Part 4.3: Eigenfunction expansions in dimension $n=2$ and $n=3$

Recall Definition 4.1 of eigenfunctions $u(x)$ to Δ. We here consider a bounded domain $D \subset \mathbb{R}^n$, $n=2$ or 3, with smooth boundary, and Dirichlet or Neumann boundary conditions.

Definition 4.18:

- Denote by $-\lambda_1 \geq -\lambda_2 \geq -\lambda_3 \geq \dots$ the Dirichlet eigenvalues of Δ, and write $u_j(x)$ for the corresponding Dirichlet eigenfunctions.
- Denote by $-\tilde{\lambda}_1 \geq -\tilde{\lambda}_2 \geq -\tilde{\lambda}_3 \geq \dots$ the Neumann eigenvalues of Δ, and write $\tilde{u}_j(x)$ for the corresponding Neumann eigenfunctions.

Proposition/Definition 4.19:

If u is an eigenfunction (Dirichlet or Neumann), with eigenvalue $-\lambda$, then

$$\lambda = \frac{\iint_D |\nabla u|^2 d\bar{x}}{\iint_D |u|^2 d\bar{x}} =: R(u)$$

$\Leftrightarrow$ Rayleigh quotient of u.

Proof: Green 1 $\Rightarrow$

$$\iint_D (\underbrace{\Delta u}_{=-\lambda u} \cdot u + |\nabla u|^2)\, d\bar{x} = \int_{\partial D} \underbrace{(\partial_n u)}_{=0 \text{ if Neumann}} \cdot \underbrace{u}_{=0 \text{ if Dirichlet}}\, ds = 0$$

∎

We note in particular that all eigenvalues $-\lambda$ are ≤ 0, so

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and}$$

$$0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$$

In fact $\lambda_1 > 0$, $\tilde{\lambda}_1 = 0$ and $\tilde{\lambda}_2 > 0$ since
$\Delta u = 0 \Rightarrow R(u) = 0 \Rightarrow u = \text{constant}$,
and if Dirichlet boundary conditions then $u = 0$.

Proposition / Definition 4.20:

If $\lambda_i \neq \lambda_j$, then $\iint_D u_i(\bar{x})\, u_j(\bar{x})\, d\bar{x} = 0$.
If $\lambda_i = \lambda_j$, then we define (by Grem-Schmidt orthogonalisation) u_i and u_j so that

$$\iint_D u_i(\bar{x})\, u_j(\bar{x})\, d\bar{x} = 0.$$

Same orthogonality result holds for the Neumann eigenfunctions.

Proof: Green 2 $\Rightarrow$

$$\iint_D \Big(\underset{\lambda_i u_i}{\cancel{\Delta u_i}}\, u_j - u_i\, \underset{\lambda_j u_j}{\cancel{\Delta u_j}}\Big) d\bar{x} = \int_{\partial D} \Big(\partial_n u_i\, \overset{0}{\cancel{u_j}} - \overset{0}{\cancel{u_i}}\, \partial_n u_j\Big) ds$$

$$\Rightarrow \underbrace{(\lambda_i - \lambda_j)}_{\neq 0} \iint_D u_i u_j\, d\bar{x} = 0 \qquad ∎$$

Note: If an eigenvalue $-\lambda$ is not simple, that is if $\dim\{u;\ \Delta u = -\lambda u\} \geq 2$, then the corresponding eigenfunctions $u_i(\bar{x})$ are not uniquely determined (not even modulo constant factors).

Example 4.21:

In one dimension, any domain is an interval $D = (0, \ell)$ (after translation), so the geometry is simple.
By Prop. 4.4:

$$\lambda_n = \left(\frac{\pi n}{\ell}\right)^2, \quad n = 1, 2, \ldots$$

with eigenfunctions $u_n(x) = \sin\left(\frac{\pi}{\ell} n x\right)$.

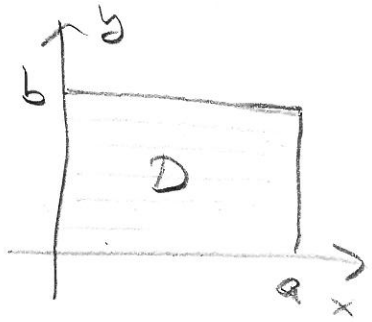
By Prop. 4.6:

$$\tilde{\lambda}_n = \left(\frac{\pi}{\ell}(n-1)\right)^2, \quad n=1,2,\dots$$

with $\tilde{u}_n(x) = \cos\left(\frac{\pi}{\ell}(n-1)x\right)$

(changing notation slightly).

