I received the Tage Erlander prize 2009 by the Royal Swedish Academy of Sciences, with the motivation “for his contributions to the development of functional calculus in harmonic analysis, which lead to the proof of Lions’ conjecture”. Below you find an attempt to explain this motivation in a relatively simple way.

The theory of differential equations is perhaps the most profound achievement of mathematical analysis. There are two kinds of such equations, the “ordinary” differential equations (ODEs) which involve functions only depending on one variable, and “partial” differential equations (PDEs) where several independent variables appear. The simplest three kinds of linear ODEs are

$$f'(t) = -af(t), \qquad f''(t) = -af(t), \qquad \text{and} \qquad f''(t) = af(t),$$

where $a > 0$ is some constant. The first is a first order ODE whose solution, with initial condition $f(0) = u$, is

$$f(t) = e^{-at}u.$$

The solutions of the two second order equations, with initial conditions $f(0) = u$, $f'(0) = v$, are

$$f(t) = \cos(\sqrt{a}t)u + \frac{\sin(\sqrt{a}t)}{\sqrt{a}}v$$

and

$$f(t) = e^{\sqrt{a}t}(u + a^{-1/2}v)/2 + e^{-\sqrt{a}t}(u - a^{-1/2}v)/2$$

respectively. Turning to PDEs, there are three fundamental types of such equations: parabolic, hyperbolic and elliptic. Standard examples are

$$\frac{\partial f}{\partial t} = \Delta f, \qquad \frac{\partial^2 f}{\partial t^2} = \Delta f(t), \qquad \text{and} \qquad \frac{\partial^2 f}{\partial t^2} = -\Delta f(t).$$

Here $f(t, x)$ is a function of t and $x = (x_1, \ldots, x_n)$, i.e. $n+1$ variables, and $\Delta = \partial_1^2 + \ldots + \partial_n^2$ is the Laplace operator. The first is the standard parabolic equation: the heat / diffusion equation, named so since it models time evolution of such phenomena. The second is the standard hyperbolic equation: the wave equation, which models time evolution of waves. The third one is the standard elliptic equation: the Laplace equation (in $n + 1$ variables), which describes equilibrium, e.g. of heat or temperature. In the latter equation, t is just another space variable like $x_1, \ldots, x_n$, and not time as in the first two equations.

The Laplace operator Δ is a negative self-adjoint operator, and behave in many ways like a negative real number $-a$. Not one value though, but all negative values simultaneously. Simply replacing a by $-\Delta$ in the solution formulas above gives the solution

$$f(t, x) = e^{t\Delta}u(x)$$

to the heat equation, and the solution

$$f(t, x) = \cos(t\sqrt{-\Delta})u(x) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}v(x)$$

to the wave equation. This is *functional calculus*: we apply the function $e^{(\cdot)t}$ to the operator Δ to obtain another operator $e^{\Delta t}$. This operator is then applied to the function $u(x)$ which describes the initial distribution of heat, to obtain the function describing the distribution of heat at time t. For the wave equation it works similarly: from the initial shape $u(x)$ and initial speed $v(x)$ of the wave, the operators $\cos(t\sqrt{-\Delta})$ and $\sin(t\sqrt{-\Delta})/\sqrt{-\Delta}$ obtained by functional calculus are applied to give the shape of the wave at time t.

Given a polynomial $p(z)$, it is clear what is meant by $p(\Delta)$. For self-adjoint operators like Δ, more general functions $f(\Delta)$ of Δ are defined through the Fourier transform and the spectral theorem. For non self-adjoint operators, which my research concerns, the key tool is instead the Cauchy integral. Much of the modern developments of this Dunford functional calculus is due to my collegue and former PhD supervisor Professor Alan McIntosh.

Much of my work deals with the application of functional calculus to elliptic equations. Here the situation is more complicated. The reason for this is that the above transference procedure from ODEs to PDEs would involve both the operators $e^{t\sqrt{-\Delta}}$ and $e^{-t\sqrt{-\Delta}}$. However, the first one is not useful for $t > 0$, since $\sqrt{-\Delta}$ behaves like infinitely large positive numbers and the exponential function grows rapidly for positive arguments. For this reason, we do not obtain a continuous operator. (For similar reason, the heat equation cannot be solved backwards in time.)

