# WRITTEN EXAM FRACTALS

**Instructions:** Choose *four* exercises and clearly indicate your choice. Only the four exercises you choose will be corrected and count as credit. You may consult anything, anywhere in the literature, but you may communicate with no-one (other than Julie in case you have questions or get stuck).

(1) Look up the definition of the Minkowski sausage, draw a picture of it, and determine the family of similitudes under which the Minkowski sausage is invariant. Use this to determine the Hausdorff dimension of the Minkowski sausage *using the theorem we proved in lecture.*

(2) Let $A$ be a subset of $\mathbb{R}^n$ with Hausdorff dimension $p$. Prove that the Hausdorff dimension of $A \times A$ in $\mathbb{R}^{2n}$ is at least $2p$. Give an example where the Hausdorff dimension actually *exceeds* $2p$.

(3) For a bounded cube $Q \subset \mathbb{R}^n$ prove using only the definition of Hausdorff measure that $H^n(Q) \in (0, \infty)$, where $n \in \mathbb{N}$, and $Q$ is assumed to be neither a point nor the empty set.

(4) (a) Compute the Julia set for $p(z) = z^2$. (b) Find an example of a rational function whose Julia set is $\hat{\mathbb{C}}$.

(5) Determine an explicit formula (not just a recursive formula) for

$$|P_c^n(0)|, \quad P_c(z) = z^2 + c.$$

(6) Let $f(z) = z^m$ for $m \in \mathbb{N}$, $m \geq 2$. Determine and classify the fixed points of $f$ and prove whether each lies in the Fatou set or the Julia set. Determine the conjugating maps at each fixed point which satisfy $\phi(f) = g(\phi)$, where $g$ is the canonical map associated to the type of fixed point.

(7) Determine the Hausdorff dimension of the set of $x \in [0, 1]$ such that $x$ has only even terms in its decimal expansion and *prove your answer.*

(8) Let $R$ be a rational map. Assume that for a point $z \in \hat{\mathbb{C}}$

$$\mathcal{O}^+(z) = \{R^n(z)\}_{n \geq 0}$$

is finite. Is it true that $\mathcal{O}^-(z)$ is also finite? Prove or give a counterexample.