Example 4.22



Consider a rectangle

$D = (0,a)\times(0,b)$ in the plane.

We search for Dirichlet eigenfunctions:

$$\begin{cases} \Delta u = -\lambda u & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

Set $u_y(x) := u(x,y)$ and use Prop. 4.5:

$$u_y(x) = \sum_{n=1}^{\infty} b_n(y)\sin\left(\frac{\pi}{a}nx\right)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(b_n''(y) - \left(\frac{\pi}{a}n\right)^2 b_n(y)\right)\sin\left(\frac{\pi}{a}nx\right) = -\lambda\sum_{n=1}^{\infty} b_n(y)\sin\left(\frac{\pi}{a}nx\right)$$

$$\Rightarrow \forall n: \quad b_n''(y) + \left(\lambda - \left(\frac{\pi}{a}n\right)^2\right)b_n(y) = 0 \qquad 0 < y < b$$

$$b_n(0) = b_n(b) = 0$$

Let $\omega := \sqrt{\lambda - \left(\frac{\pi}{a}n\right)^2}$. Then

$$b_n(y) = \underbrace{\cancel{A}}_{=0 \text{ since } b_n(0)=0}\cos(\omega y) + B\sin(\omega y),$$

and $0 = b_n(b) = \underbrace{B}_{\neq 0}\sin(\omega b) \Rightarrow \omega = \frac{\pi}{b}m, \; m \in \mathbb{Z}_+.$

$$\Rightarrow \lambda = \omega^2 + \left(\frac{\pi}{a}m\right)^2 = \pi^2\left(\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right)$$

Thus the eigenvalues are of the form

$$\lambda_k = \pi^2\left(\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right), \quad n,m = 1,2,\dots$$

with eigenfunctions

$$u_k(x,y) = \sin\left(\frac{\pi}{a}nx\right)\sin\left(\frac{\pi}{b}my\right).$$

A similar argument based on Prop. 4.7 gives the Neumann eigenvalues

$$\tilde{\lambda}_k = \pi^2\left(\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right), \quad n, m = \underline{\underline{0}}, 1, 2, \dots$$

with eigenfunctions

$$\tilde{u}_k(x,y) = \cos\left(\frac{\pi}{a} n x\right) \cos\left(\frac{\pi}{b} m x\right). \quad \blacksquare$$

For general domains $D \subset \mathbb{R}^n$, $n = 2, 3$, one cannot compute explicitly the eigenvalues (and even less the eigenfunctions). We next study methods for estimating the eigenvalues λ_n.

<u>Example 4.23</u>: (Rayleigh-Ritz)

Consider a Dirichlet eigenvalue for a domain D:

$$\begin{cases} \Delta u = -\lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

By Green 1, this is equivalent to

$$\int_D (\nabla u, \nabla v)\, d\bar{x} = \lambda \int_D u v \, d\bar{x}, \quad \text{for all } v \text{ in } D \text{ with } v = 0 \text{ on } \partial D.$$

Similar to part 4.2, we can make a FEM-approximation:

Let $w_1, \dots, w_N$ be given functions in D all with $w_j = 0$ on ∂D.

Assume that $u(x) = \sum_{j=0}^{N} c_j w_j(x)$.

(If we look for approximation to the first N eigenvalues/vectors, the ansatz $w_1, \dots, w_N$ should resemble these.)

Testing with $v = w_i$ for $i = 1, \dots, N$, we get the system of equations

$$\sum_{j=1}^{N} c_j \underbrace{\int_D (\nabla w_j, \nabla w_i)\, d\bar{x}}_{=: a_{ij}} = \lambda \sum_{j=1}^{N} \underbrace{\int_D w_j w_i}_{=: b_{ij}}$$

Let A and B be the symmetric matrices with entries a_{ij} and b_{ij} respectively, and let $c = [c_1, \dots, c_N]^t$. Then we have

$$(A - \lambda B) c = 0$$

The Rayleigh-Ritz approximation to the N first Dirichlet eigenvalues $\lambda_1, \dots, \lambda_N$ are the zero of

$$\det (A - \lambda B).$$

The corresponding approximation of the eigenvector is $w = \sum c_j w_j$, where c solves $(A - \lambda B) c = 0$.

We now turn to another characterisation of the Dirichlet eigenvalues, referred to as the "minimum principle" and the "maximum principle" in the course book. This will be our main tool in the remainder of this part of the course.

