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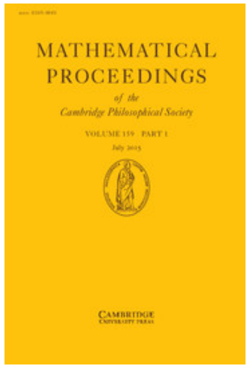

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A heat trace anomaly on polygons

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Abstract

Let Ω_0 be a polygon in $\mathbb{R}^2$, or more generally a compact surface with piecewise smooth boundary and corners. Suppose that Ω_ϵ is a family of surfaces with $\mathcal{C}^\infty$ boundary which converges to Ω_0 smoothly away from the corners, and in a precise way at the vertices to be described in the paper. Fedosov [**6**], Kac [**8**] and McKean–Singer [**13**] recognised that certain heat trace coefficients, in particular the coefficient of t^0, are not continuous as $\epsilon \searrow 0$. We describe this anomaly using renormalized heat invariants of an auxiliary smooth domain Z which models the corner formation. The result applies to both Dirichlet and Neumann boundary conditions. We also include a discussion of what one might expect in higher dimensions.

1. *Introduction*

Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary, or more generally, any two dimensional compact Riemannian manifold with smooth boundary. The Laplace operator with Dirichlet boundary conditions has discrete spectrum $\{\lambda_i\}$ and corresponding eigenfunctions $\{\phi_i\}$. The fundamental solution to the Cauchy problem for the heat equation has Schwartz kernel

$$H^\Omega(t, z, z') = \sum_{i=1}^{\infty} e^{-\lambda_i t} \phi_i(z)\phi_i(z');$$

this converges in $\mathcal{C}^\infty((0, \infty) \times \overline{\Omega} \times \overline{\Omega})$ and is smooth up to $t = 0$ away from the diagonal of $\Omega \times \Omega$. The so-called heat trace is the function

$$\operatorname{Tr} H^\Omega = \sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_\Omega H^\Omega(t, z, z)\, dz; \tag{1·1}$$

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this has an asymptotic expansion as $t \searrow 0$ of the form

$$\operatorname{Tr} H^{\Omega} \sim \sum_{j=0}^{\infty} a_j t^{-1+\frac{j}{2}}. \tag{1·2}$$

Each coefficient a_j is a sum of two terms: an integral over Ω of some universal polynomial in the Gauss curvature K of the metric and its covariant derivatives, and an integral over $\partial\Omega$ of another universal polynomial in the geodesic curvature κ of the boundary and its derivatives. Precise formulæ for these polynomials are complicated (and mostly unknown) when j is large, but the first few are quite simple:

$$a_0 = \frac{1}{4\pi}\int_{\Omega} 1\, dA = \frac{1}{4\pi}|\Omega|, \qquad a_1 = -\frac{1}{8\sqrt{\pi}}\int_{\partial\Omega} 1\, ds = -\frac{1}{8\sqrt{\pi}}|\partial\Omega|$$

and

$$a_2 = \frac{1}{12\pi}\left(\int_{\Omega} K\, dA + \int_{\partial\Omega} \kappa\, ds\right) = \frac{1}{6}\chi(\Omega). \tag{1·3}$$

Here and elsewhere, $|\cdot|$ refers to either area of a domain or length of its boundary, as appropriate.

Almost all of this remains true if the boundary of Ω is piecewise smooth. More precisely, assume that $\partial\Omega$ is a finite union of smooth arcs, γ_i, $i = 1, \ldots, k$, where (counting indices mod k) γ_i meets γ_{i+1} at the vertex p_i with an interior angle $\alpha_i \in (0, 2\pi)$. In fact, the only modification in the statements above is that the heat trace coefficients may now include contributions from the vertices. The formulæ for a_0 and a_1 are the same as before, but now

$$a_2 = \frac{1}{12\pi}\left(\int_{\Omega} K\, dA + \sum_{j=1}^{k}\int_{\gamma_j} \kappa\, ds\right) + \sum_{j=1}^{k}\frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j}. \tag{1·4}$$

The term in parentheses on the right now equals $2\pi\chi(\Omega) - \sum_{j=1}^{k}(\pi - \alpha_j)$. That the coefficient a_2 contains an extra contribution from the vertices was already known to Fedosov [**6**] (who was studying Riesz means of the eigenvalues on polyhedra of arbitrary dimension) and to Kac [**8**], although the explicit expression here was obtained by Dan Ray (this is referenced by Kac and also later by Cheeger [**2**], but apparently Ray did not publish his result). A particularly transparent derivation of this corner term appears in a paper by van den Berg and Srisatkunarajah [**1**].

The heat trace anomaly in the title of our paper is the discrepancy between the heat coefficients in the smooth and polygonal settings. More specifically, it refers to the fact that at least one heat invariant is not continuous with respect to Lipschitz convergence of domains. To phrase this more precisely, let Ω_ϵ be a family of surfaces with *smooth* boundary which converge to a piecewise smoothly bounded domain Ω_0 as $\epsilon \to 0$. We think of Ω_ϵ as Ω_0 with each corner 'rounded out' slightly, but will give a precise formulation in the next paragraph. Denoting the heat trace coefficients for Ω_ϵ by $a_j(\epsilon)$, it will be clear from this definition that

$$\begin{aligned}\lim_{\epsilon\to 0} a_2(\epsilon) &= \lim_{\epsilon\to 0}\frac{1}{12\pi}\left(\int_{\Omega_\epsilon} K_\epsilon\, dA_\epsilon + \int_{\partial\Omega_\epsilon}\kappa_\epsilon\, ds\right)\\ &= \frac{1}{12\pi}\left(\int_{\Omega_0} K_0\, dA_0 + \sum_{i=1}^{k}\int_{\gamma_i}\kappa_0\, ds + \sum_{i=1}^{k}(\pi - \alpha_i)\right),\end{aligned}$$

where K_ϵ and κ_ϵ are the Gauss curvatures of g_ϵ and the geodesic curvatures of $\partial\Omega_\epsilon$ for every

$\epsilon \geqslant 0$, respectively. The anomaly is simply that this formula does not agree with the expression (1·4). The aim of this paper is to provide a simple explanation for the disagreement between these two expressions.

We now explain the desingularisation more precisely. For simplicity, suppose that Ω_0 and Ω_ϵ all lie in some slightly larger ambient open surface $\widetilde{\Omega}$, and that the metrics g_ϵ on Ω_ϵ are all extended to metrics (still denoted g_ϵ) on this larger domain. We assume that this family of metrics converges smoothly on $\widetilde{\Omega}$. Let p be a vertex of Ω_0 and consider the portion of Ω_ϵ in some ball of fixed size around p, $B_c(p) \bigcap \Omega_\epsilon$. Our main assumption is that the family of pointed spaces $(B_c(p) \bigcap \Omega_\epsilon, \epsilon^{-2} g_\epsilon, p)$ converges in pointed Gromov–Hausdorff norm, and smoothly, to a noncompact region $Z \subset \mathbb{R}^2$ with smooth boundary, such that at infinity, Z is asymptotic to a cone with vertex at 0 and with opening angle α, the same angle as at the vertex p in (Ω_0, g). Note that this is actually pointed Gromov–Hausdorff convergence for the ambient space $(\widetilde{\Omega}, g_\epsilon, p)$.

