# Part 5.1: Systems of linear first order PDEs

We have so far studied

- the heat equation: $\partial_t u = k\Delta u$,
- the wave equation: $\partial_t^2 u = c^2 \Delta u$, and
- the Laplace equation: $\Delta u = 0$.

Each of these is a single, linear second order PDE. In this part, we study another equally important type of PDE: systems of first order PDEs. By this we mean that not only do we have several independent variables $x_1, x_2, \dots, x_n$, but also several unknown functions $u_1(\bar{x}), \dots, u_N(\bar{x})$, which are assumed to satisfy $m \geq 1$ number of linear first order PDEs.

Ex. 5.1: The case of a scalar linear first order PDE, that is $m=1$. Assume for simplicity $n=2$ and consider

$$a_1(x_1,x_2)\,\partial_{x_1} u + a_2(x_1,x_2)\,\partial_{x_2} u = 0.$$

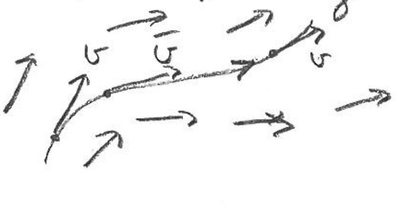
To solve this, introduce the vector field

$$\bar{v}(\bar{x}) := (a_1(\bar{x}), a_2(\bar{x})).$$

Then the PDE is $\underbrace{\partial_{\bar{v}} u}_{\text{directional derivative}} = \bar{v}(\bar{x}) \cdot \nabla u(\bar{x}) = 0.$

A characteristic curve $\bar{\gamma}$ of this PDE is a solution path to the system of ODEs

$$\frac{d\bar{\gamma}(t)}{dt} = \bar{v}(\bar{\gamma}(t))$$




$u$ is now a solution to the PDE if and only if $u$ is constant on any characteristic curve.

Ex 5.2: From complex analysis, we know that Cauchy-Riemann's equations

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

for two functions $u(x,y)$ and $v(x,y)$ in the plane, describe an analytic function.

This first order system is closely related to the Laplace equation in the plane: each of the two functions is harmonic:

$$\partial_x^2 u + \partial_y^2 u = \partial_x(\partial_y v) + \partial_y(-\partial_x v) = 0,$$
$$\partial_x^2 v + \partial_y^2 v = \partial_x(-\partial_y u) + \partial_y(\partial_x u) = 0.$$

Note carefully that the converse is not true: $u$ and $v$ must be harmonic functions which are coupled in a certain way (we say that they are harmonic conjugate functions) for $u+iv$ to be analytic.

The following is another relation between analytic and harmonic functions.

Prop. 5.3: Let $D \subset \mathbb{R}^2 = \mathbb{C}$ be a simply connected domain. Then the following are equivalent for two functions $u, v: D \to \mathbb{R}$ in $D$:

(1) $u+iv$ is analytic

(2) there exists a harmonic function $w: D \to \mathbb{R}$ such that

$$(u, -v) = \nabla w.$$

Thus analytic functions are complex conjugates of gradients of harmonic functions.

Proof: The curl of $(u,-v)$ is

$\partial_x(-v) - \partial_y u = 0$ by the second CR equation.

By vector calculus, this is equivalent to $(u,-v) = \nabla w$ for some $w$.

The first CR equation now say

$$\partial_x^2 w + \partial_y^2 w = \partial_x u + \partial_y(-v) = 0. \quad \blacksquare$$

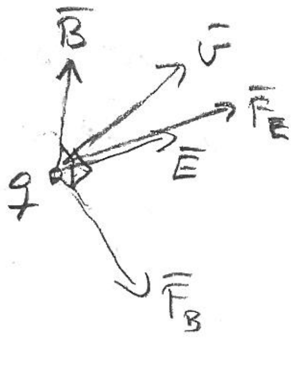
The goal in this part 5.1 is now to show that the Maxwell equations for the electromagnetic field can be viewed as a higher dimensional version of Cauchy-Riemann's equations. All these are special cases of Dirac type equations, but we shall limit ourselves to Maxwell's equations.

## Maxwell's equations

The electric and magnetic fields $\bar{E}(\bar{x},t)$ and $\bar{B}(\bar{x},t)$ are two vector field in space $\mathbb{R}^3$, depending on time $t$ and position $\bar{x} = (x_1, x_2, x_3)$.

