RESEARCH PRESENTATION TOPIC IDEAS

1. Research presentation

One possible way to pass the course is to give a research presentation, based on research you have done (possibly with a partner) concerning a topic related to the topics covered in the course. In your presentation you should include a mathematical statement, like a theorem, or a collection of examples which you will prove satisfy certain statements. You should also include the proof of this mathematical statement (or statements) and be able to explain it clearly and completely. Here are a few ideas for research topics:

(1) A journey into the realm of non-measurable sets. Give examples of non-measurable sets, with respect to different measures. Prove some general statements, like for certain measures, can you give a necessary and sufficient condition for sets to be measurable? What do the non-measurable sets of $\mathbb{R}^n$ look like? How do they depend on the choice of measure on $\mathbb{R}^n$? Feel free to get a little wild here, after all we are talking about non-measurable stuff... On the other hand, can you cook up very tame examples of non-measurable sets?

(2) Flowers and other pretty formations in complex dynamics (attracting petals, repelling arms, and the like). Here you could prove a statement like computing the pattern of attracting petals for $z+z^4$ and $-z+z^4$, and computing some examples of repelling arms.

(3) A brief history of dimension. What are some different notions of dimension? (eg box dimension, Minkowski dimension...) How do these notations relate to Hausdorff dimension? Prove various results about these notions of dimension, and give a history of how Hausdorff dimension has become something of the "industry standard." (At least it seems that way to me, but maybe I am wrong... Find out!)

(4) Douady's rabbit.

(5) Iterated function system fractals: present several examples. Prove that they satisfy the open set condition (if they do satisfy it), and use our results to determine their Hausdorff dimensions. Give some examples of IFSs where the dimension is merely conjectured, but it is not clear how to prove it...

(6) State and prove Siegel's theorem. Investigate current research on Siegel disks and their generalizations.

(7) What the heck is Haar measure? Are there some other interesting measures that are in common use, which we have overlooked? Give a survey of the most important measures, and in the particular case of Haar measure, prove some awesome theorems about it. Why is it so cool, and why should I really know more about it?

(8) The Mandelbrot set (actually, this could be enough material for two different research projects & presentations). The Mandelbrot set has all kinds of cool little bits and pieces. One interesting fact is that shapes of certain

Julia sets, J_c, are reflected in the shape of the Mandelbrot set near the point c. Why is this? What re airplane sets? What is the main antenna? What are the hyperbolic components of the Mandelbrot set?

(9) Optimal transport. What is the optimal transport problem? What is the history of the problem? What is its solution? Give the proof of the existence of the solution to the optimal transport problem. Is this solution actually used in practice? Why or why not?

(10) * Consider a smooth function φ defined on $\mathbb{R}$. The first part of this question is: let $C \subset [0,1]$ be a d-dimensional Cantor set. Let μ be d-dimensional Hausdorff measure. Prove a formula of the type

$$\int_C \varphi'(x) d\mu = \text{ something expressed in terms of the values of } \varphi(x) \text{ on } C \text{ or a subset of } C.$$

What if you use instead of d-dimensional Hausdorff measure the canonically associated invariant measure whose existence we proved in the course? Is it true in that case? Prove or give counter examples.

The real goal is to prove a more general result which says that if $\mathbb{F}$ is a d-dimensional IFS fractal as we have studied in the course, and if φ is a smooth function on $\mathbb{F}$, then we have

$$\int_{\mathbb{F}} \nabla_\eta \varphi d\mu = \text{ something expressed in terms of the values of } \varphi \text{ on } \mathbb{F} \text{ or a subset of } \mathbb{F}.$$

Above, ∇_η is the directional derivative of φ in some direction. The motivation comes from looking at

$$\int_{\mathbb{F}} e^{i\lambda x \cdot \xi} f(x) d\mu(x).$$

We want an estimate of this for large λ. The function f above is totally harmless. The reason for this question is that we would like to show an estimate of the type:

$$\int_{\mathbb{F}} \nabla_\eta \left(e^{i\lambda x\cdot\xi} f(x) d\mu(x) \right) = \mathcal{O}(e^{i\lambda x\cdot\xi} f(x)) = \mathcal{O}(1).$$

Since

$$\int_{\mathbb{F}} \nabla_\eta \left(e^{i\lambda x\cdot\xi} f(x) d\mu(x) \right) = \int_{\mathbb{F}} f(x) \nabla_\eta e^{i\lambda x\cdot\xi} d\mu + \int_{\mathbb{F}} e^{i\lambda x\cdot\xi} \nabla_\eta f(x) d\mu,$$

we would get the estimate

$$\mathcal{O}(1) - \int_{\mathbb{F}} e^{i\lambda x\cdot\xi} \nabla_\eta f(x) d\mu = i\lambda\xi\cdot\eta \int_{\mathbb{F}} f(x) e^{i\lambda x\cdot\xi} d\mu.$$

We could then move $\lambda\xi\cdot\eta$ to the other side (as long as ξ and η are not orthogonal). This would give the estimate

$$\frac{\mathcal{O}(1) - \int_{\mathbb{F}} e^{i\lambda x\cdot\xi} \nabla_\eta f(x) d\mu}{i\lambda\xi\cdot\eta} = \int_{\mathbb{F}} f(x) e^{i\lambda x\cdot\xi} d\mu.$$

This is precisely the estimate needed to complete a certain research project...

(11) * The Golden Mean Siegel disk. Investigate the article: "Scaling ratios and triangles in Siegel disks." Is it correct? Why or why not?