Note that this definition implies that the distance between p and $\partial\Omega_\epsilon$ is bounded above by a constant times ϵ, and that g_ϵ is a small perturbation, which decreases with ϵ, of the rescaling of the standard flat metric on $Z \bigcap B_{c/\epsilon}$. For convenience we assume in the rest of this paper that the constant c equals 1. Thus the basic assumption is the existence of a smoothly bounded asymptotically conic region Z in the plane such that $\epsilon^{-1}(\Omega_\epsilon \bigcap B_1(p))$ converges to Z.

This definition is a special case of a more general desingularisation construction explored carefully in [**14**] and [**15**], as well as [**16**], for the case of degeneration to spaces with isolated conic singularities, and in greater generality still in [**11**]. The aim in these first three papers, as here, is to analyse the behaviour of the heat kernel under this degeneration process. That analysis is quite involved, but yields much sharper results than can be obtained by the present more naive methods. However, one motivation for the present paper is to show how some very simple rescaling arguments, which are only slight generalisations of ones used (in substantially more sophisticated ways) by Cheeger [**2**], already yield some interesting results.

Now consider the function

$$G(t, \epsilon) = \operatorname{Tr} H^{\Omega_\epsilon} = \int_{\Omega_\epsilon} H^{\Omega_\epsilon}(t, z, z)\, dz, \tag{1·5}$$

which is smooth on the interior of the quadrant $Q = \{t \geqslant 0, \epsilon_0 > \epsilon \geqslant 0\}$; our main theorem concerns its precise regularity at the corner $t = \epsilon = 0$. This will be decribed in terms of its regularity on the parabolic blowup of Q which we denote Q_0. This space is diffeomorphic to Q away from the origin, but has an extra ‘front face’ F replacing the point $(0, 0)$ which encodes all the directions of approach to this point along parabolic trajectories. It is described more carefully in Section 2 below. One of the aims of this paper, in fact, is to advertise the utility and naturality of this blowup construction.

THEOREM 1·6. *Let $(\Omega_\epsilon, g_\epsilon)$ be a family of smooth surfaces with Riemannian metrics which converge in the manner described above to a surface with piecewise smooth boundary (Ω_0, g_0). Then the function $G(t, \epsilon)$ lifts to Q_0 to be polyhomogeneous at all boundaries and corners of this space.*

See Remark 1·9 below for a more precise explanation of what is actually proven here; as explained there, the polyhomogeneity at the right face, where $\epsilon \to 0$, turns out to be harder to establish than the corresponding property at the other faces. This has been verified fully in recent work by Sher [**16**], so we have phrased this theorem to cover all that is known, but the proofs here will only focus on the polyhomogeneous behaviour at the left and front faces and then demonstrate the continuity at the right face, which is sufficient to explain the heat trace anomaly.

Recall that polyhomogeneity means simply that the lift of G has asymptotic expansions at the boundary hypersurfaces and product type expansions at the corners. The existence of such expansions brings this phenomenon into better focus; indeed, the heat trace anomaly is simply the fact that the limit as $\epsilon \searrow 0$ of the second asymptotic coefficient $a_2(\epsilon)$ in the expansion as $t \searrow 0$ is not the same as the second asymptotic coefficient of the heat expansion for Ω_0. The front face F of Q_0 separates where these limits are taken (first $t \to 0$ then $\epsilon \to 0$ vs. the other way around) and this extra face allows for the existence of a function which interpolates between these two values. (Note that in all of these statements we only use the continuity at the right face, rather than any expansion there; the key point is the expansion at the left and front faces.) Our second main result describes this function.

THEOREM 1·7. *There is a function $C_2(\tau)$ defined along the front face of Q_0, which is smooth in the rescaled time variable $\tau = t/\epsilon^2$, and satisfies*

$$\lim_{\tau\searrow 0} C_2(\tau) = \frac{\chi(\Omega_0)}{6} \qquad \textit{and}$$

$$\lim_{\tau\nearrow\infty} C_2(\tau) = \frac{\chi(\Omega_0)}{6} + \sum_{j=1}^{k} \frac{\pi^2-\alpha_j^2}{24\pi\alpha_j} - \frac{1}{12\pi}\sum_{j=1}^{k}(\pi-\alpha_j).$$

Its explicit form involves the finite part of a divergent expansion:

$$C_2(\tau) = \frac{\chi(\Omega_0)}{6} + \sum_{j=1}^{k} \operatorname*{f.p.}_{\epsilon=0} \int_{\{z\in Z_j : |z|\leqslant 1/\epsilon\}} H^{Z_j}(\tau, z, z)\, dz - \frac{1}{12\pi}\sum_{j=1}^{k}(\pi-\alpha_j),$$

where Z_j is a noncompact region in the plane which models the collapse at the jth corner.

Remark 1·8. *When Ω_0 is a triangle (or indeed, any polygon in the plane), the first and third terms in the formula for* $\lim_{\tau\to\infty} C_2(\tau)$ *cancel, and we obtain Ray's original formula*

$$\lim_{\tau\to\infty} C_2(\tau) = a_2(0) = \sum_{j=1}^{k} \frac{\pi^2-\alpha_j^2}{24\pi\alpha_j}.$$

This interpolating function $C_2(\tau)$ therefore 'explains' the heat trace anomaly, or alternately, the anomaly is caused by the renormalised heat trace on the complete space Z_j. We also discuss some of the other coefficients in the asymptotic expansions for the lift of G at the various boundary faces and corners of Q_0.

Finally, we note that the behaviour of spectral quantities under 'self-similar smoothing of corners' in two-dimensional domains has been considered elsewhere. In particular, Dauge, Tordeux and Vial [**3**] have carried out an extensive analysis of the asymptotic behaviour of solutions of $\Delta u = f$ on such a family of domains.

This paper is organised as follows. In Section 2, we recall some preliminary facts about parabolic blowups and scaling properties of heat kernels and the standard parametrix construction for heat kernels. The proofs of the two theorems are then presented in Section 3. In Section 4 we indicate the minor modifications needed to prove the analogous result for Neumann boundary conditions; the statement of the main theorem in that setting will be given there. We discuss what can be done toward a result of this type in higher dimensions in Section 5. The results then are less explicit, but the proofs carry over fairly directly.

Remark 1·9. In this paper we do not actually prove the polyhomogeneity of $\mathcal{G}$ at right face of Q_0, i.e. where $\epsilon \to 0$ for $t > 0$, although this had been claimed in an earlier version of this paper. This turns out to be more subtle than we anticipated, and requires an extensive analysis of the asymptotics of the heat kernel on the region Z at large times *and* as the space variables tend to infinity. The relevant information can be obtained using known facts about the low frequency asymptotics of the resolvent. This is developed carefully in the thesis of D. Sher [**16**] (where various interesting applications of this result, in two and higher dimensions, are developed). As noted earlier, the statement of Theorem 1·6 includes the full regularity, including the results from [**16**], but we prove here only that $\mathcal{G}$ is continuous at this right face (and that its restriction to that face has an asymptotic expansion as $t \to 0$). This does not effect the essential point of this paper, which is the existence of the expansion at the front face and the behaviour of the function $C_2(\tau)$.