Consider a point charge $q$ at position $\bar{x}$, with velocity $\bar{v}$. Then $\bar{E}$ and $\bar{B}$ act on $q$ by the Lorentz force

$$\bar{F} = \underbrace{q\bar{E}}_{\bar{F}_E} + \underbrace{q\,\bar{v}\times\bar{B}}_{\bar{F}_B}.$$



Maxwell's equations are <u>eight</u> linear PDEs for six functions $E_1, E_2, E_3, B_1, B_2, B_3$, depending on four variables $x_1, x_2, x_3, t$:

$$\begin{cases} \nabla \cdot \bar{B} = 0 & \text{(magnetic Gauss' law)} \\ \partial_t \bar{B} + \nabla \times \bar{E} = 0 & \text{(Faraday's law)} \\ \varepsilon_0 \partial_t \bar{E} - \frac{1}{\mu_0} \nabla \times \bar{B} = -\bar{J} & \text{(Ampère-Maxwell's law)} \\ \varepsilon_0 \nabla \cdot \bar{E} = \rho & \text{(Gauss' law)} \end{cases}$$

These are the converse to the Lorentz force: Maxwell's equations describe how the time evolution ($\partial_t \bar{E}$ and $\partial_t \bar{B}$) of the electromagnetic field is determined by a given charge density $\rho(\bar{x}, t)$ and electric current density $\bar{J}(\bar{x}, t)$.
($\varepsilon_0 \approx 8.85 \cdot 10^{-12}$ C/Vm and $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am are numerical constants, using SI units)

A derivation of Maxwell's equations like in Part 1 belong to a physics course. Here we shall focus on the relation to the wave equation.
Analogous to Prop. 5.3, we have the following.

<u>Prop. 5.4</u>: Let $D \subset \mathbb{R}^3$ be a domain (which is topologically equivalent to a ball), and let $\rho(\bar{x}, t)$ and $\bar{J}(\bar{x}, t)$ be given sources in $D$ as above, which are assumed to satisfy the continuity equation

$$\partial_t \rho + \nabla \cdot \bar{J} = 0.$$

(conservation of charge)

Then the following are equivalent for a pair of vector fields $\bar{E}(\bar{x},t)$ and $\bar{B}(\bar{x},t)$ in $D$:

(1) They satisfy Maxwell's equations.

(2) There exists a vector field $\bar{A}(\bar{x},t)$ in $D$ satisfying the wave equation (component wise)

$$\partial_t^2 \bar{A} = \frac{1}{\varepsilon_0\mu_0}\Delta\bar{A} + \frac{1}{\varepsilon_0}\bar{J}$$

and a function $\Phi(\bar{x},t)$ in $D$ satisfying

$$\begin{cases} \partial_t \Phi = -\frac{1}{\varepsilon_0\mu_0}\nabla\cdot\bar{A} & \text{for all } t, \\ \Delta\Phi = -\nabla\cdot(\partial_t\bar{A}) - \frac{1}{\varepsilon_0}\rho & \text{for some } t=t_0, \end{cases}$$

such that
$$\begin{cases} \bar{E} = -\nabla\Phi - \partial_t\bar{A}, \\ \bar{B} = \nabla\times\bar{A}. \end{cases}$$

Proof: (1) $\Rightarrow$ (2)

Consider first magnetic Gauss and Faraday.

$\nabla\cdot\bar{B} = 0 \iff \bar{B} = \nabla\times\bar{A}_0$ for some $\bar{A}_0$,

$\partial_t\bar{B} + \nabla\times\bar{E} = 0 \iff \nabla\times(\partial_t\bar{A}_0 + \bar{E}) = 0$

$\iff \partial_t\bar{A}_0 + \bar{E} = -\nabla\Phi_0$ for some $\Phi_0$

using vector calculus.

We now "change gauge": let $u(\bar{x},t)$ be a scalar function in $D$ and define

$$\begin{cases} \bar{A} := \bar{A}_0 - \nabla u \\ \Phi := \Phi_0 + \partial_t u \end{cases}$$

Then $\nabla\times\bar{A} = \bar{B} - \nabla\times\nabla u = \bar{B}$ and

$$-\nabla\Phi - \partial_t\bar{A} = \bar{E} - \nabla\partial_t u + \partial_t\nabla u = \bar{E},$$

for any $u$.