Proposition 4.24:

Fix $n \geq 1$ and consider the n'th Dirichlet eigenvalue λ_n. Let

$$m := \min \{ R(w) \,;\, w|_{\partial D} = 0 \text{ and } \int_D w \overset{\text{(eigenfunctions)}}{\overset{\downarrow}{v_j}} = 0 \text{ for } j = 1, \dots, n-1 \};$$

(i) If this minimum is attained for some u, then u is an eigenfunction to Δ with eigen-value $-m$.

(ii) The minimum value m equals λ_n.

(iii) Let $Y_1, \dots, Y_{n-1}$ be any functions in D (they may be nonzero on ∂D, discontinuous, ...).

Let $m_* := \min\{R(w);\ w|_{\partial D} = 0,\ \int_D w Y_j = 0$ for $j = 1, 2, \dots, n-1\}$.

Then $m_* \leq m = \lambda_n$.

Before the proof, we remark that it is a fact, but not obvious, that the minima m and m_* are attained. If $n = 1$, the constraints $\int_D w v_j = 0$ and $\int_D w Y_j = 0$ are void.

Proof:

(i) We need to show

$$\int_D (\nabla u, \nabla v)\, d\bar{x} = m \int_D u v\, d\bar{x} \quad \text{for all } v \text{ with } v|_{\partial D} = 0.$$

Let first $v = v_j$ for some $j = 1, \dots, n-1$.

Then by Green 1,

$$-\int_D u \underbrace{\Delta v_j}_{= -\lambda_j v_j} = m \int_D u v_j.$$

But $\int_D u v_j = 0$ by assumption,

Thus we may assume $\int_D v v_j\, d\bar{x} = 0$, $j = 1, \dots, n-1$.

Let $w := u + \varepsilon v$ for some $\varepsilon > 0$, Then, since u is the minimum and u and w both satisfy the required constraints, we have

$$R(w) \geq R(u)$$

In particular, if

$$f(\varepsilon) := \frac{\int_D |\nabla(u+\varepsilon v)|^2 \, d\bar{x}}{\int_D |u+\varepsilon v|^2 \, d\bar{x}}$$

has derivative $f'(0) = 0$.

$$\Rightarrow \frac{d}{d\varepsilon} \left. \frac{\int |\nabla u|^2 + 2\varepsilon \int (\nabla u, \nabla v) + \varepsilon^2 \int |\nabla v|^2}{\int u^2 + 2\varepsilon \int uv + \varepsilon^2 \int v^2} \right|_{\varepsilon=0}$$

$$= \frac{2\int (\nabla u, \nabla v)}{\int u^2} - \frac{2\int uv}{(\int u^2)^2} \int |\nabla u|^2 = 0$$

$$\Leftrightarrow \int (\nabla u, \nabla v) = \underbrace{\frac{\int |\nabla u|^2}{\int u^2}}_{=m} \int uv$$

(ii) By Prop. 4.19, we have

$$m \leq R(v_n) = \lambda_n$$

If $m = \lambda_j < \lambda_n$ for some $j = 1, \dots, n-1$, then a minimizer u is an eigenfunction with eigen-val. λ_j by (i). At the same time, $\int_D uv \, d\bar{x} = 0$ for any such eigenfunction v, by the constraints in the definition of m. In particular, $\int_D u^2 \, d\bar{x} = 0$, so $u \equiv 0$, a contradiction.

It follows that $m = \lambda_n$.

(iii) Let $y_1, \dots, y_{n-1}$ be given.

Consider $w(\bar{x}) = c_1 \underset{\uparrow}{v_1}(\bar{x}) + \dots + c_n v_n(\bar{x})$.

(the eigen functions)

We can, and do, choose the coefficients $c_j \not\equiv 0$ so that

$$\int_D w y_j = \sum_{i=1}^{n} \left(\int_D v_i y_j \right) c_i = 0 \quad \text{for } j=1,\dots,n-1.$$

(n unknowns, n-1 equations!)