2. *Preliminaries*

In this section we collect the requisite facts and tools: the behaviour of the heat kernel under scaling of the underlying space, a review of parabolic blowups and polyhomogeneity, and a slight modification of the standard parametrix construction for heat kernels.

2·1. *Heat kernels and dilations*

The heat kernel transforms naturally under dilations of the domain, or equivalently, of the metric. Let (M, g) be any complete Riemannian manifold with smooth (or piecewise smooth) boundary, and denote by $H^M(t, z, z')$ the minimal heat kernel for the Laplacian with Dirichlet boundary conditions on M. This is a smooth function on the interior of $\mathbb{R}^+ \times M \times M$ with well-known regularity properties at the various boundaries and corners.

We first relate this heat kernel to the one for the same manifold M but with rescaled metric $g_\lambda = \lambda^2 g$, $\lambda \in \mathbb{R}^+$. This will be applied when $M \subset \mathbb{R}^2$, g is the induced Euclidean metric, and we relate its heat kernel to the one for λM, the image of M under the dilation $D_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, $z \mapsto \lambda z$. The pullback of the Euclidean metric from λM to M is simply $\lambda^2 g$.

PROPOSITION 2·1. *The heat kernels on M and λM are related by the formula*

$$H^{\lambda M}(\lambda^2 t, \lambda z, \lambda z')\lambda^2 = H^M(t, z, z').$$

Implicit in this formula, we are parametrising points in λM with points in M via D_λ. To prove this proposition, observe that the heat operator $\partial_t - \Delta_z$ on M transforms homogeneously with respect to the parabolic dilation $(t, z) \mapsto (\lambda^2 t, \lambda z)$. Hence, the expression on the left satisfies the heat equation; the additional λ^2 is the Jacobian factor accounting for the fact that $H^{\lambda M}(0, w, w') = \delta(w - w')$ is homogeneous of order -2 in two dimensions.

2·2. *Parabolic blowup*

The parabolic dilation $D_\lambda(t, \epsilon) = (\lambda^2 t, \lambda\epsilon)$ motivates the introduction of the parabolic blowup Q_0 of the quadrant $Q := [0, \infty)_t \times [0, \epsilon_0)_\epsilon$ at $(0, 0)$. This space is defined as follows. As a set, Q_0 is the disjoint union of $Q \setminus \{(0, 0)\}$ and the orbit space $F = (Q \setminus \{(0, 0)\}) / \sim$, where $(t, \epsilon) \sim (t', \epsilon')$ if $(t', \epsilon') = D_\lambda(t, \epsilon)$ for some $\lambda > 0$. More concretely, F is diffeomorphic to a closed quarter-circle; it is also identified with the set of all equivalence classes of parametrised curves $\gamma(s) = (t(s), \epsilon(s))$ with $\lim_{s \searrow 0} \gamma(s) = (0, 0)$, and where $\lim_{s \searrow 0} \epsilon(s)^2/t(s)$ exists (or possibly equals $+\infty$); the identification between curves is given by

$$\gamma \sim \tilde{\gamma} \quad \Longleftrightarrow \quad \lim_{s \to 0} \frac{\epsilon(s)^2}{t(s)} \Big/ \frac{\tilde{\epsilon}(s)^2}{\tilde{t}(s)} = 1.$$

The curves $t = \tau\epsilon^2$ (parametrised by $s \mapsto (\tau s^2, s)$), $\tau \geqslant 0$, provide representatives of each equivalence class except the one represented by the t axis. There is a unique minimal $\mathcal{C}^\infty$ structure on Q_0 for which the lifts of smooth functions from Q and the parabolic polar coordinates $r = \sqrt{t + \epsilon^2}$, t/r^2 and ϵ^2/r^2 are all smooth. We label the faces of Q_0 as follows: F is the new front face, and L and R are the left and right side faces (the lifts of $t = 0$ and $\epsilon = 0$, respectively). There is a smooth 'blowdown' map $\beta : Q_0 \to Q$ defined in the obvious way.

It is usually more convenient to use projective rather than polar coordinates. There are two such systems,

$$(\tau, \epsilon), \quad \tau = t/\epsilon^2 \qquad\qquad \text{and} \quad (t, \eta), \quad \eta = \epsilon/\sqrt{t},$$

which are valid away from R and L, respectively. Thus, for example, τ is an 'angular' variable which vanishes on L, and in this coordinate system $F = \{\epsilon = 0\}$.

Parabolic blowups are described in detail and greater generality in [**12**].

2·3. *Polyhomogeneous conormal functions*

Let M be a manifold with corners. A class of functions which is the natural replacement for (or at least just as good as) the class of smooth functions is the class of polyhomogeneous conormal functions. We refer to [**10**] for a detailed exposition, but review a few facts about these here.

First recall the space $\mathcal{V}_b$ of all smooth vector fields on M which are tangent to all boundaries of M. If $H_1, \ldots, H_k$ are boundary hypersurfaces of M meeting at a corner of codimension k, with boundary defining functions $x_1, \ldots, x_k$, respectively, and local coordinates $y = (y_1, \ldots, y_{n-k})$ on the corner, then $\mathcal{V}_b$ is spanned over $\mathcal{C}^\infty(M)$ locally near this corner by $\{x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{y_1}, \ldots, \partial_{y_{n-k}}\}$.

A function (or distribution) u is said to be conormal if it has stable regularity with respect to $\mathcal{V}_b$. In other words, there exists a k-tuple of real numbers $\mu_1, \ldots, \mu_k$ so that

$$V_1 \ldots V_\ell\, u \in x_1^{\mu_1} \ldots x_k^{\mu_k} L^\infty(M), \qquad \forall\, \ell \quad \text{and}\ \forall\, V_j \in \mathcal{V}_b.$$

(In particular, the μ_i are independent of ℓ and the V_j.) Examples include monomials $x_1^{s_1} \ldots x_k^{s_k}$ for $s_j \in \mathbb{C}$, as well as products of arbitrary powers of $|\log x_j|$. (This definition is slightly inaccurate since it omits the distributions supported at the boundary, i.e. delta sections and their derivatives, which are also conormal, but suffices here.) The special subclass with which we are interested consists of the functions with asymptotic expansions in terms of powers of the boundary defining functions and nonnegative integer powers of the logs of these defining functions, with coefficients which are smooth in all other variables. The expansions are formalised using the notion of an index set I. This consists of a countable sequence of pairs $(\alpha, N) \in \mathbb{C} \times \{\mathbb{N} \cup \{0\}\}$ such that for each $A \in \mathbb{R}$, $\operatorname{Re}\alpha > A$ for all but a finite number of these pairs. By definition, the conormal function u has a polyhomogeneous expansion near a corner of codimension k if there are k index sets $I_1, \ldots, I_k$ so that

$$u \sim \sum_{(\alpha_j, N_j) \in I_j} \sum_{\ell_j \leqslant N_j} x_1^{\alpha_1} (\log x_1)^{\ell_1} \cdots x_k^{\alpha_k} (\log x_k)^{\ell_k} a_{\alpha,\ell}(y),$$

where each coefficient function $a_{\alpha,\ell}$ is $\mathcal{C}^\infty$. Note that since u is already assumed to be conormal, this expansion may be differentiated.

