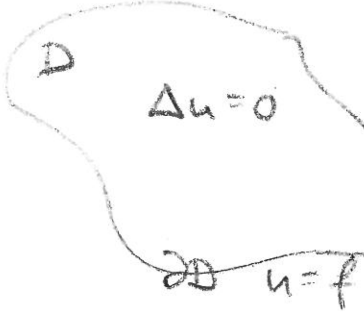
Part 4.2: Numerical solution of PDEs



Consider the Dirichlet problem for Δ in a domain $D \subset \mathbb{R}^n$. If the geometry of D is very simple, we have seen in part 3 how to solve for u by hand. However, for "real-life geometries", we need to resort to numerical methods to find u approximatively. We shall study two such methods here:

- FEM: the finite element method
- BEM: the boundary element method.

① FEM

We first reformulate the homogeneous Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases} \qquad (D_0)$$

as an inhomogeneous Dirichlet problem

$$\begin{cases} -\Delta v = g & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \qquad (D_1)$$

To this end, assume that $f(x)$ is defined for all $x \in \overline{D}$ (that is, we assume that $f: \partial D \to \mathbb{R}$ extends to a function $f: \overline{D} \to \mathbb{R}$) and is sufficiently smooth. Then (D_0) is equivalent to

$$\begin{cases} -\Delta(u-f) = \Delta f & \text{in } D, \\ u - f = 0 & \text{on } \partial D, \end{cases}$$

which is (D_1) with $v = u - f$ and $g = \Delta f$.
Given f, we compute g and solve (D_1) for v, giving the solution $u = v + f$ to (D_0).
The following "weak formulation" of (D_1) is used for FEM.

<u>Prop 4.13</u>: Let $H := \{w : \bar{D} \to \mathbb{R};\ w = 0 \text{ on } \partial D\}$.
Then the following are equivalent for $v \in H$:

- $-\Delta v = g$,
- $\iint_D (\nabla v, \nabla w)\, d\bar{x} = \iint_D g\, w\, d\bar{x}$, for all $w \in H$.

<u>Proof</u>:
Green 1 $\Rightarrow$

$$\iint_D (\Delta v \cdot w + \nabla v \cdot \nabla w)\, d\bar{x} = \int_{\partial D} \frac{\partial v}{\partial n} \cdot \cancelto{0}{w}\, ds$$

$$\Rightarrow \iint_D (\nabla v \cdot \nabla w)\, d\bar{x} - \iint_D g w\, d\bar{x} = -\iint_D (\Delta v + g) w\, d\bar{x}.$$

The right hand side is $= 0$ for all $w \in H$ if and only if $-\Delta v = g$. ∎

The FEM is the following:

- Choose "trial functions" $v_1, v_2, \dots, v_N$
- Search for an approximate solution of the form
$$\tilde{v}(\bar{x}) = V_1 v_1(\bar{x}) + V_2 v_2(\bar{x}) + \_\_ + V_N \cdot v_N(\bar{x}),$$
that is a linear combination of the trial functions.
- Require that $\tilde{v}(\bar{x})$ solves
$$\iint_D (\nabla \tilde{v}, \nabla \tilde{w})\, d\bar{x} = \iint_D g \tilde{w}\, d\bar{x} \quad \text{for}$$
all $\tilde{w}$ being linear combinations

$$\tilde{w}(\bar{x}) = W_1 v_1(\bar{x}) + W_2 v_2(\bar{x}) + \dots + W_N v_N(\bar{x})$$

of the trial functions as well.

We have

$$\sum_{i=1}^{N}\sum_{j=1}^{N} V_j W_i \underbrace{\iint_D (\nabla v_j, \nabla v_i)\, d\bar{x}}_{=: M_{ij}} = \sum_{i=1}^{N} W_i \underbrace{\iint_D g\, v_i(\bar{x})\, d\bar{x}}_{=: G_i}$$

$$\iff \sum_{i=1}^{N} W_i \left(\sum_{j=1}^{N} M_{ij} V_j - G_i \right) = 0$$

Since W_i are arbitrary, this is equivalent to the matrix equation

$$MV = G,$$

where M is the symmetric matrix with elements $M_{ij} = \iint_D (\nabla v_j, \nabla v_i)\, d\bar{x}$ and G is the column vector with entries $G_i = \iint_D g(\bar{x})\, v_i(\bar{x})\, d\bar{x}$.