Then

$$m_* \le R(w) = \frac{\int_D |\nabla w|^2}{\int_D w^2}.$$

By orthogonality (Prop. 4.20):

$$\int_D w^2 = c_1^2 \int_D v_1^2 + \dots + c_n^2 \int_D v_n^2$$

$$\int_D |\nabla w|^2 = -\int_D \Delta w \cdot w = c_1^2 \lambda_1 \int_D v_1^2 + \dots + c_n^2 \lambda_n \int_D v_n^2$$

$$\Rightarrow \quad R(w) = \sum_{j=1}^{n} \left(\frac{c_j^2 \int_D v_j^2}{\int_D w^2} \right) \lambda_j$$

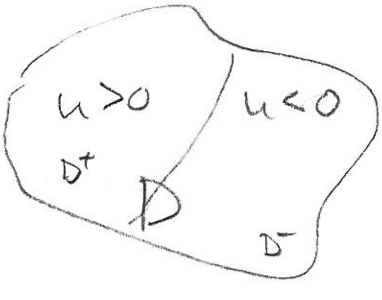
$$\le \lambda_n \sum_{j=1}^{n} \frac{c_j^2 \int_D v_j^2}{\int_D w^2} = \lambda_n \quad \blacksquare$$

Example 4.25: (≈ 11.6.7)

Consider the first eigenvalue:

$$\lambda_1 = \min \{ R(w);\ w|_{\partial D} = 0 \}$$

Claim: The first Dirichlet eigenfunction $u_1(\vec{x})$ (the minimizer) is non-zero in all D.



If not, then $u > 0$ in $D^+ \neq \emptyset$ and $u < 0$ in $D^- \neq \emptyset$. Let

$$|u| := \begin{cases} u & \text{in } D^+ \\ -u & \text{in } D^- . \end{cases}$$

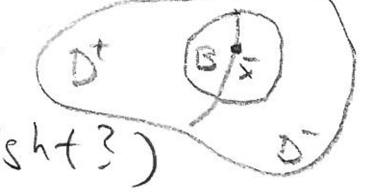
Then $R(|u|) = R(u) = \lambda_1$, so by Prop. 4.24 (i)

$|u|$ is also an eigenfunction. Now consider $v := |u| + u \begin{cases} >0 & \text{in } D^+ \\ =0 & \text{in } D^- \end{cases}$, which also is a Dirichlet eigenfunction: $\Delta v = -\lambda_1 v$. To show that this is impossible, let $\bar{x} \in \partial D_1 \cap \partial D_2$ and $B(\bar{x}, r) \subset D$ a ball.

Green 3 $\Rightarrow$

$$0 = v(\bar{x}) = \int_{\partial B} \underbrace{v(\xi)}_{\geq 0} \underbrace{\frac{\partial G_B}{\partial n}(\xi, \bar{x})}_{>0} \, dS(\xi) + \iint_B \underbrace{G_B(\xi, \bar{x})}_{<0} \underbrace{\Delta v(\xi)}_{= -\lambda_1 v \leq 0} d\xi$$

The right hand side is seen to be > 0, a contradiction.



(We generalised the maximum principle, right?)

Example 4.26 (evolution of eigenfunctions)

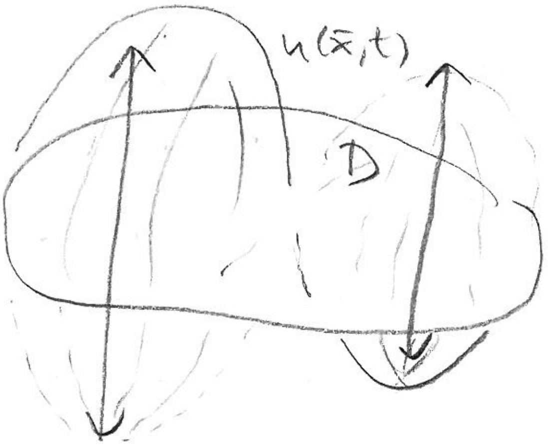
Assume $v(\bar{x})$ is an eigenfunction, with eigenvalue $-\lambda$. Consider the initial value problem

$$\begin{cases} \partial_t^2 u(\bar{x}, t) = c^2 \Delta u(\bar{x}, t) \\ u(\bar{x}, 0) = v(\bar{x}) \\ \partial_t u(\bar{x}, 0) = 0 \end{cases}$$

for the wave equation.

Since $\Delta v = -\lambda v$, it follows by solving the ODE $\partial_t^2 u = -c^2 \lambda u$, that the solution is

$$u(\bar{x}, t) = \cos(c\sqrt{\lambda}\, t) \cdot v(\bar{x})$$



Thus, wave evolution of an eigenfunction gives rise to a standing wave $u(\bar{x}, t)$, where shape of $u(\cdot, t)$ is unchanged, except for a scaling by the cos-factor.

The musical interpretation is a pure note, containing only one frequency $\omega = c\sqrt{\lambda}$.