To be specific, a polyhomogeneous function u on Q_0 can be described as follows: near L, u has an expansion in powers of t with coefficients smooth in ϵ; near F in terms of either of the projective coordinate systems, it has an expansion in powers of ϵ with coefficients smooth in τ, or equivalently, in powers of t with coefficients smooth in η; near the corner $L \cap F$ it has an expansion in powers of τ and ϵ, with coefficients now simply numbers. The polyhomogeneous functions on Q and Q_0 which appear below are quite simple. None of them have log terms in their expansions, and the exponents are integers and half-integers.

The final point here is that if u is polyhomogeneous conormal on Q, then its lift $\beta^* u$ to Q_0 is also polyhomogeneous conormal and

$$u \sim \sum a_{jk} t^j \epsilon^k \Longrightarrow \beta^* u \sim \sum a_{jk} (\tau \epsilon^2)^j \epsilon^k = \sum a_{jk} \tau^j \epsilon^{2j+k}.$$

On the other hand, if w is polyhomogeneous on Q_0, then its pushforward to Q is always conormal, but only rarely polyhomogeneous.

2·4. *Parametrix construction*

We conclude this section by reviewing a parametrix construction for the heat kernel, which is useful because it accurately captures the asymptotics of the true heat kernel as $t \searrow 0$. The construction here is slightly nonstandard, but is well suited for the calculations below.

Let M be a complete Riemannian manifold, possibly with boundary, and suppose that $M = M_1 \cup M_2$ where M_1 and M_2 are two manifolds with boundary with $M_1 \cap M_2 = \Sigma$ a hypersurface. If M has boundary, assume that Σ intersects ∂M transversely, and M_1 and M_2 are manifolds with corners of codimension two. Suppose further that M_j lies in a slightly larger complete manifold M_j', again possibly with boundary, such that for some neighbourhood $\mathcal{U}$ of Σ, $M_j' \cap \mathcal{U} = M \cap \mathcal{U}$.

Taking the heat kernels on each M_j' as given, define

$$\tilde{H}^M(t, z, z') = \sum_{j=1}^{2} \chi_j(z) H^{M_j'}(t, z, z') \chi_j(z'),$$

where χ_j is the characteristic function of M_j in M. In the more customary parametrix construction, the M_j are relatively open in M, and $M_1 \cap M_2$ is also open; the $H^{M_j'}$ are pasted together using smooth cutoff functions $\{\psi_j\}$ and $\{\tilde{\psi}_j\}$ with $\psi_1 + \psi_2 = 1$, where $\operatorname{supp} \psi_j \subset \{\tilde{\psi}_j = 1\}$, and $\operatorname{supp} \tilde{\psi}_j \subset M_j'$. We are using sharp (discontinuous) cutoffs rather than smooth ones, however, so that we can identify certain asymptotic coefficients in the calculations to follow.

LEMMA 2·2. *Let $H^M(t, z, z')$ denote the true heat kernel on M, and set*

$$K(t, z) = \tilde{H}^M(t, z, z) - H^M(t, z, z).$$

Then $K(t, z) = \mathcal{O}(t^\infty)$ as $t \searrow 0$.

Proof. Rewrite

$$\tilde{H}^M(t,z,z) = \chi_1(z)\left(H^{M_1'}(t,z,z) - H^M(t,z,z)\right) \\ + \chi_2(z)\left(H^{M_2'}(t,z,z) - H^M(t,z,z)\right) + H^M(t,z,z).$$

By assumption, M_j' agrees with M in a neighbourhood of M_j, so that $H^{M_j'}(t,z,z) - H^M(t,z,z) = \mathcal{O}(t^\infty)$ on the support of χ_j (remember that the small t expansions of these operators are local), and this proves the claim.

Remark 2·3. *We actually need a slightly stronger version, where the first space depends on a parameter ϵ. Suppose that $(M_1'(\epsilon), g_1(\epsilon))$ is a family of spaces which is isometric to $(M_1'(0), g_1)$ for all z with dist$(z,\Sigma) \leqslant c$ for some $c > 0$. Assume also that the heat kernels H^ϵ on this family of spaces satisfy a Gaussian estimate $H^\epsilon(t,z,w) \leqslant Ct^{-\dim M_1/2}\exp(-dist\,(z,w)^2/Ct)$, again uniformly in ϵ. Then the conclusion of this lemma holds uniformly in ϵ, i.e. for each $N > 0$, $|K(t,z)| \leqslant C_N t^N$ with C_N independent of ϵ. We shall explain below that this uniform Gaussian estimate holds in our setting.*

3. *Proofs of main theorems*

We have now assembled all the requisite facts and can proceed with the proofs of the main theorems.

As in the introduction, let $G(t,\epsilon) = \operatorname{Tr} H^{\Omega_\epsilon}$. If $\beta : Q_0 \to Q$ is the blowdown map, then let $\mathcal{G} = \beta^* G$. We need to analyze the behaviour of $\mathcal{G}$ near each of the faces and corners of Q_0, and for that we first use the coordinates (τ,ϵ) introduced in Section 2·2.

We make a simplifying assumption about the geometry in order to focus on the essential parts of the proof. For each i, let S_{α_i} denote the sector in $\mathbb{R}^2$ with opening angle α_i. Choose a smoothly bounded region Z_i in the plane which coincides with S_{α_i} outside $B_{1/2}(0)$, and let $Z_i^\epsilon = B_{1/\epsilon}(0) \bigcap Z_i$. Then we assume that near each vertex p_i, the restriction of the metric g_ϵ to $B_1(p_i) \bigcap \Omega_\epsilon$ is isometric to the dilation by the factor ϵ of the region Z_i^ϵ, which obviously lies in the unit ball. The result remains true in the generality stated earlier, but the proof requires a few more technical steps which are both standard and not particularly germane to the main ideas here. Furthermore, for notational convenience, we assume that there is only a single vertex p and denote the corresponding smooth model region and sector by Z and S, respectively.

Proof of Theorem 1·6. We first construct a particular family of parametrices for the heat kernel on Ω_ϵ. For any $0 \leqslant \epsilon < \epsilon_0$, decompose

$$\Omega_\epsilon = \Omega_{\epsilon,1} \bigcup \Omega',$$

where $\Omega_{\epsilon,1} = \Omega_\epsilon \bigcap B_1(p)$, and $\Omega' = \Omega_\epsilon \setminus (\Omega_\epsilon \bigcap B_1(p))$. Note that Ω' is independent of ϵ. Lemma 2·2 shows that

$$H^{\Omega_\epsilon}(t,z,z) = \chi_1(z) H^{\epsilon Z}(t,z,z) + \chi_2(z) H^{\Omega_0}(t,z,z) + K(t,z), \tag{3·1}$$

where χ_1 is the characteristic function of $|z| \leqslant 1$, $\chi_2 = 1 - \chi_1$, and K is the error term from Lemma 2·2, hence

$$G(t,\epsilon) = \int_{|z|\leqslant 1} H^{\epsilon Z}(t,z,z)\,dz + \int_{\Omega'} H^{\Omega_0}(t,z,z)\,dz + \int_{\Omega_\epsilon} K(t,z)\,dz.$$

We denote the sum on the right hand side by $\mathrm{I} + \mathrm{II} + \mathrm{III}$, and analyse the lifts of these terms successively.

