) \right) \sin\left(\frac{\pi}{\ell} n x\right).$$

Identify coefficients:

$$b_n''(y) = \left(\frac{\pi n}{\ell}\right)^2 b_n(y)$$

Since $u_0(x) = 0$, we need $b_n(0) = 0$ for all n.

$$\Rightarrow b_n(y) = B_n \underbrace{\frac{1}{2}\left(e^{\frac{\pi n}{\ell} y} - e^{-\frac{\pi n}{\ell} y}\right)}_{=: \sinh\left(\frac{\pi n}{\ell} y\right)}$$

Solution:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{\pi n}{\ell} y\right) \cdot \sin\left(\frac{\pi}{\ell} n x\right).$$

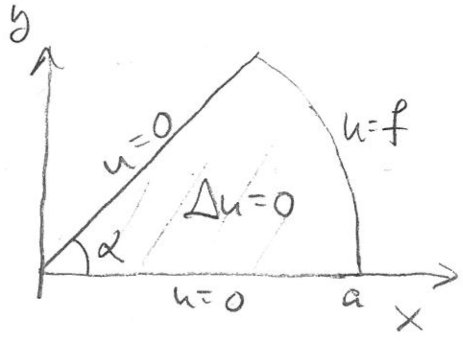
The boundary data φ gives B_n:

$$B_n \sinh\left(\frac{\pi n}{\ell} \cdot b\right) = \frac{2}{\ell} \int_0^{\ell} \varphi(x) \sin\left(\frac{\pi}{\ell} n x\right) dx$$

in this case.

<u>Ex. 4.12:</u> We

We want to solve the following Dirichlet problem for Laplace on the wedge:



$$\begin{cases} \Delta u = 0 & , 0 < r < a, \ 0 < \varphi < \alpha \\ u = 0 & , \varphi = 0 \text{ or } \varphi = \alpha \\ u = f(\varphi) & , r = a \end{cases}$$

Consider, for each fixed r, the one variable function

$$u_r(\varphi) := u(r,\varphi), \qquad \varphi \in (0,\alpha).$$

Apply Prop. 4.5 with $f = u_r$:

$$u_r(\varphi) = \sum_{n=1}^{\infty} b_n(r) \sin\left(\frac{\pi}{\alpha} n \varphi\right)$$

By Prop. 3.3

$$0 = \Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\varphi^2 u.$$

$$\Rightarrow \quad 0 = \sum_{n=1}^{\infty}\left(\frac{1}{r}(r b_n'(r))' + \frac{1}{r^2}\left(-\left(\frac{\pi n}{\alpha}\right)^2\right) b_n(r)\right) \sin\left(\frac{\pi}{\alpha} n \varphi\right).$$

Identify coefficients:

$$b_n''(r) + \frac{1}{r} b_n'(r) - \left(\frac{\pi n}{\alpha}\right)^2 \frac{1}{r^2} b_n(r) = 0$$

This is a socalled Euler ODE (coefficient constant $\cdot\, r^k$ in front of $b_n^{(k)}(r)$).

Trick to solve such ODEs: change variable to

$$x = \ln r.$$

$$\Rightarrow \frac{db_n}{dr} = \frac{db_n}{dx} \cdot \frac{1}{r}$$

$$\Rightarrow \frac{d^2 b_n}{dr^2} = \frac{d^2 b_n}{dx^2} \cdot \frac{1}{r^2} - \frac{db_n}{dx}\frac{1}{r^2}$$

$$\therefore \left(\frac{d^2 b_n}{dx^2}\frac{1}{r^2} - \frac{db_n}{dx}\frac{1}{r^2}\right) + \frac{1}{r}\left(\frac{db_n}{dx}\cdot\frac{1}{r}\right) - \left(\frac{\pi n}{\alpha}\right)^2 \frac{1}{r^2} b_n = 0.$$

$$\Leftrightarrow \quad \frac{d^2 b_n}{dx^2} - \left(\frac{\pi n}{\alpha}\right)^2 b_n = 0$$

$$\Rightarrow b_n(x) = A_n e^{\frac{\pi n}{\alpha}x} + B_n e^{-\frac{\pi n}{\alpha}x}$$

$$\Rightarrow b_n(r) = A_n r^{\frac{\pi n}{\alpha}} + B_n r^{-\frac{\pi n}{\alpha}}$$

We want solutions $u(x,y)$ that are good at the origin $(0,0)$. Thus $B_n = 0$ for all n.

Solution

$$u(r,\varphi) = \sum_{n=1}^{\infty} A_n r^{\frac{\pi n}{\alpha}} \sin\left(\frac{\pi}{\alpha} n \varphi\right).$$

Boundary data f gives A_n:

$$A \cdot a^{\frac{\pi n}{\alpha}} = \frac{2}{\alpha} \int_0^{\alpha} f(\varphi) \sin\left(\frac{\pi}{\alpha} n \cdot \varphi\right) d\varphi.$$

Summary of the algorithm used:

1. Introduce appropriate family of one-variable functions
2. Depending on type of boundary condition, expand each function in series, using Prop. 4.3, 4.5, 4.7 or 4.9.
3. Insert into PDE to obtain ODEs for the coefficients. Solve these ODEs.
4. Use Fourier integral formulas to find initial data for the ODEs in 3.

The reader is encouraged to see the similarity between this technique and the Fourier transform technique used in Theorems 2.1 and 2.2, in the case of an unbounded domain.