Note for the oscillation, that for higher eigenfunctions, if $v(\bar{x}) = 0$ at a point $\bar{x}$ initially, then $u(\bar{x}, t) = 0$ for all t.

Our goal for the remainder of this part is to prove the following asymptotic formula:

Theorem 4.27:

Consider Dirichlet eigenvalues λ_n for large n.

- If $D \subset \mathbb{R}^1$, then $\lim\limits_{n \to \infty} \frac{\lambda_n}{n^2} = \left(\frac{\pi}{l}\right)^2$
- If $D \subset \mathbb{R}^2$, then $\lim\limits_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{A}$
- If $D \subset \mathbb{R}^3$, then $\lim\limits_{n \to \infty} \frac{\lambda_n}{n^{2/3}} = \left(\frac{6\pi^2}{V}\right)^{2/3}$

when l, A and V denotes length, area and volume of D respectively.

The case $n=1$ follows trivially from Example 4.21, since in fact $\lambda_n = \left(\frac{\pi}{l} n\right)^2$.

In dimension $n=2$, the geometry of D can be much more complicated, and so the proof of Thm 4.27. We only consider $n=2$, since the proof for $n=3$ is similar.

Before the proof of Thm 4.27, we make the following observation:

If we know the Dirichlet eigenvalues of Δ on D, then we know the area A of D (large eigenvalues $\Leftrightarrow$ small area, right?)

In fact, for this we only need to know c in

$$\lambda_n \approx c \cdot n \quad , \text{as } n \to \infty.$$

A natural question arises:

If you know all λ_n's, can you determine the shape of D completely? (not only its area)

The mathematician Mark Kac wrote an article "Can you hear the shape of a drum?" in American Mathematical Monthly on this question. (See wikipedia.)

For the proof of Thm 4.27, we next prove a number of results we need.

Define

$$N(\lambda) := \text{number of eigenvalues} \leq \lambda.$$

Thus $N(\lambda_n) = n$.

We need to show

$$\frac{N(\lambda)}{\lambda} \xrightarrow{\lambda \to \infty} \frac{A}{4\pi} \quad , \text{since then}$$

$$\frac{\lambda_n}{n} = \frac{\lambda_n}{N(\lambda_n)} \xrightarrow{n \to \infty} \frac{4\pi}{A}.$$

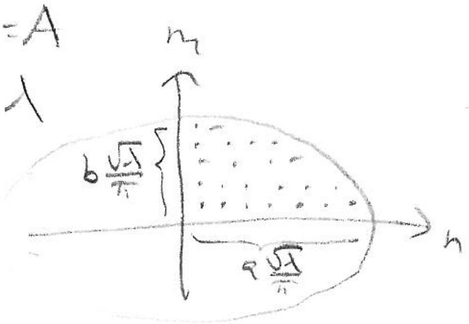
Example 4.28:

We show that Thm 4.27 is true when $n=2$ and $D=(0,a)\times(0,b)$ is a rectangle.
By Ex. 4.22,
$N(\lambda)$ is the number of points (n,m), $n,m=1,2,\dots$, in the plane, such that

$$\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2 \le \frac{\lambda}{\pi^2}$$

Approximating this number by the area of the quarter-ellipse, we have

$$N(\lambda) \approx \frac{1}{4}\pi\left(a\frac{\sqrt{\lambda}}{\pi}\right)\cdot\left(b\frac{\sqrt{\lambda}}{\pi}\right) = \frac{\overset{=A}{\boxed{ab}}}{4\pi}\cdot\lambda$$



We need the following analogue of Prop. 4.24 for the Neumann eigenvalues.

Prop. 4.29:

Fix $n\ge 1$ and consider the n'th Neumann eigenvalue $\tilde\lambda_n$. Let

$$\tilde m := \min\{R(w);\ \int_D w\tilde v_j\,d\bar x=0 \text{ for } j=1,\dots,n-1\}.$$

(i) If this minimum is attained for some u, then u is a Neumann eigenfunction to Δ with eigenvalue $-\tilde m$.

(ii) The minimum value $\tilde m$ equals $\tilde\lambda_n$.

(iii) Let $y_1,\dots,y_{n-1}$ be any functions in D (they may be non-zero on ∂D, discontinuous, ..).
Let $\tilde m_* := \min\{R(w);\ \int_D w\, y_j\,d\bar x=0$ for $j=1,\dots,n-1$.
Then $\tilde m_* \le \tilde m = \tilde\lambda_n$.