The discussion now is primarily directed toward establishing polyhomogeneity of $\mathcal{G}$ at L and F. We show later that $\mathcal{G}$ is at least continuous up to R, but as noted in the introduction, a lengthier analysis carried out by Sher [**16**] shows that it is polyhomogeneous at that face too.

By Proposition 2·1, $H^{\epsilon Z}(t,z,z') = \epsilon^{-2}H^{Z}(t/\epsilon^2, z/\epsilon, z'/\epsilon)$, so setting $z = z' = \epsilon w$, we see that

$$\beta^* \mathrm{I} = \int_{|w| \leqslant 1/\epsilon} H^{Z}(\tau, w, w)\, dw.$$

This will be the principal term, and we defer its analysis for the moment.

Next, II is independent of ϵ, and it is polyhomogeneous as $t \searrow 0$, with expansion given by integrating the standard heat coefficients $a_j(z)$ over this restricted domain. Hence its lift to Q_0 is clearly polyhomogeneous (at all faces).

Finally, by Lemma 2·2, III depends on ϵ but decays rapidly in t uniformly in ϵ. Thus, $\beta^*\mathrm{III}$ is polyhomogeneous at $L \cup F$.

We now examine $\beta^*\mathrm{I}$ more closely. Choose a smoothly bounded compact region W which agrees with Z in $|w| \leqslant 2$, so that $Z = (W \cap B_1) \cup (S \setminus B_1)$. Using Lemma 2·2 again, write

$$H^{Z}(t,z,z) = \chi_1(z) H^{W}(t,z,z) + \chi_2(z) H^{S}(t,z,z) + K_1(t,z), \tag{3·2}$$

where K_1 is the corresponding error term. Then

$$\beta^*\mathrm{I} = \int_{|w| \leqslant 1} H^{W}(\tau,w,w)\,dw + \int_{1 \leqslant |w| \leqslant 1/\epsilon} H^{S}(\tau,w,w)\,dw + \int_{|w| \leqslant 1/\epsilon} K_1(\tau,w)\,dw,$$

which we write as $\mathrm{I}_i + \mathrm{I}_{ii} + \mathrm{I}_{iii}$.

We first prove polyhomogeneity of these terms at L and F, away from the right face R of Q_0. In this region, we use the coordinates (τ, ϵ); thus $(\tau,\epsilon) = (0,0)$ corresponds to the corner $\mathrm{L} \cap \mathrm{F}$, while $\tau \to \infty$ corresponds to the face R.

The term I_i has an expansion as $\tau \searrow 0$, decays rapidly as $\tau \to \infty$, and is independent of ϵ, so $\beta^*\mathrm{I}_i$ is polyhomogeneous at all faces.

To analyse I_{ii}, set

$$D(R) := \int_{|w| \leqslant R} H^{S}(1,w,w)\,dw.$$

By Proposition 2·1, $\mathrm{I}_{ii}(\epsilon,\tau) = D(1/\epsilon\sqrt{\tau}) - D(1/\sqrt{\tau})$, so it will suffice to show that D has an expansion in powers of $1/R$ as $R \to \infty$. For this, we appeal to a calculation by van den Berg and Srisatkunarajah [**1**], who prove that

$$D(R) = \frac{\alpha R^2}{8\pi} - \frac{R^2}{2\pi}\int_0^1 e^{-R^2 y^2}\sqrt{1-y^2}\,dy + \frac{\pi^2 - \alpha^2}{24\pi\alpha} + \mathcal{O}(e^{-cR^2}), \tag{3·3}$$

for some $c > 0$ independent of R. We remark that it is not hard to obtain polyhomogeneity at the right face too; looking at (3·3), only the structure of the second term on the right is not completely obvious. For that, we may as well replace the upper limit of integration by 1/2 since the integral from 1/2 to 1 decreases exponentially in R. Using the Taylor series for $\sqrt{1-y^2}$ at $y=0$, we find that

$$\frac{R^2}{2\pi}\int_0^{1/2} e^{-R^2y^2}\left(1 - \frac{1}{2}y^2 - \frac{1}{4}y^4 - \dots\right)dy \sim \frac{R}{4\sqrt{\pi}} - \frac{1}{16\sqrt{\pi}R} + \mathcal{O}(R^{-3}).$$

We turn finally to the term I_{iii}. By Lemma 2·2 again, K_1 decreases rapidly as $\tau \to 0$, so this term is also polyhomogeneous at L. Next, by the explicit form of the error term in the proof of that lemma, and using the dilation properties of H^Z and H^S, $K_1(\tau, z) = \mathcal{O}(|z|^{-\infty})$ uniformly for $\tau \leqslant T < \infty$, for any $T < \infty$; in other words, the difference between H^Z and H^S decays rapidly in z uniformly up to any finite time. Hence its integral over $|z| \leqslant 1/\epsilon$ is bounded independently of ϵ. This gives the polyhomogeneous expansion of I_{iii} at L and along F away from R.

This completes the proof of polyhomogeneity of $\beta^*\mathrm{I}$ for τ in any bounded set, and away from R. Note that we have actually established polyhomogeneity of $\mathcal{G}$ at the right face for all terms except I_{iii} and III.

To finish the proof, we demonstrate that $\mathcal{G}$ is continuous up to the interior of R, and is also continuous at the corner $\mathrm{R} \bigcap \mathrm{F}$ once we subtract off the singular part of the expansion at F.

Consider the continuity up to R away from the corner. For each $\epsilon \geqslant 0$,

$$\operatorname{Tr} H^{\Omega_\epsilon}(t) = \sum_{j=1}^{\infty} e^{-\lambda_j(\epsilon)t}.$$

We claim that there is a constant $C > 0$ which is independent of ϵ such that

$$\lambda_j(\epsilon) \geqslant Cj$$

for all j. This can be proved in several ways. One argument, which works only for Dirichlet boundary conditions, is to consider a slightly larger domain Ω' which contains Ω_ϵ for all $\epsilon \geqslant 0$, so that by domain monotonicity,

$$\lambda_j(\epsilon) \geqslant \lambda_j(\Omega'), \quad \forall \quad j \in \mathbb{N}.$$

By the Weyl asymptotic formula for the eigenvalues,

$$\lim_{j\to\infty} \frac{\lambda_j(\Omega')}{4\pi j/|\Omega'|} = 1,$$

where $|\Omega'|$ denotes the area of Ω'. Consequently, since all Dirichlet eigenvalues of Ω' are positive, there exists a constant $C > 0$ such that

$$\lambda_j(\Omega') \geqslant Cj, \quad \forall \quad j \in \mathbb{N}.$$

It follows from domain monotonicity that

$$\lambda_j(\epsilon) \geqslant \lambda_j(\Omega') \geqslant Cj, \quad \forall \quad j \in \mathbb{N}.$$

There are other ways to prove this which work equally well for Neumann boundary conditions (and other generalisations). For example, it suffices to show that $\operatorname{Tr} H^{\Omega_\epsilon}(t) \leqslant C/t$ for some constant C which is independent of ϵ, see [**16**] for details.