We chose $u$ so that

$$0 = \partial_t \Phi + \frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \bar{A} = \partial_t \Phi_0 + \partial_t^2 u + \frac{1}{\varepsilon_0 \mu_0} (\nabla \cdot \bar{A}_0 - \Delta u)$$

$$\Longleftrightarrow \quad \partial_t u = c^2 \Delta u + f \quad \text{, where}$$

$$f := -\partial_t \Phi_0 - c^2 \nabla \cdot \bar{A}_0$$

$$c := \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

We know that this wave equation with a source $f$ has a solution.

Next consider Ampère-Maxwell's law:

$$\varepsilon_0 \partial_t \bar{E} - \frac{1}{\mu_0} \nabla \times \bar{B} = -\bar{J}$$

$$\Longleftrightarrow \quad -\varepsilon_0 \partial_t \nabla \Phi - \varepsilon_0 \partial_t^2 \bar{A} - \frac{1}{\mu_0} \nabla \times (\nabla \times \bar{A}) = -\bar{J}$$

From vector calculus: $\Delta \bar{A} = \nabla(\nabla \cdot \bar{A}) - \nabla \times (\nabla \times \bar{A})$

componentwise!

$$\Rightarrow \quad -\varepsilon_0 \partial_t^2 \bar{A} + \frac{1}{\mu_0} \Delta \bar{A} = -\bar{J} \quad \text{, since } \partial_t \Phi + \frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \bar{A} = 0$$

Finally Gauss' law:

$$\varepsilon_0 \nabla \cdot \bar{E} = \rho \quad \Longleftrightarrow \quad -\varepsilon_0 \Delta \Phi - \varepsilon_0 \nabla \cdot (\partial_t \bar{A}) = \rho$$

(2) $\Rightarrow$ (1):

Now let $\bar{A}, \Phi$ have the stated properties, and let $\bar{E} := -\nabla \Phi - \partial_t \bar{A}$ and $\bar{B} := \nabla \times \bar{A}$.

We must show that $\bar{E}, \bar{B}$ satisfy Maxwell's equations.

Magnetic Gauss and Faraday are immediate.

Ampère-Maxwell $\Longleftrightarrow$

$$\underbrace{-\varepsilon_0 \partial_t^2 \bar{A} + \frac{1}{\mu_0} \Delta \bar{A}}_{= -\bar{J}} - \varepsilon_0 \nabla \Big( \underbrace{\partial_t \Phi + \frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \bar{A}}_{= 0} \Big) = -\bar{J}$$

holds.

Gauss' law $\Leftrightarrow$ $-\varepsilon_0 \Delta \Phi - \varepsilon_0 \nabla \cdot (\partial_t \bar{A}) = \rho$.

This is assumed to hold at $t = t_0$, so it suffices to show

$$-\varepsilon_0 \partial_t \left( \Delta \Phi + \nabla \cdot \partial_t \bar{A} \right) = \partial_t \rho$$

$$\Leftrightarrow -\varepsilon_0 \Delta \left( -\frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \bar{A} \right) - \varepsilon_0 \nabla \cdot (\partial_t^2 \bar{A}) = -\nabla \cdot \bar{J}$$

$$\Leftrightarrow \nabla \cdot \underbrace{\left( \partial_t^2 \bar{A} - \frac{1}{\varepsilon_0 \mu_0} \Delta \bar{A} - \frac{1}{\varepsilon_0} \bar{J} \right)}_{=0} = 0$$

$\blacksquare$

The point of this proposition is to show that the Maxwell system of PDEs is <u>equivalent</u> to 3 <u>independent</u> wave equations:

$$\partial_t^2 A_i = c^2 \Delta A_i + \frac{1}{\varepsilon_0} J_i \quad , \quad i = 1, 2, 3$$

for the components of the vector potential $\bar{A}$ of the electromagnetic field.

Given such a wave solution $\bar{A}(\bar{x}, t)$

$\Delta \Phi = f$ can be solve at $t = t_0$ with source $f = -\nabla \cdot (\partial_t \bar{A}) - \frac{1}{\varepsilon_0} \rho$

given $\Phi(\bar{x}, t_0)$, uniquely modulo harmonic functions.