Proof: The proof is similar to that of Prop. 4.24, and we only point out the main differences.

(i) It suffices to show

$$\iint_D (\nabla u, \nabla v)\, d\bar{x} = \tilde{m} \iint_D u v \, d\bar{x} \quad \text{for all } v$$

(possibly $v|_{\partial D} \neq 0$).

Indeed, Green 1 gives

$$\int_{\partial D} (\partial_n u) v \, ds = \iint_D (\Delta u + \tilde{m} u) v \, d\bar{x}$$

Choosing v first with $v|_{\partial D} = 0$, gives $\Delta u + \tilde{m} u$. Next chosing v arbitrary on ∂D, gives $\partial_n u = 0$.

Note: In Prop. 4.24 we assumed Dirichlet boundary conditions for u. Here we prove Neumann boundary conditions for u.

(ii) and (iii) are the same as in Prop. 4.24. ■

Prop. 4.30: The Neumann eigenvalues are smaller than the Dirichlet eigenvalues:

$$\tilde{\lambda}_n \leq \lambda_n \quad \text{for } n = 1, 2, \ldots$$

Proof: By Props. 4.24 and 4.29:

$$\tilde{\lambda}_n = \max_{y_j} \underbrace{\min\{R(w);\ \textstyle\int w y_j = 0\}}_{\leq \min\{R(w);\ w|_{\partial D} = 0 \text{ and } \int w y_j = 0\}}$$

$$\leq \lambda_n \quad ■$$

Prop. 4.31: The Dirichle eigenvalues decreases as the domain increases:

$$D_1 \subset D_2 \Rightarrow \lambda_n(D_2) \leq \lambda_n(D_1) \text{ for all } n=1,2,\dots$$

Proof: Let $v_1, \dots, v_{n-1}$ denote eigenfunctions on D_2. By Prop. 4.24

$$\lambda_n(D_2) = \min\left\{ \frac{\int_{D_2}|\nabla w|^2}{\int_{D_2}|w|^2} \,;\, w|_{\partial D_2}=0 \,,\, \int_{D_2} w v_j = 0 \;\; j=1,\dots,n-1 \right\}.$$

Next consider D_1, and let

$$y_j := v_j|_{D_1}$$

If w_1 is a function on D_1 with $w_1|_{\partial D_1}=0$, we extend it by zero to w_2 on D_2:

$$w_2(\bar{x}) := \begin{cases} w_1(\bar{x}) & , \bar{x}\in D_1 , \\ 0 & , \bar{x} \in D_2 \setminus D_1 . \end{cases}$$

$$\Rightarrow m_*(D_1) = \min\left\{ \frac{\int_{D_1}|\nabla w_1|^2}{\int_{D_1} w_1^2} \,;\, w_1|_{\partial D_1} = 0 \,,\, \int_{D_1} w_1 v_j = 0 \right\}$$

$$\geq \min\left\{ \frac{\int_{D_2}|\nabla w_2|^2}{\int_{D_2} w_2^2} \,;\, w_2|_{\partial D_2} = 0 \,,\, \int_{D_2} w_2 v_j = 0 \right\} = \lambda_n(D_2)$$

$$\Rightarrow \lambda_n(D_1) \overset{4.24\,(iii)}{\geq} m_*(D_1) = \lambda_n(D_2) \;\blacksquare$$

The next two results consider a domain D, which is split into N disjoint subdomains D_j

$$D = D_1 \cup \dots \cup D_N ,$$

modulo the boundaries.

On D we have Dirichlet eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \dots$$

and on each D_j, we have

$\lambda_1(D_j') \le \lambda_2(D_j') \le \lambda_3(D_j') \le \dots$

Taking the union of these N sequences, and reorder the eigenvalues in increasing order, we obtain a sequence

$\mu_1 \le \mu_2 \le \mu_3 \le \dots$

where each μ_i is some eigenvalue $\lambda_j(D_k)$ on some subdomain.

We do the same for the Neumann eigenvalues:

$\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \dots$ for D, and

$\tilde{\mu}_1 \le \tilde{\mu}_2 \le \dots$ for the subdomains.

Prop 4.32: $\lambda_n \le \mu_n$ for each $n = 1, 2, \dots$

Proof: Let $v_1, \dots, v_n$ be functions on D such that each v_j, for some k is an eigenfunction on D_k with eigenvalue μ_j, and $v_j|_{D \setminus D_k} = 0$.