<u>Definition 4.14:</u> M is called the <u>stiffness matrix</u> for the problem and trial functions considered.

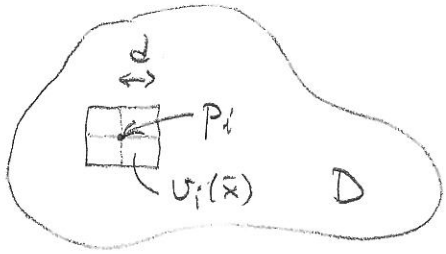
We summarize the FEM:

- choose trial functions $v_i(\bar{x})$
- compute M and G.
- solve the $N \times N$ equation system $MV = G$
- an approximate solution to (D_1) is $\tilde{v}(\bar{x}) = V_1 v_1(\bar{x}) + \dots + V_N v_N(\bar{x})$.

Example 4.15: We compute the stiffness matrix for "bilinear" trial functions on squares, in dimension $n=2$.

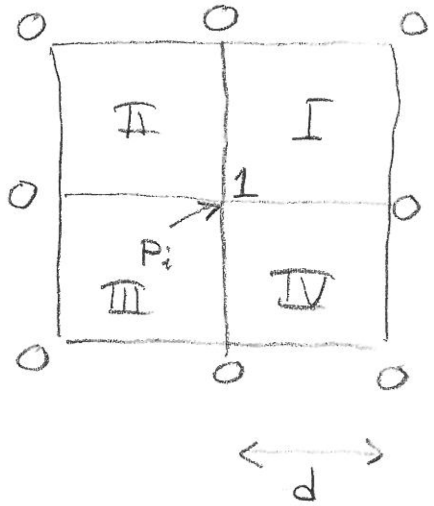
Fix $0<d\ll 1$ and consider mesh size d.

Let $p_i=(kd,\ell d)$ for some $k,\ell\in\mathbb{Z}$, be such that $\operatorname{dist}(p,\partial D)>\sqrt{2}\,d$.



Define the trial function

$$v_i(\bar{x}) := \begin{cases} \left[1-\left(\frac{x}{d}-k\right)\right]\left[1-\left(\frac{y}{d}-\ell\right)\right] & \text{in I} \\ \left[1+\left(\frac{x}{d}-k\right)\right]\left[1-\left(\frac{y}{d}-\ell\right)\right] & \text{in II} \\ \left[1+\left(\frac{x}{d}-k\right)\right]\left[1+\left(\frac{y}{d}-\ell\right)\right] & \text{in III} \\ \left[1-\left(\frac{x}{d}-k\right)\right]\left[1+\left(\frac{y}{d}-\ell\right)\right] & \text{in IV} \\ 0 & \text{elsewhere.} \end{cases}$$



This means that

- v_i is linear on the 4 lines into p_i, for example $v_i(kd,(\ell+t)d)=1-t$ for $t\in(0,1)$.
- v_i is continuous, but ∇v jumps at the boundaries of I, II, III, IV.
- $\Delta v_i=0$ except at the boundaries of I–IV, where it is undefined.

We compute the stiffness matrix M:

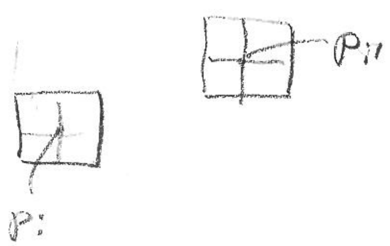
consider $p_i=(kd,\ell d)$ and $p_{i'}=(k'd,\ell' d)$.

Four cases:

- $\max(|k-k'|,|\ell'-\ell|)\geq 2$.