In any case, fixing $\delta > 0$, this implies that there exists an N so that

$$\sum_{j=N}^{\infty} e^{-\lambda_j(\epsilon)t} < \frac{\delta}{2}, \quad \text{for all } \epsilon \geqslant 0.$$

It is also known, see [**11**] and [**15**], that $\lambda_j(\epsilon) \to \lambda_j(0)$ for $j = 1, \dots, N$. Hence there exists $\epsilon_0 > 0$ such that

$$\left| \sum_{j=1}^{N} e^{-\lambda_j(\epsilon)t} - \sum_{j=1}^{N} e^{-\lambda_j t} \right| < \frac{\delta}{2},$$

for $\epsilon < \epsilon_0$. Since these estimates hold uniformly for t in any compact interval $[t_0, t_1] \subset (0, \infty)$, we obtain continuity of G up to R away from the corner.

We now analyse each of the terms β^*I, β^*II and β^*III at R $\cap$ F separately. The second of these is independent of ϵ, so its continuity along R up to the corner is obvious. Lemma 2·2 and the remark after it show that III decays rapidly as $t \to 0$, uniformly in ϵ, hence β^*III decays rapidly at R $\cap$ F.

It remains to consider $\beta^* I$ in its entirety, which we do using a somewhat different subdivision. First, using the coordinates η and t,

$$\beta^* I = \int_{|z| \leqslant 1/\sqrt{t}} H^{\eta Z}(1, z, z)\, dz.$$

Now, observe that $\eta Z \setminus (B_\delta(0) \cap \eta Z)$ is identified with $S \setminus (B_\delta(0) \cap S)$ provided $\delta \geqslant \eta$, and that for such δ, we have $|B_\delta(0) \cap \eta Z|, |B_\delta(0) \cap S| = \mathcal{O}(\delta^2)$, and in addition,

$$\int_{B_\delta(0) \cap \eta Z} H^{\eta Z}(1, z, z)\, dz = \mathcal{O}(\delta^2), \quad \int_{B_\delta(0) \cap S} H^S(1, z, z)\, dz = \mathcal{O}(\delta^2). \tag{3·4}$$

These last two estimates follow by rescaling by $1/\delta$ to obtain integrals over $B_1(0) \cap (\eta/\delta)Z$ and $B_1(0) \cap S$, respectively, at time $1/\delta^2$ and then invoking the standard $1/t$ decay rate of the heat kernel on compact sets as $1/\delta^2 \nearrow \infty$. (This rate of decay is uniform for $\eta/\delta \leqslant 1$.)

Applying (3·3) with $R = t^{-1/2}$ and the second part of (3·4) gives

$$\int_{\{\delta \leqslant |z| \leqslant 1/\sqrt{t}\}} H^S(1, z, z)\, dz = \frac{\alpha}{8\pi t} - \frac{1}{4\sqrt{\pi t}} + \frac{\pi^2 - \alpha^2}{24\pi\alpha} + \mathcal{O}(\sqrt{t} + \delta^2)$$

at this corner. To compare this with

$$\int_{\{\delta \leqslant |z| \leqslant 1/\sqrt{t}\}} H^{\eta Z}(1, z, z)\, dz,$$

we proceed as follows. First, on any compact subset $\delta \leqslant |z|, |w| \leqslant R$,

$$H^{\eta Z}(1, z, w) \longrightarrow H^S(1, z, w) \text{ as } \eta \longrightarrow 0.$$

To prove this, observe that so long as $H^{\eta Z}$ has a limit, then by uniqueness of the heat kernel, this limit must equal H^S. By parabolic Schauder estimates, to guarantee convergence it suffices to show that for any sequence (z_j, w_j) in this compact set and $\eta_j \to 0$, $H^{\eta_j}(1, z_j, w_j)$ is bounded away from 0 and ∞. Convergence to zero is ruled out by the parabolic Harnack inequality and uniqueness, so we need only consider the case that $H^{\eta_j Z}(1, z_j, w_j) = A_j \to \infty$. But if this were to occur, then $A_j^{-1} H^{\eta_j}(t, z, w_j)$ would converge to a nontrivial solution of the heat equation with vanishing initial conditions at $t = 0$, which is again impossible.

To conclude, we must finally show that

$$\int_{\{R \leqslant |z| \leqslant 1/\sqrt{t}\}} (H^{\eta Z}(1, z, z) - H^S(1, z, z))\, dz$$

can be made as small as desired by taking R sufficiently large. For this we invoke the (uniform in η) Gaussian upper bounds

$$H^{\eta Z}(t, z, w), H^S(t, z, w) \leqslant C t^{-1} e^{-|z-w|^2/Ct}$$

which were already mentioned at the end of Section 2. It follows from these by direct and straightforward estimation that the integrand above is bounded by $Ce^{-|z|^2/C}$. We sketch the argument. Consider the parametrix for $H^{\eta Z}$,

$$\widetilde{H}^{\eta Z}(t,z,w) = \tilde{\chi}_1(z)H^{W_\eta}(t,z,w)\chi_1(w) + \tilde{\chi}_2(z)H^S(t,z,w)\chi_2(w),$$

where W_η is a compact domain which agrees with ηZ for $|z| \leqslant 1$ and is independent of η in the region $|z| \geqslant 1$, where $\{\chi_1, \chi_2\}$ is a partition of unity relative to the open cover $\{|z| < 3/2\} \cup \{|z| > 1/2\}$, and where $\tilde{\chi}_j = 1$ on the support of χ_j, with $\tilde{\chi}_j'$ of compact support. The error term $E(t,z,w) = \widetilde{H}^{\eta Z}(t,z,w) - H^{\eta Z}(t,z,w)$ equals

$$E(t,z,w) = \int_0^t \int_{\eta Z} H^{\eta Z}(s,z,v)\big([\tilde{\chi}_1, \Delta]H^{W_\eta}(t-s,v,w)\chi_1(w) \\ +[\tilde{\chi}_2,\Delta]H^S(t-s,v,w)\chi_2(w)\big)\, dvds.$$

Using that $\operatorname{supp}\tilde{\chi}_2'$ is compact, we obtain $E(1,z,z) \leqslant Ce^{-|z|^2/C}$, uniformly in η, hence $H^{\eta Z}(1,z,z) - H^S(1,z,z) = E(1,z,z)$ decays rapidly as $|z| \to \infty$, as claimed.

The uniform Gaussian upper bounds of these heat kernels employed here is well known, see [**4**, Chapter 3 and 5].

Collecting these various estimates yields

$$\beta^* I = \frac{\alpha}{8\pi t} - \frac{1}{4\sqrt{\pi t}} + \frac{\pi^2-\alpha^2}{24\pi\alpha} + o(1),$$

as $(\eta, t) \to (0,0)$.

Finally, combining terms and summing over all corners (as explained more carefully in the proof of Theorem 1·7 at the end of this section), we have proved that

$$\beta^* I(\eta,t) + \beta^* II(\eta,t) = \frac{|\Omega_0|}{4\pi t} - \frac{|\partial\Omega_0|}{8\sqrt{\pi t}} + \sum_{j=1}^n \frac{\pi^2-\alpha_j^2}{24\pi\alpha_j} + o(1)$$

near $\mathrm{F} \cap \mathrm{R}$. This completes our analysis of $\mathcal{G}_0$ on Q_0.

Proof of Theorem 1·7. This consists of examining the terms in the expansion of $\mathcal{G}$ at the various boundary faces.