Then we solve for $\Phi(\bar{x}, t)$, $t \neq t_0$:

$$\Phi(\bar{x}, t) = \Phi(\bar{x}, t_0) + \int_{t_0}^{t} \left( -\frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \bar{A}(\bar{x}, s) \right) ds$$

Prop. 5.4: When Maxwell's equations hold, then in particular we also have wave equations

$$\partial_t^2 \Phi = c^2 \Delta \Phi + \frac{1}{\varepsilon_0^2 \mu_0} \rho$$

$$\partial_t^2 \bar{E} = c^2 \Delta \bar{E} - \frac{1}{\varepsilon_0^2 \mu_0} \nabla \rho - \frac{1}{\varepsilon_0} \partial_t \bar{J}$$

$$\partial_t^2 \bar{B} = c^2 \Delta \bar{B} + \frac{1}{\varepsilon_0} \nabla \times \bar{J}.$$

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Prove this from Prop. 5.4.

Note that the wave equations for $\bar{A}$ and $\Phi$ are not independent.

Similarly

$$\left.\begin{aligned} \partial_t^2 \bar{E} &= c^2 \Delta \bar{E} - \frac{1}{\varepsilon_0^2 \mu_0} \nabla \rho - \frac{1}{\varepsilon_0} \partial_t \bar{J} \\ \partial_t^2 \bar{B} &= c^2 \Delta \bar{B} + \frac{1}{\varepsilon_0} \nabla \times \bar{J} \end{aligned}\right\} \not\Longrightarrow \text{Maxwell's equations for } \bar{E}, \bar{B}.$$

On the left are six independent wave equations for $E_1, E_2, E_3, B_1, B_2, B_3$, which shows that these propagates like waves with propagation speed

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 2{,}998 \cdot 10^8 \text{ m/s}.$$

However Maxwell's equations also involve a certain <u>coupling</u> between the components of $\bar{E}, \bar{B}$. (compare with conjugate harmonic function.)

# Static solutions to Maxwell:

Assume that $\rho = \rho(\bar{x})$ and $\bar{J} = \bar{J}(\bar{x})$ are time-independent, and therefore also $\bar{E} = \bar{E}(\bar{x})$ and $\bar{B} = \bar{B}(\bar{x})$.

Then
$$\begin{cases} \nabla \cdot \bar{B} = 0 \\ \nabla \times \bar{E} = 0 \\ \nabla \times \bar{B} = \mu_0 \bar{J} \\ \nabla \times \bar{E} = \frac{1}{\varepsilon_0} \rho \end{cases} \qquad \text{and} \quad \nabla \cdot \bar{J} = 0$$

In this case, the fields $\bar{E}$ and $\bar{B}$ are independent.

Consider $\bar{E}$: Faraday tell that:

$\bar{E} = -\nabla \Phi$ for some $\Phi$, and

then Gauss becomes: $\Delta \Phi = -\frac{1}{\varepsilon_0} \rho$.

$\Phi$ is the potential function of the electric field.

We have the following electrostatic interpretations for objects from Part 3:

- The fundamental solution $\Phi$ in $\mathbb{R}^3$ to $\Delta$, is the potential from a (negative) point charge at $\bar{x} = \bar{0}$, with ground at infinity ($\lim_{\bar{x} \to \infty} \Phi = 0$)

- The Green's function $G_D(\bar{x}, \bar{x}_0)$ for a domain $D \subset \mathbb{R}^3$ is the potential from a (negative) point charge at $\bar{x} = \bar{x}_0$, with $\partial D$ grounded ($\Phi|_{\partial D} = 0$).

# Part 5.2: Examples of non-linear PDEs

## Euler-Lagrange equations

One important class of non-linear PDEs arise upon generalising Dirichlet's principle Prop. 3.15. We illustrate this "calculus of variations" by considering the following problem.

- Let $F(x_1, x_2, z, \xi_1, \xi_2)$ be a function defined for $(x_1, x_2) \in D$ and $z, \xi_1, \xi_2 \in \mathbb{R}$, where $D \subset \mathbb{R}^2$ is a domain in the plane.
- Let $g: \partial D \to \mathbb{R}$ be a given function, and consider all functions $u: \overline{D} \to \mathbb{R}$ with $u|_{\partial D} = g$.
- Problem: find a minimizer $u$ to the energy functional

$$E[u] := \iint_D F(x_1, x_2, u(x_1,x_2), \partial_{x_1} u(x_1,x_2), \partial_{x_2} u(x_1,x_2))\, dx_1 dx_2$$

under the constraint $u|_{\partial D} = g$.

