Let (Z, g) be an (n dimensional) asymptotically conic space whose metric satisfies

$$g = dr^2 + r^2 h(r, y) \text{ on } Z \setminus K \cong [r_0, \infty)_r \times Y,$$

for some compact subset $K \subset Z$. The Laplace operator on Z has a canonical self-adjoint extension to $\mathcal{L}^2(Z, dV_g)$ because Z is complete. Then, the "normal operator at infinity" (see references for b-operators and more general "edge operators" - Melrose and/or Mazzeo) is

$$-\partial_r^2 - \frac{n-1}{r}\partial_r + \Delta_h,$$

where Δ_h is the limiting Laplace operator on Y. Using separation of variables, any harmonic function (or form) f has an expansion as $r \to \infty$ of the form

$$\sum_{j=1}^{k} (c_j^+ r^{\alpha_j^+} + c_j^- r^{\alpha_j^-}) f_j(y), \tag{0.1}$$

where f_j is an eigenform of Δ_h satisfying

$$\Delta_h f_j = \lambda_j f_j.$$

Note that Δ and Δ_h are non-negative operators by our sign convention. We compute the α_j by computing solutions of the quadratic equation

$$\alpha^2 + (n-2)\alpha - \lambda_j = 0,$$

so

$$\alpha_j^{\pm} = \frac{2-n}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda_j}.$$

Thus, the α_j lie in the range

$$\left(-\infty, \frac{2-n}{2} - \sqrt{\frac{(n-2)^2}{4} + \lambda_0}\right] \cup \left[\frac{2-n}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda_0}, \infty\right), \tag{0.2}$$

where λ_0 is the bottom of the spectrum of Δ_h on Y.

The condition from "Case 1" is that $-\delta > \frac{2-n}{2}$. In Case 2, we end up with a harmonic form f on Z which satisfies

$$|f| \leq C r^{-\delta} \text{ as } r \to \infty,$$

for some constant C which is independent of $r \to \infty$. Taking $-\delta$ close to $\frac{2-n}{2}$, since we know the range (0.2) in which the exponents in the expansion (0.1) lie, we know that in fact,

$$|f| \leq C r^{-\gamma}, \quad -\gamma \leq \frac{2-n}{2} - \sqrt{\frac{(n-2)^2}{4} + \lambda_0}.$$

However, this does not give us the necessary contradiction, that $f \in \mathcal{L}^2$, unless we add some hypotheses.

The volume form on Z is asymptotic to $r^{n-1} dr dy$ for large r, thus we need f to decay better than $r^{-n/2}$ for f to be $\mathcal{L}^2$. Thus, we need to know that

$$\frac{2-n}{2} - \sqrt{\frac{(n-2)^2}{4} + \lambda_0} < \frac{-n}{2}.$$

This is true if we add the following hypothesis to the statement of the theorem.

- Assume Z has no $\mathcal{L}^2$ harmonic forms and
$$\lambda_0(Y, \Delta_h) > 1 - \frac{(n-2)^2}{4}.$$
Note that this is a priori satisfied if $n \geq 5$.