For elliptic equations, we think of the variable t as being the coordinate transversal to the boundary $t = 0$. The function $u(x)$ describes the boundary values of $f(t, x)$, the Dirichlet data, and the function $v(x)$ describes the normal derivatives of $f(t, x)$ at the boundary, the Neumann data. We see that under the condition $v = -\sqrt{-\Delta}u$, we can solve the equation for positive t, with solution

$$f(t, x) = e^{-t\sqrt{-\Delta}}u(x),$$

since the bad term vanishes. Similarly, if $v = \sqrt{-\Delta}u$, we can solve the equation for negative t, with solution

$$f(t, x) = e^{t\sqrt{-\Delta}}u(x).$$

Thus, we see that for "half" of the boundary data $\{u(x), v(x)\}$, we can find a solution on one side of the boundary, whereas for the remaining "half", we find a solution on the other side of the boundary. This *splitting* in two subspaces of the space of functions is fundamental in the theory of elliptic equations, and the subspaces are referred to as *Hardy subspaces*. The reason why the boundary data here consist of pairs of functions is that the PDE is of second order. This structure of elliptic boundary value problems becomes more transparent when working with first order elliptic systems of PDEs, where this type of splitting of the function space is seen more clearly. For example, it was in the investigation of analytic functions (which solve the elliptic Cauchy–Riemann system) where Hardy originally discovered this type of subspaces.

The boundary value problem discussed above dealt with the flat surface $t = 0$. The extension to more general smooth surfaces is relatively straightforward, although more technical. However, real life geometries are seldom smooth. Corners and edges often appear.

And even if a surface appears smooth, irregularities may appear if we zoom in on the surface and look at smaller scales. My work is focused on understanding functional calculi as above, for boundary surfaces which are irregular on all scales, so called *Lipschitz regular surfaces*. By a change of variables, one may assume that the surface is flat, but then the operator Δ is replaced by a partial differential operator involving coefficients without any smoothness *whatsoever*. In order to understand such operators, techniques from the field of harmonic analysis are needed. A key result of fundamental importance is the *Kato estimate*, conjectured by T. Kato in the 1960's and proved by my collegues A. McIntosh (the Australian National University) and P. Auscher (Université de Paris-Sud), with collaborators S. Hofmann, M. Lacey and P. Tchamitchian in 2001. This estimate states that

$$\int_{\mathbf{R}^n} |\sqrt{-\mathrm{div} A(x)\nabla} u(x)|^2 dx \approx \int_{\mathbf{R}^n} |\nabla u(x)|^2 dx,$$

for any bounded and accretive matrix coefficient function $A(x)$. When A is the identity matrix, then $\mathrm{div} A(x)\nabla = \Delta$. The Kato estimate means that in a certain way the square root of the second order operator always behaves like a first order operator. Less transparent is the fundamental fact that the Kato estimate really is a statement about continuity of projections onto Hardy subspaces for elliptic PDEs as above. Indeed, if we measure the size of the pair of functions $\{u(x), v(x)\}$ according to $(\int_{\mathbf{R}^n}(|\nabla u(x)|^2 + |v(x)|^2)dx)^{1/2}$, then continuity of the two projections onto the Hardy subspaces given by $v = -\sqrt{-\Delta} u$ and $v = \sqrt{-\Delta} u$ is seen to be exactly the Kato estimate.

My work on elliptic boundary value problems has lead to a deepened understanding of this estimate and related functional calculi, and to powerful applications to the theory of elliptic PDEs. More fundamental than scalar PDEs as above are *systems of PDEs*, which not only involve one function $f(t, x)$ but several functions coupled by the equations. Many of the interesting differential equations appearing in physics are first order systems of PDEs, for example Maxwell's equations in electrodynamics and the Dirac equation in quantum electrodynamics. The most exciting aspects of my work is that it applies equally well to systems of PDEs. As compared to scalar equations, elliptic systems of PDEs have previously not been very well understood. The reason is that earlier methods frequently made use of the order structure of the real numbers through various comparison principles. On the other hand, the above Kato estimate is really a statement about systems of PDEs, although not completely obvious. As an analogy with an elementary example, consider the theory of summability of infinite series $\sum_{k=1}^{\infty} a_k$. It is well known that the theory for positive series (all $a_k \geq 0$) is simpler than understanding series with general sign-changing terms, where no comparison theorems are available.

Finally let me comment on the conjecture by J.-L. Lions from 1962, which is mentioned in the motivation for the Tage Erlander prize. This concerns the validity of the Kato square root estimate on domains, where general mixed boundary conditions are imposed on the function $u(x)$. On part of the boundary Dirichlet data are specified whereas on the remaining part Neumann data are specified. This conjecture was settled to the positive in joint work with Alan McIntosh and Stephen Keith at the Australian National University.