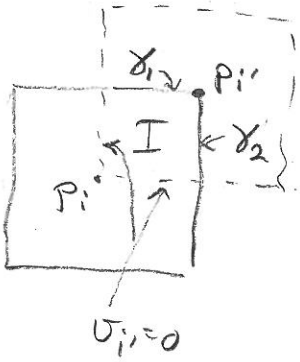
  Then clearly

  $$M_{i,i'}=\iint_D(\nabla v_i,\nabla v_{i'})\,d\bar{x}=0.$$



• $|k-k'| = 1 = |\ell'-\ell|$.

<u>Claim</u>: $M_{i,i'} = -\frac{1}{3}$.



By symmetry, we can assume $k'=k+1$ and $\ell'=\ell+1$.

$\Rightarrow M_{i,i'} = \iint_I (\nabla v_i, \nabla v_{i'}) =$ /Green 1/

$$= \int_{\partial I} \frac{\partial v_i}{\partial n} v_{i'} - \iint_I \underbrace{(\Delta v_i)}_{=0} v_{i'} = \int_{\gamma_1} \frac{\partial v_i}{\partial y} v_{i'} + \int_{\gamma_2} \frac{\partial v_i}{\partial x} v_{i'}$$

$$= \int_{\gamma_1} \left[\left(\frac{x}{d}-k-1\right)\frac{1}{d}\right] \underbrace{\left(1+\frac{x}{d}-k'\right)}_{=\frac{x}{d}-k} + \int_{\gamma_2} \left[\left(\frac{y}{d}-\ell-1\right)\frac{1}{d}\right] \underbrace{\left(1+\frac{y}{d}-\ell'\right)}_{\frac{y}{d}-\ell}$$

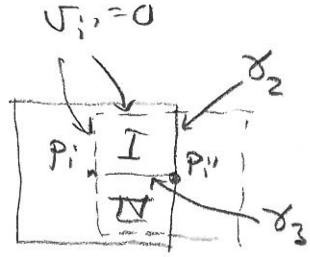
/ $x=(k+t)d$
$y=(\ell+s)d$ /

$$= \int_0^1 (t-1)\frac{1}{d}\, t\, d \cdot dt + \int_0^1 (s-1)\frac{1}{d}\, s\, d \cdot ds$$

$$= 2\int_0^1 (t^2-t)\,dt = 2\left(\frac{1}{3}-\frac{1}{2}\right) = -\frac{1}{3}.$$

• $|k-k'|=1$ and $|\ell-\ell'|=0$, or vice verse.

<u>Claim</u>: $M_{i,i'} = -\frac{1}{3}$



By symmetry, we can assume $k'=k+1$ and $\ell'=\ell$.

$\Rightarrow M_{i,i'} =$ /symmetry/ $= 2\iint_I (\nabla v_i, \nabla v_{i'})$

$=$ /Green 1/ $= 2\int_{\partial I} \frac{\partial v_i}{\partial n} v_{i'} = 2\int_{\gamma_2} \frac{\partial v_i}{\partial x} v_{i'} - 2\int_{\gamma_3} \frac{\partial v_i}{\partial y} v_{i'}$

$$= 2\int_{\gamma_2} \frac{1}{d}\left(\frac{y}{d}-\ell-1\right) \underbrace{\left(1-\left(\frac{y}{d}-\ell'\right)\right)}_{1+\ell-\frac{y}{d}} - 2\int_{\gamma_3} \frac{1}{d}\left(\frac{x}{d}-k-1\right) \underbrace{\left(1+\left(\frac{x}{d}-k'\right)\right)}_{\frac{x}{d}-k}$$

/ $y=(\ell+s)d$
$x=(k+t)d$ /

$$= 2\int_0^1 \frac{1}{d}\underbrace{(s-1)(1-s)}_{2s-1-s^2}\, d\cdot ds - 2\int_0^1 \frac{1}{d}\underbrace{(t-1)t}_{t^2-t} \cdot d\,dt$$