First, at L, away from F we may use the variables (t, ϵ), and

$$\mathcal{G}(t,\epsilon) \sim \sum_{j=0}^{\infty} a_j(\epsilon)t^{-1+j/2}.$$

Near $\mathrm{L} \cap \mathrm{F}$, we substitute $t = \epsilon^2\tau$ to get

$$\mathcal{G}(\tau,\epsilon) \sim \sum_{j=0}^{\infty} a_j(\epsilon)\tau^{-1+j/2}\epsilon^{-2+j}. \tag{3·5}$$

The coefficients $a_j(\epsilon)$ are polyhomogeneous as $\epsilon \to 0$ by Theorem 1·6.

Near $F \cap R$ we use the coordinates t and $\eta = \epsilon/\sqrt{t}$ to see that

$$\mathcal{G}(t,\eta) \sim \sum_{j=-1}^{\infty} B_j(\eta)t^{j/2},$$

where each B_j is continuous to $\eta = 0$. Along R, $\mathcal{G}(t,0) = \operatorname{Tr} H^{\Omega_0}$.

Finally, near F, we use the coordinates (τ, ϵ), so the expansion is in powers of ϵ, and by (3·5) it is

$$\mathcal{G}(\tau, \epsilon) \sim \sum_{j=0}^{\infty} C_j(\tau)\epsilon^{-2+j}.$$

We shall identify the coefficients C_0, C_1 and C_2.

By our analysis of the terms I_i, I_{ii}, I_{iii}, II and III, we see that only I_i, I_{ii} and II contribute to the coefficients of ϵ^{-2} and ϵ^{-1}. Substituting directly from the expansions of these terms (using the McKean–Singer asymptotics on W and Ω' for I_i and II, respectively, and the first terms in the expansion of $D(1/\epsilon\sqrt{\tau})$ for I_{ii}) and then using the definition of the finite part at $\epsilon = 0$ of I, we have

$$\begin{aligned}\mathcal{G}(\tau, \epsilon) \sim \frac{1}{\epsilon^2\tau}\left(\frac{|\Omega'|}{4\pi} + \frac{\alpha}{8\pi}\right) - \frac{1}{\epsilon\tau^{1/2}}\left(\frac{|\partial\Omega'|}{8\sqrt{\pi}} + \frac{1}{4\sqrt{\pi}}\right) \\ + \frac{1}{12\pi}\left(\int_{\Omega'} K\,dA + \int_{\partial\Omega'} \kappa\,ds\right) + \underset{\epsilon=0}{\text{f.p.}} \int_{|w|\leqslant 1/\epsilon} H^Z(\tau, w, w)\,dw + \mathcal{O}(\epsilon).\end{aligned}$$

In other words,

$$C_0(\tau) = \frac{1}{\tau}\left(\frac{|\Omega'|}{4\pi} + \frac{\alpha}{8\pi}\right),$$

$$C_1(\tau) = -\frac{1}{\sqrt{\tau}}\left(\frac{|\partial\Omega'|}{8\sqrt{\pi}} + \frac{1}{4\sqrt{\pi}}\right)$$

and

$$C_2(\tau) = \frac{1}{12\pi}\left(\int_{\Omega'} K dA + \int_{\partial\Omega'} \kappa ds\right) + \underset{\epsilon=0}{\text{f.p.}} \int_{|w|\leqslant 1/\epsilon} H^Z(\tau, w, w)\,dw.$$

This simplifies by the following observations: first, the area of a circular sector of opening α and radius 1, i.e. $|\Omega_0 \cap B_1|$, equals $\alpha/2$, so the coefficient of $\epsilon^{-2}\tau^{-1}$ is just $|\Omega_0|/4\pi$; similarly, the sides of this circular sector are straight lines, so $|\partial\Omega_0 \cap B_1| = 2$, which means that the next coefficient is $-|\partial\Omega_0|/8\sqrt{\pi}$; finally, since g_0 is flat in $\Omega_0 \cap B_1$, $K \equiv 0$ there, so using that the contribution from 'turning the corner' at p in the boundary integral is $\pi - \alpha$, we find that

$$\int_{\Omega'} K\,dA + \int_{\partial\Omega'} \kappa\,ds = 2\pi\chi(\Omega_0) - (\pi - \alpha).$$

This means that

$$C_2(\tau) = \underset{\epsilon=0}{\text{f.p.}} \int_{|w|\leqslant 1/\epsilon} H^Z(\tau, w, w)\,dw + \frac{1}{6}\chi(\Omega_0) - \frac{\pi - \alpha}{12\pi}. \tag{3·6}$$

We conclude by calculating its behaviour for small and large τ. Using the small τ asymptotics, we see that

$$\begin{aligned}\int_{|w|\leqslant 1/\epsilon} H^Z(\tau, w, w)\,dw \sim \frac{|Z \cap B_{1/\epsilon}|}{4\pi}\tau^{-1} \\ - \frac{|\partial Z \cap B_{1/\epsilon}|}{8\sqrt{\pi}}\tau^{-1/2} + \frac{1}{12\pi}\int_{\partial Z} \kappa\,ds + \mathcal{O}(\epsilon\tau^{1/2});\end{aligned}$$

hence the finite part of this integral is equal (up to the factor 12π) to the integral of curvature on the boundary of Z, which is the total turning angle $\pi - \alpha$, so finally, the limit of C_2 as $\tau \to 0$ is $\chi(\Omega_0)/6$, as claimed.

Finally, we use the dilation one more time to calculate that

$$\int_{|w|\leqslant 1/\epsilon} H^Z(\tau, w, w)\, dw = \int_{|w|\leqslant 1/\epsilon\sqrt{\tau}} H^{Z/\sqrt{\tau}}(1, w, w)\, dw.$$

Noting that $\epsilon\sqrt{\tau} = \sqrt{t}$, and since $Z/\sqrt{\tau}$ converges to the sector S as $\tau \to \infty$, we can use the expansion (3·3) to see that the finite part is indeed $(\pi^2 - \alpha^2)/24\pi\alpha$. Therefore, in general, with an arbitrary number of vertices,

$$\lim_{\tau\to\infty} C_2(\tau) = \frac{\chi(\Omega_0)}{6} + \sum_{j=1}^{k} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j} - \frac{1}{12\pi}\sum_{j=1}^{k}(\pi - \alpha_j);$$

in particular, if Ω_0 is a polygon, its Euler characteristic is 1, so the first and third terms cancel.

This completes the proof.

4. *Neumann boundary conditions*

We now briefly discuss the minor modifications needed to prove the analogues of Theorems 1·6 and 1·7 assuming Neumann rather than Dirichlet boundary conditions.

A cursory inspection of the proof shows that the only real issue is to find an analogue of the van den Berg–Srisatkunarajah formula (3·3) in this setting. This does not seem to appear explicitly in the literature, but fortunately, a recent paper by Kokotov [**9**] contains the corresponding formula for the complete cone $C_{2\alpha}$ of angle 2α. Let H^C denote the heat kernel on this cone. Then by [**9**, Proposition 1], there exists $c > 0$ such that for every $R > 0$,

$$\int_{|z|\leqslant R} H^C(1, z, z)\, dz = \frac{\alpha R^2}{4\pi} + \frac{1}{12}\left(\frac{4\pi^2 - (2\alpha)^2}{2\pi(2\alpha)}\right) + \mathcal{O}(e^{-cR^2}). \tag{4·1}$$

This formula is stated in [**9**] for fixed radius R and for the heat kernel at time t as $t \to 0$, but because of the usual scaling properties, it holds equally well for fixed t, say $t = 1$, and as the radius $R \to \infty$; indeed, the quantity on the left depends only on the ratio R/t^2. The coefficients in this expansion have been written in a nonreduced form in order to emphasize the dependence on the angle 2α.