Prop. 5.5: Assume that $u: \overline{D} \to \mathbb{R}$ minimizes $E[u]$ among all $u$ such that $u|_{\partial D} = g$. Then $u$ satisfies the Euler-Lagrange PDE

$$\partial_{x_1}\Big((\partial_{\xi_1} F)(\bar{x}, u(\bar{x}), \nabla u(\bar{x}))\Big) + \partial_{x_2}\Big((\partial_{\xi_2} F)(\bar{x}, u(\bar{x}), \nabla u(\bar{x}))\Big) = (\partial_z F)(\bar{x}, u(\bar{x}), \nabla u(\bar{x})).$$

Proof: We proceed as in Prop. 3.15.

Let $u$ be a minimum.

Let $v : \overline{D} \to \mathbb{R}$ be any function such that $v|_{\partial D} = 0$.

Consider $u + \varepsilon v$ for any value of $\varepsilon \in \mathbb{R}$.

By assumption

$$f(\varepsilon) := E[u + \varepsilon v]$$

has a minimum at $\varepsilon = 0$, so

$$0 = f'(0) = \iint_D \Big( \partial_z F(\bar{x}, u(\bar{x}), \nabla u(\bar{x})) \cdot v(\bar{x})$$

$$+ \partial_{\xi_1} F(\bar{x}, u(\bar{x}), \nabla u(\bar{x}))\, \partial_{x_1} v(\bar{x}) + \partial_{\xi_2} F(\bar{x}, u(\bar{x}), \nabla u(\bar{x}))\, \partial_{x_2} v(\bar{x}) \Big)\, dx_1\, dx_2$$

$= /$ Gauss' theorem using $v|_{\partial D} = 0$ $/$

$$= \iint_D \Big( \partial_z F - \partial_{x_1}(\partial_{\xi_1} F) - \partial_{x_2}(\partial_{\xi_2} F) \Big)\, v\, d\bar{x}$$

Since this holds for any $v$ as above, we must have the PDE

$$\partial_z F - \partial_{x_1}(\partial_{\xi_1} F) - \partial_{x_2}(\partial_{\xi_2} F) = 0. \quad \blacksquare$$

Ex 5.6: $F(\bar{x}, z, \bar{\xi}) = |\bar{\xi}|^2$ yields the E-L PDE

$$\partial_{x_1}(2 u'_{x_1}) + \partial_{x_2}(2 u'_{x_2}) = 0 \iff \Delta u = 0,$$

recovering Prop. 3.15.

Ex. 5.7. The minimizing problem

$$\min_{u|_{\partial D} = g} \iint_D |\nabla u|^p\, d\bar{x}$$

yield a harmonic function as in Ex. 5.6 for $p = 2$.

For $p \neq 2$, we consider $F(\bar{x}, z, \bar{\xi}) = |\bar{\xi}|^p$

which will yield the E-L PDE

$$\partial_{x_1}\left(p u'_{x_1} |\nabla u|^{p-2}\right) + \partial_{x_2}\left(p u'_{y} |\nabla u|^{p-2}\right) = 0$$

$$\Longleftrightarrow \quad \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right) = 0.$$

This is the socalled <u>p-Laplace equation</u>, which for $p \neq 2$ is a non-linear PDE.

Loosely speaking, p-harmonic functions are flatter the larger $p$ is, since $E[u]$ then "punishes" $u$ with $|\nabla u|$ large.

This equation has been in image processing, where $u(x_1, x_2)$ represents the picture (intensity at $(x_1, x_2)$).

<u>Ex 5.8</u>: Consider the graph of a function $u(x,y)$ over a domain $D$. Its area is

$$A[u] := \iint_D \sqrt{1 + |\nabla u|^2}\, dx dy.$$

We wish to find $u$ with minimal such area and prescribed boundary values $u|_{\partial D} = g$.

With $F(\bar{x}, z, \bar{\xi}) := \sqrt{1 + |\bar{\xi}|^2}$ we get the Euler-Lagrange PDE

$$\partial_{x_1}\left(\frac{2 u'_{x_1}}{2\sqrt{1+|\nabla u|^2}}\right) + \partial_{x_2}\left(\frac{2 u'_{x_2}}{2\sqrt{1+|\nabla u|^2}}\right) = 0$$

$$\Longleftrightarrow \quad \operatorname{div}\left(\frac{1}{\sqrt{1+|\nabla u|^2}} \nabla u\right) = 0$$

This is the <u>minimal surface equation</u>, the solution to which describes the shape of a soap bubble formed at equilibrium with a wire of shape $g$.