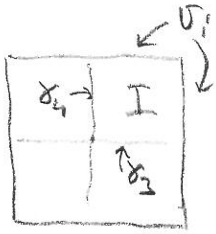
$$= 2\int_0^1 (3s-1-2s^2)\,ds = 2\left(\frac{3}{2}-1-\frac{2}{3}\right) = -\frac{1}{3}.$$

- $k'=k$ and $l'=l$

<u>Claim</u>: $M_{i,i} = \frac{8}{3}$

$$M_{i,i} = /\text{symmetry}/ = 4\iint_I (\nabla v_i, \nabla v_i) = /\text{Green } 1/$$

$$= 4\int_{\partial I} \frac{\partial v_i}{\partial n} v_i = -4\int_{\gamma_3} \frac{\partial v_i}{\partial y} v_i - 4\int_{\gamma_4} \frac{\partial v_i}{\partial x} v_i$$



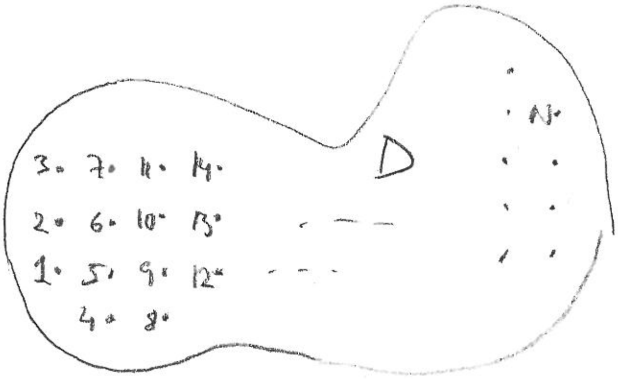
$$= -4\int_{\gamma_3} \frac{1}{d}\left(\frac{x}{d}-k-1\right)\left(1-\left(\frac{x}{d}-k\right)\right)$$

$$- 4\int_{\gamma_4} \frac{1}{d}\left(\frac{y}{d}-l-1\right)\left(1-\left(\frac{y}{d}-l\right)\right)$$

$$= -8\int_0^1 (t-1)(1-t) = 8\int_0^1 s^2\,ds = \frac{8}{3}.$$

<u>Summary</u>:

- list the points p_i of the form (kd, ld), which are interior to D as above, in order $i = 1, 2, \ldots, N$, for example as in picture.



- In the matrix M, the element $M_{i,j}$ in column i is $\neq 0$ only if j corresponds to the one of at most 8 neighbours to i.

Thus M is sparse: it is an $N\times N$ matrix ($N \sim 1/d^2$), but with at most $9\cdot N$ non-zero elements.

- For the computation of

$G_i = \iint_D g(\bar{x})\, v_i(\bar{x})$, we can as a first approx.

use $G_i \approx g(p_i) \cdot \iint_D v_i(\bar{x}) = 4 g(p_i) \iint_I v_i(\bar{x})$

$= 4 d^2 g(p_i) \int_0^1 \int_0^1 (1-x)(1-y)\, dx\, dy = d^2 g(p_i).$

(2) BEM

As in part 3, we can solve (D_0) by solving an integral equation of ∂D:

Prop 4.16: Given Dirichlet boundary data $f : \partial D \to \mathbb{R}$, solve the integral equation

$$f(\bar{x}) = \frac{1}{2} v(\bar{x}) + \int_{\partial D} v(\bar{y}) \frac{\partial \Phi}{\partial n}(\bar{y} - \bar{x})\, dS(\bar{y}), \quad x \in \partial D,$$

for $v : \partial D \to \mathbb{R}$. Then the solution to (D_0) is

$$u(\bar{x}) = \int_{\partial D} v(\bar{y}) \frac{\partial \Phi}{\partial n}(\bar{y} - \bar{x})\, dS(\bar{y}), \quad x \in D.$$

The BEM is the following:

- Write ∂D as a disjoint union of N subsets γ_i:
  $\partial D = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_N$, and pick $p_i \in \gamma_i$
- Search for $v(\bar{x})$ among functions
  $v(\bar{x}) = v_i$ on γ_i, $i = 1, \ldots, N$,
  which are piecewise constant.
- Require, as a first approx. that

$$f(p_i) = \frac{1}{2} v_i + \sum_{j \neq i} v_j \cdot n(p_j) \cdot \nabla \Phi(p_j - p_i) \cdot \underbrace{|\gamma_j|}_{\text{length of } \gamma_j}$$

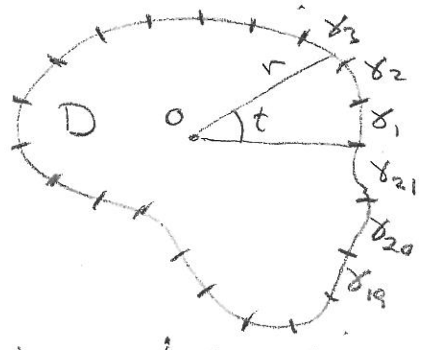
for all $i = 1, \ldots, N$.

• The BEM solution

$$u(\bar{x}) = \sum_{j=1}^{N} v_j \, n(p_j) \cdot \nabla\Phi(p_j - \bar{x}) \cdot |\gamma_j|$$

<u>Example 4.17:</u>

Let $h : \mathbb{R} \to \mathbb{R}$ be a positive 2π-periodic function and let $D \subset \mathbb{R}^2$ be the domain with ∂D described by h in polar coordinates.



Fix $N \in \mathbb{Z}_+$ and consider mesh size $\sim 1/N$. More precisely, let

$$\gamma_i := \{ h(t) \cdot (\cos t, \sin t) \in \partial D \ ; \ \tfrac{i-1}{N} < t < \tfrac{i}{N} \}, \quad i = 1, \dots, N$$

and

$$p_i := h(\tfrac{i}{N}) \cdot \left(\cos(\tfrac{i}{N}), \sin(\tfrac{i}{N}) \right)$$

As a first approx., we have

• $|\gamma_i| \approx |p_i - p_{i-1}|$

• $n(p_j) \approx (-i) \cdot \dfrac{p_j - p_{j-1}}{|p_j - p_{j-1}|}$

imaginary unit.

• $\nabla\Phi(p_j - p_i) = \dfrac{1}{2\pi} \dfrac{p_j - p_i}{|p_j - p_i|^2}$

$$\Rightarrow \quad n(p_j) \cdot \nabla\Phi(p_j - p_i) = \frac{-i}{2\pi} \frac{(p_j - p_{j-1}) \cdot (p_j - p_i)}{|p_j - p_{j-1}| \cdot |p_j - p_i|^2}$$

Define the (non-symmetric) matrix M with entries

$$M_{ij} := \begin{cases} 1/2, & j = i \\ -\dfrac{i}{2\pi} \dfrac{(p_j - p_{j-1}) \cdot (p_j - p_i)}{|p_j - p_{j-1}| \cdot |p_j - p_i|^2}, & j \neq i \end{cases}$$

Solve the $N \times N$ system of equations

$F = MV$, with $F = [f(p_1), \dots, f(p_N)]^t$, for $V = [v_1, \dots, v_N]^t$.

<u>A comparison between FEM & BEM:</u>

- FEM deals directly with the PDE, and thus requires at least continuity of the trial functions. BEM deals with an integral equation, and thus trial functions can be allowed to be discontinuous. (step functions simply.)

- With a mesh size ~~~~ d in the plane $\mathbb{R}^2$, FEM require solving an $(1/d^2)\times(1/d^2)$ system, BEM require solving an $\underbrace{(1/d)\times(1/d)}_{\text{much smaller!}}$ system.

  Sparseness of the stiffness-matrix for FEM needs to be exploited if to compete with BEM.

- In the plane, discretization of ∂D = a curve, is geometrically simpler than discretization of D.

- FEM deals naturally with the Poisson equation (D_1), whereas BEM deals naturally with the Laplace equation (D_0).

- BEM gives an exact solution to the PDE $\Delta u = 0$ (but approx. boundary conditions).
  FEM gives exact boundary conditions (but approx. PDE $\Delta u \cong 0$).