We now observe that the cone $C_{2\alpha}$ is the union of two copies of the sector S_α with the boundary rays identified. Alternately, let σ be the obvious reflection on the cone $C_{2\alpha}$; then a region isometric to the sector S_α is a fundamental domain for this action, and its image $\sigma(S_\alpha)$ is the other half of the cone. In any case, using this, the formula for the Neumann heat kernel follows directly from (3·3) and (4·1). Indeed, let $L^2(C_{2\alpha}) = L^2_+ \oplus L^2_-$ be the decomposition into functions which are even and odd with respect to τ. If $u \in H^2(C_{2\alpha}) \cap L^2_+$, then u has vanishing normal derivative at ∂S_α, while if $u \in H^1(C_{2\alpha}) \cap L^2_-$ then u vanishes at ∂S_α. Since the Laplacian commutes with σ, the heat kernel has a 2-by-2 block decomposition: the upper left and lower right on-diagonal blocks are canonically identified with the Neumann and Dirichlet heat kernels of S_α, and we denote these by H^S_{N} and H^S_{D}, respectively. Therefore,

$$\int_{|z|\leqslant R} \left(H^S_{\mathrm{D}}(1, z, z) + H^S_{\mathrm{N}}(1, z, z)\right) dz = \int_{|z|\leqslant R} H^C(1, z, z)\, dz,$$

whence

$$\int_{|z|\leqslant R} H_{\mathrm{N}}^{S}(1,z,z)\,dz \sim \frac{\alpha R^2}{4\pi} + \frac{R}{4\sqrt{\pi}} + \frac{\pi^2-\alpha^2}{24\pi\alpha} + \cdots \tag{4·2}$$

In other words, in this asymptotic formula, only the signs of the odd powers of R are reversed from those in the corresponding formula for the Dirichlet heat kernel.

It is now a simple matter to track through the various arguments in this paper to obtain that $\mathcal{G}_{\mathrm{N}}(\tau,\epsilon)$, the pullback to Q_0 of the trace of the heat kernel for the Laplacian with Neumann boundary conditions on Ω_ϵ, is polyhomogeneous at $\mathrm{L}\cup\mathrm{F}$ and has the expansion

$$\mathcal{G}_{\mathrm{N}}(\tau,\epsilon) \sim \frac{1}{\epsilon^2\tau}\frac{|\Omega_0|}{4\pi} + \frac{1}{\epsilon\tau^{1/2}}\frac{|\partial\Omega_0|}{8\sqrt{\pi}} + \\ \frac{1}{12\pi}\left(\int_{\Omega'} K\,dA + \int_{\partial\Omega'}\kappa\,ds\right) + \underset{\epsilon=0}{\mathrm{f.p.}}\int_{|w|\leqslant 1/\epsilon} H_{\mathrm{N}}^{Z}(\tau,w,w)\,dw + \mathcal{O}(\epsilon)$$

at F. In particular, the coefficient $C_2(\tau)$ of ϵ^0 is exactly the same as in the Dirichlet case. The proof of continuity at R proceeds as before too, noting that the Gaussian upper bounds hold for the Neumann heat kernel as well. We leave the details, which are all straightforward, to the reader.

5. *Higher dimensions and other generalizations*

We have focused in this paper on two-dimensional domains in order to emphasize the simplicity of the arguments and to take advantage of the explicit nature of the formulæ. There are various analogues of these results in higher dimensions, which we now describe briefly. These generalizations should have some interesting applications, which will be developed elsewhere (in particular, see [**16**]).

One direction is to consider a family of Riemannian metrics g_ϵ on a compact manifold M^n of any dimension such that (M, g_ϵ) degenerates to a space (M_0, g_0) which has isolated conic singularities. We assume that this degeneration is modelled on the rescalings of a complete asymptotically conic space (Z, g_Z), i.e. such that suitable neighbourhoods of (M, g_ϵ) are (asymptotically equivalent to) rescalings of truncations of (Z, g_Z). The behaviour of the entire heat kernel for this type of degeneration was studied in detail in [**14**, **15**]. The analysis in those papers is considerably more intricate than what is done here, but also gives information about the entire heat kernel, not just its behaviour along the diagonal. (Note that [**14**] and [**15**] only treat the case where M has no boundary, so strictly speaking do not apply to domains in $\mathbb{R}^2$, as considered here, but the techniques there can certainly be adapted to cover our setting.) One consequence of those results is that the trace of the heat kernel for Δ_{g_ϵ} lifts to Q_0 to be polyhomogeneous conormal at L and F. Continuity up to $\epsilon = 0$ also follows from what is written there; the stronger assertions about full polyhomogeneity are not verified there, however.) In other words, by specializing the results of those papers to two dimensions, one can recover the results here. One of the motivations for the present paper, however, is to develop simpler methods to obtain this information about the trace directly without understanding the entire and considerably more complicated structure of the family of heat kernels. The difficulty in obtaining polyhomogeneity of the heat trace at the faces covering $\epsilon = 0$ is that the behaviour there involves the heat kernel on Z as the rescaled time $\tau \to \infty$, so one must analyze H^Z as both the time and spatial variables tend to infinity separately, even if one is only interested in the behaviour along the diagonal. One way to manage this difficulty is to represent the heat kernel in terms of the resolvent $R(\lambda)$

and use detailed information about the asymptotics of the resolvent as $\lambda \to 0$ obtained by Guillarmou and Hassell [**7**]; see Sher [**16**] for this analysis.

In higher dimensions, the coefficients in the expansions at the faces L and R can be still be determined somewhat explicitly. In particular, the coefficient $C_n(\tau)$ of ϵ^0 at the front face of Q_0 is now equal to the sum

$$\int_{M_0} q_n \, dV + \underset{\epsilon=0}{\text{f.p.}} \int_{|w| \leqslant 1/\epsilon} H^Z(\tau, w, w) \, dw,$$

where q_n is the standard heat invariant integrand for the metric g_0. Unfortunately, there is probably no explicit formula for the limit as $\tau \to \infty$ of this regularised trace, except in special cases. Work in progress of L. Friedlander give some partial results for three dimensional piecewise linear sectors.

There should be a similar generalisation of the ideas here to the setting of resolution blowups of iterated edge spaces (or smoothly stratified spaces), as introduced in [**11**]; understanding the structure of the full heat kernel for those degenerating families is likely to present enormous difficulties, whereas it may be possible to obtain information about just the traces more simply.

A special and very interesting case would be to find an analogue of Theorem 1·7 for smoothings of Euclidean polyhedra in arbitrary dimension. The description of a family of 'self-similar' smoothings of an arbitrary polyhedron is not difficult and closely follows the scheme presented in [**11**]. However, in order to make this formula explicit, one would need an analogue of (3·3) or (4·1) for higher dimensional polyhedral sectors, which does not seem to be available. (Analogous results *are* known for other spectral invariants, however; see [**2**] and [**6**].)

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