As usual, important mathematical questions about PDEs concerns well posedness. Let us limit our short discussion here to existence.

A basic existence result proved in the book "Evans: partial differential equations" states roughly the following.

If: • $F(\bar{x}, z, t\bar{\xi}_1 + (1-t)\bar{\xi}_2) \leq tF(\bar{x}, z, \bar{\xi}_1) + (1-t)F(\bar{x}, z, \bar{\xi}_2)$
for all $\bar{x} \in D$, $z \in \mathbb{R}$, $t \in (0,1)$, $\bar{\xi}_1, \bar{\xi}_2 \in \mathbb{R}^2$
($F$ is convex in the variable $\bar{\xi}$), and

• the exists $R < \infty$ and $p > 1$ such that
$F(\bar{x}, z, \bar{\xi}) \geq |\bar{\xi}|^p$
for all $|\bar{\xi}| \geq R$, $\bar{x} \in D$, $z \in \mathbb{R}$
($F$ is coersive in the variable $\bar{\xi}$)

then there exists a minimum $u$ for the functional

$$E[u] = \iint_D F(\bar{x}, u, \nabla u)\, d\bar{x},$$

with $\iint_D |\nabla u|^p\, d\bar{x} < \infty$.

Ex 5.9

• Since the function $f(x) = |x|^p$ is convex for $p \geq 1$, we conclude from above that a p-harmonic function $u$ with prescribed Dirichlet data $u|_{\partial D} = g$ exists if $p > 1$. (But maybe not for $p = 1$?)

• The function $f(x) = \sqrt{1+x^2}$ is convex, but $f(x) = |x| + \mathcal{O}(1)$ as $x \to \infty$ so coersivity fails. Thus existence for the minimal surface equation is not clear from above. The problem whether a solution to the minimal surface equation always exists was called the Plateau problem. It solved to the positive by Jesse Douglas, who received the first Fields medal ($\approx$ math. Nobel prize) for this in 1936 (shared with L. Ahlfors).

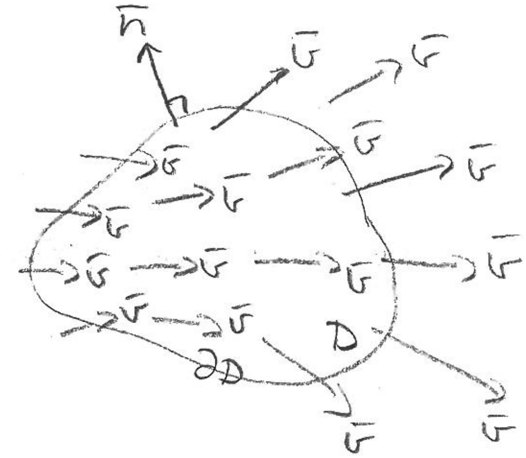
## Navier-Stokes equations

This is a system of non-linear PDEs which describes the flow of a fluid (liquid or gas), and which we now derive.

- The main unknown variable is the velocity vector field $\bar{v}(\bar{x},t)$, which represents the velocity of the fluid at position $\bar{x}$ and time $t$.

- We assume that the liquid in particular is incompressible, so that its density $\rho$ is constant (in space and time).

Consider a region $D$ in space.




(1) Conservation of mass

$$\underbrace{\partial_t \iiint_D \rho \, d\bar{x}}_{\substack{\text{rate of change} \\ \text{of mass in } D}} = \underbrace{-\int_{\partial D} \rho(\bar{n}\cdot\bar{v})\, dS}_{\text{mass flow into } D}$$

$$\Rightarrow \iiint_D \left(\cancel{\partial_t \rho}^{\,0} + \nabla\cdot(\rho\bar{v})\right) d\bar{x} = 0 \quad \text{by Gauss' thm.}$$

Since $D$ is arbitrary, we have $\nabla\cdot\bar{v} = 0$.