Fix $y_1, \dots, y_{n-1}$ on D, and let

$w(\bar{x}) := c_1 v_1(\bar{x}) + \dots + c_n v_n(\bar{x})$ on D be such that

$$\int_D w y_j = \sum_{k=1}^{n} \left(\int_D v_k y_j \right) c_k = 0 \quad \text{for } j = 1, \dots, n-1.$$

Since we have n unknowns and $n-1$ equations, we can choose $c_k \neq 0$.

$$\Rightarrow R(w) = \frac{\int_D |\nabla w|^2}{\int_D w^2} = \text{/ disjointness or Prop. 4.20 /}$$

$$= \frac{c_1^2 \mu_1 \int_D v_1^2 + \dots + c_n^2 \mu_n \int_D v_n^2}{c_1^2 \int_D v_1^2 + \dots + c_n^2 \int_D v_n^2} \le \mu_n \frac{c_1^2 \int_D v_1^2 + \dots + c_n^2 \int_D v_n^2}{c_1^2 \int_D v_1^2 + \dots + c_n^2 \int_D v_n^2}$$

$$= \mu_n.$$

Thus

$\lambda_n = \max_{y_j} \underbrace{m_*}_{\le \mu_n} \le \mu_n$ by Prop. 4.24. ■

Prop. 4.33: $\tilde{\lambda}_n \ge \tilde{\mu}_n$ for each $n=1,2,\dots$

Proof: Let $y_1, \dots, y_{n-1}$ be functions on D such that each y_j is a Neumann eigenfunction of D_k for some k, with eigenvalue $\tilde{\mu}_j$.

Let w be any function on D with $\int_D w y_j \, d\vec{x} = 0$, $j = 1, \dots, n-1$.

$$\Rightarrow R(w) = \frac{\int_{D_1} |\nabla w|^2 + \dots + \int_{D_N} |\nabla w|^2}{\int_{D_1} w^2 + \dots + \int_{D_N} w^2}$$

By construction, $w|_{D_k}$ is orthogonal to any eigenfunction on D_k with eigenvalue $\le \tilde{\mu}_{n-1}$.

Thus $\dfrac{\int_{D_k} |\nabla w|^2}{\int_{D_k} w^2} \ge \tilde{\mu}_n$ by Prop. 4.29.

$$\Rightarrow R(w) = \sum_{k=1}^{N} \left(\frac{\int_{D_k} w^2}{\int_D w^2} \right) \underbrace{\frac{\int_{D_k} |\nabla w|^2}{\int_{D_k} w^2}}_{\ge \tilde{\mu}_n} \ge \tilde{\mu}_n$$

Thus $\tilde{\lambda}_n = \max_{y_j} \underbrace{m_*}_{\ge \tilde{\mu}_n} \ge \tilde{\mu}_n$. ■

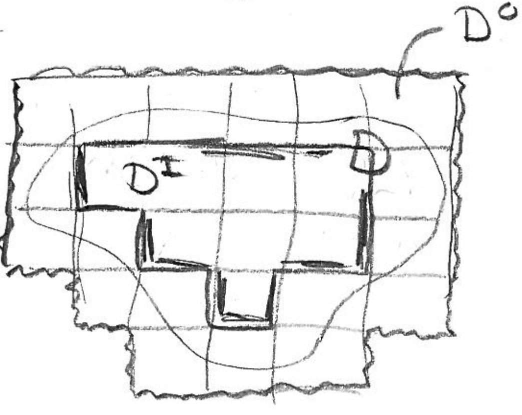
We are now in position to prove Thm 4.27.

Proof of Thm. 4.27:

We need to show $\frac{N(\lambda)}{\lambda} \longrightarrow \frac{A(D)}{4\pi}$ as $\lambda \to \infty$.

Let $\varepsilon > 0$, and let $D_1, D_2, \dots, D_{N_2}$ be disjoint rectangles such that

$$\underbrace{\bigcup_{k=1}^{N_1} D_k}_{=:D^I} \subset D \subset \underbrace{\bigcup_{k=1}^{N_2} D_k}_{=:D^O}$$



modulo boundaries, for some $1 \le N_1 < N_2$, and with

$$A(D^O \setminus D) \le \varepsilon \text{ and } A(D \setminus D^I) \le \varepsilon.$$

- On each D_k, we have $\frac{N(\lambda, D_k)}{\lambda} \longrightarrow \frac{A(D_k)}{4\pi}$ as $\lambda \to \infty$, by Ex. 4.28.

  By Prop. 4.32: $\frac{N(\lambda, D^I)}{\lambda} \ge \sum_{k=1}^{N_1} \frac{N(\lambda, D_k)}{\lambda} \longrightarrow \frac{A(D^I)}{4\pi}$

- Write

  $\tilde{N}(\lambda, D) :=$ number of Neumann eigenvalues $\le \lambda$ on D.

  Similarly to Ex. 4.28:

  $$\frac{\tilde{N}(\lambda, D_k)}{\lambda} \longrightarrow \frac{A(D_k)}{4\pi} \text{ as } \lambda \to \infty$$

  for each rectangle D_k.

  Prop. 4.33: $\frac{\tilde{N}(\lambda, D^O)}{\lambda} \le \sum_{k=1}^{N_2} \frac{\tilde{N}(\lambda, D_k)}{\lambda} \longrightarrow \frac{A(D^O)}{4\pi}$

- By Props 4.31 and 4.30:

  $$N(\lambda, D^I) \le N(\lambda, D) \le N(\lambda, D^O) \le \tilde{N}(\lambda, D^O).$$

We conclude that

$$\underline{\lim_{\lambda\to\infty}} \frac{N(\lambda,D)}{\lambda} \geq \frac{A(D^I)}{4\pi} \quad \text{and}$$

$$\overline{\lim_{\lambda\to\infty}} \frac{N(\lambda,D)}{\lambda} \leq \frac{A(D^0)}{4\pi}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{\lambda\to\infty} \frac{N(\lambda,D)}{\lambda} = \frac{A(D)}{4\pi} \text{ follows. } \blacksquare$$

We end part 4 by extending the completeness results from prop 4.5 in dimension 1, to higher dimension.

Thm 4.34: Let $D \subset \mathbb{R}^n$, $n=2,3$, be a domain with Dirichlet eigenfunctions $u_1(\bar{x}), u_2(\bar{x}), \ldots$ and eigenvalues $\lambda_1, \lambda_2, \ldots$.

Given any function f on D, define

$$c_k := \frac{\int_D f u_k \, d\bar{x}}{\int_D u_k^2 \, d\bar{x}}.$$

Then $$\int_D \left| f(\bar{x}) - \sum_{k=1}^{N} c_k u_k(\bar{x}) \right|^2 d\bar{x} \to 0 \text{ as } N \to \infty.$$

Remark: we view c_k as generalised Fourier coefficients of f on D.

<u>Proof</u>: We may assume that f is smooth and $f|_{\partial D} = 0$, since any square integrable function can be approximated by such.

Let $r_N(\bar{x}) := f(\bar{x}) - \sum_{k=1}^{N} c_k u_k(\bar{x})$

$$\Rightarrow \int_D r_N u_j = \int_D f u_j - \sum_{k=1}^{N} \frac{\int_D f u_k}{\int_D u_k^2} \underbrace{\int_D u_k u_j}_{=0 \text{ if } k \neq j} = 0 \text{ if } j \leq N$$

By Prop. 4.24

$$\lambda_{N+1} \leq \frac{\int_D |\nabla r_N|^2}{\int_D r_N^2},$$

so $\int_D r_N^2 \leq \frac{1}{\lambda_{N+1}} \int_D |\nabla r_N|^2 \to 0$ as $N \to \infty$ as wanted

since $\lambda_{N+1} \to \infty$ by Thm 4.27

<u>if</u> we can prove that $\int_D |\nabla r_N|^2$ is bounded as $N \to \infty$.

$$\int_D |\nabla r_N|^2 = \underbrace{\int_D (\nabla f, \nabla r_N)} - \sum_{k=1}^{N} c_k \int_D (\nabla u_k, \nabla r_N)$$

$$= \int_D |\nabla f|^2 - \sum_{k=1}^{N} c_k \int_D (\nabla f, \nabla u_k)$$

$=$ / Green 1 /

$$= \int_D |\nabla f|^2 + \sum_{k=1}^{N} c_k \int_D (f + r_N, \underbrace{\Delta u_k}_{= -\lambda_k u_k})$$

$$= \int_D |\nabla f|^2 - \sum_{k=1}^{N} c_k \lambda_k \left(2 c_k \int_D u_k^2 - c_k \int u_k^2 \right)$$

$$= \int_D |\nabla f|^2 - \sum_{k=1}^{N} c_k^2 \lambda_k \underbrace{\int_D u_k^2}_{\geq 0} \leq \underset{\uparrow}{\int_D |\nabla f|^2}$$

independent of N ∎