(2) Conservation of momentum

/ Newton's second law: $\partial_t (m\bar{v}) = \bar{f}$ /

$$\underbrace{\partial_t \left( \iint_D \rho \bar{v} \, d\bar{x} \right)}_{\text{rate of change of momentum in } D} + \underbrace{\int_{\partial D} \rho \bar{v} \, (\bar{v} \cdot \bar{n}) \, dS}_{\text{flux of momentum across } \partial D} =$$

$$= \underbrace{\int_{\partial D} p \, (-\bar{n}) \, dS}_{\text{pressure force on } D} + \underbrace{\int_{\partial D} \mu \frac{\partial \bar{v}}{\partial n} \, dS}_{\text{viscous force on } D} + \underbrace{\iint_D \bar{f} \, d\bar{x}}_{\text{external forces on } D}$$

First some explanations:

- The increase of momentum of "fluid particles" in $D$ comes both from change of velocity inside $D$ and flow of particles across $\partial D$. Thus the two terms on the left hand side.
- Pressure $p$ acts at the boundary with a force along $-\bar{n}$.
- Viscous means "sticky" and the parameter $\mu \geq 0$ represents the strength of the viscosity. The interpretation of the viscosity term $\frac{\partial \bar{v}}{\partial n}$ is as follows.

If near a point on $\partial D$, the velocity is $\bar{v}_1$ at distance $\varepsilon$ into $D$ and $\bar{v}_2$ at distance $\varepsilon$ outside $D$, then the fluid

outside $D$ will "drag the fluid inside $D$ along" if the fluid is viscous.

The viscous force is modelled by

$$\mu \cdot \frac{\bar{v}_2 - \bar{v}_1}{2\varepsilon} \approx \mu \frac{\partial \bar{v}}{\partial n}$$

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Applying Gauss' theorem, we obtain the PDE

$$\rho \, \partial_t \bar{v} + \left( \rho \bar{v} \, \nabla \cdot \bar{v} + \rho (\bar{v} \cdot \nabla) \bar{v} \right) = -\nabla p + \mu \Delta \bar{v} + \bar{f},$$

where $(\bar{v} \cdot \nabla) \bar{v}$ is the directional derivative, in direction $\bar{v}$, of the vector field $\bar{v}$ (component wise)

and $\Delta \bar{v} = \begin{bmatrix} \Delta v_1 \\ \Delta v_2 \\ \Delta v_3 \end{bmatrix}$ is the Laplace operator acting component wise.

Definition 5.10:

- The Navier-Stokes equations are

$$\begin{cases} \rho \left( \partial_t \bar{v} + (\bar{v} \cdot \nabla) \bar{v} \right) = -\nabla p + \mu \Delta \bar{v} + \bar{f} \\ \nabla \cdot \bar{v} = 0 \end{cases}$$

- For inviscous fluids ($\mu = 0$), we have the Euler equations

$$\begin{cases} \rho \left( \partial_t \bar{v} + (\bar{v} \cdot \nabla) \bar{v} \right) = -\nabla p + \bar{f} \\ \nabla \cdot \bar{v} = 0 \end{cases}$$

- In the limit of small, steady state velocities, we have for viscous fluids the linear Stokes equations

$$\begin{cases} 0 = -\nabla p + \mu \Delta \bar{v} + \bar{f} \\ \nabla \cdot \bar{v} = 0 \end{cases}$$

Remarks:

- The key term in NS and Euler equations is the non-linear term $(\bar{v}\cdot\nabla)\bar{v}$. It is called the convective term, and represents the acceleration of the fluid due to its steady-state flow.

  These non-linear equations seem to model real fluid flows well, including complex chaotic behaviour like turbulence.

- The viscous term $\Delta\bar{v}$ present in NS equations, acts like a "diffusion of momentum", and in some sense smoothens the solutions $\bar{v}$ similarly to the heat equation. In this sense NS equations are more well behaved than Eulers equations.

- The pressure term $\nabla p$ is "a reaction force needed to leave volume unchanged". One can eliminate it by equivalently formulating the momentum equations to say that

  $$\rho(\partial_t\bar{v}+(\bar{v}\cdot\nabla)\bar{v})-\mu\Delta\bar{v}-\bar{f}$$

  should be a curl-free vector field for each $t$.

The most important mathematical question concerning the very difficult non-linear NS and Euler equations is whether solutions exists to the initial value problem.

- For two-dimensional flows ($\bar{U}(\cdot,t): \mathbb{R}^2 \to \mathbb{R}^2$) it was shown by Ladyzhenskaya in the 1960's that a solution to NS and Euler equations exists for all $t>0$, given any "good" initial data.

- For "real" three-dimensional flows, it is only known that a solution to NS and Euler equations exists for $0 \le t < T$, where $T$ may depend on the initial data.
  Whether a solution actually exists for all $0 \le t < \infty$, for any "good" initial data, is not known. The Clay mathematics institute offers a US $1.000.000 prize for a solution to this very important problem.