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ÉTALE LATTICES OVER QUADRATIC INTEGERS

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Abstract. We construct lattices with quadratic structure over the integers in quadratic number fields having the property that the rank of the quadratic structure is constant and equal to the rank of the lattice in all reductions modulo maximal ideals. We characterize the case in which such lattices are free. The construction gives a representative of the genus of such lattices as an orthogonal sum of "standard" pieces of ranks 1 – 4 and covers the case of the discriminant of the real quadratic number field congruent to 1 modulo 8 for which a general construction was not known.

1. INTRODUCTION

The purpose of the paper is to study quadratic forms with "good reduction" modulo all maximal ideals in the rings of integers in number fields. The quadratic forms x_1x_2 and $x_1x_2 + x_3^2$ are such examples over the rational integers (and, in fact, over any ring when the definitions are suitably chosen) – reductions of these quadratic forms modulo all prime numbers p give forms of respectively ranks 2 and 3 over all fields $\mathbb{Z}/(p)$. Thus, by a "good reduction" we mean unchanged rank of the form when its coefficients are reduced modulo any maximal ideal in the ring over which the form is defined. By analogy with the theory of étale algebras, and in fact, because of some close relations of such forms to these algebras, we use the term étale form (or étale lattice). The quadratic forms x_1x_2 and $x_1x_2 + x_3^2$ are étale over the rational integers, while the unimodular form $x_1^2 + x_2^2$ is not étale, since its reduction modulo 2 gives a form of rank 1 over the field $\mathbb{Z}/(2)$. For general formal definitions see Section 2. The étale forms (étale lattices) were introduced by Deligne [D], who used the term "ordinaire". Later Swan studied such forms in [Sw], where he also comments on terminological problems related to their definition (see Remark 2.3).

Over the rational integers, it is possible to get a satisfactory characterization of étale quadratic forms using well-known results on classification of quadratic forms with discriminants 1 and 2 as presented in e.g. [CS] (see Section 8). The aim of the present paper is to characterize étale forms, or more exactly, étale lattices over the integers in quadratic number fields. Investigations of this type as regards even dimensional totally definite integral quadratic forms over the integers in real quadratic fields were carried out by Maass in [Ma]. He gave a description of such forms and constructed particular representatives of their genus over the integers in the fields $\mathbb{Q}(\sqrt{\Delta})$ when $\Delta > 0$ is square-free and $\Delta \not\equiv 1 \pmod 8$. A characterization of such forms by their natural invariants and more general similar results concerning invariants of lattices over the integers in arbitrary algebraic number fields were later obtained in [Ch] (see Remark 9.10).

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In the present paper, we find some simple necessary and sufficient conditions on the classical invariants of quadratic spaces over the quadratic number fields, which guarantee that the space is étale, that is, it contains an étale lattice (see Theorems 9.2, 9.7 (a) and 9.9 (a)). This part could be also deduced from the results in [Ch] (however, see Remark 9.10). All étale lattices on an arbitrary étale quadratic space form one genus and for all quadratic number fields, we prove the existence of a particular lattice belonging to this genus also in the real case when $\Delta \equiv 1 \pmod 8$ (and the space is totally definite), which was not covered in [Ma] (see Theorems 9.2, 9.7 (b) and 9.9 (b)). Our construction is different from that in [Ma] and the method we use gives a much more uniform treatment of all the cases. Combining some results of [BB] with the results of the present paper, it also gives a natural explanation and characterization of the case considered by Maass when an étale quadratic space covers a free étale lattice giving a "genuin" étale quadratic form.

Our lattices decompose into an orthogonal sum of some "standard pieces" of ranks depending on the arithmetical properties of the field. In the final Section 10, we give an explicit construction of these standard pieces of ranks $1, 2, 3$ and 4. Over the rational integers the standard pieces have ranks $1, 2, 7$ and 8, where the last two ranks come from the lattices E_7 and E_8. The results of Section 10 for rank 4 depend on a general algorithm for construction of minimal over-orders of a given non-maximal quaternion order in an arbitrary quaternion algebra over the quotient field of a Dedekind ring of characteristic 0. This algorithm is presented in an Appendix. The preliminary Sections 2–7 present the necessary definitions (Section 2), auxiliary results (Sections 3–5, 7) and the characterization of étale spaces over local fields (Section 6).

We express our thanks to Laura Fainsilber who turned our attention to the reference [CS] concerning the étale lattices over the rational integers.

2. ÉTALE LATTICES

Let R be a commutative ring with a unity element. Denote by (L, q) a finitely generated projective R-module L with a quadratic mapping $q : L \to R$. This means that,

$$b_q = b : L \times L \to R, \text{ where } b(x, y) = q(x + y) - q(x) - q(y)$$

is an R-bilinear mapping and $q(ax) = a^2 q(x)$ for $x, y \in L$ and $a \in R$. We call (L, q) a quadratic R-module. In the case when R is a domain and L has rank $\mathrm{rk}(L) = n$ over R, we say that (L, q) is a quadratic module of rank n.

Let $\varphi : R \to R'$ be a ring homomorphism. It is well known (and easy to prove) that there exists a unique quadratic mapping $q_{R'} : L \otimes_R R' \to R'$ such that $q_{R'}(l \otimes a') = \varphi(q(l))a'^2$ for each $l \in L$ and $a' \in R'$. Sometimes we denote $q_{R'}$ by $q \otimes R'$. In particular, if $R' = R/\mathfrak{a}$ for an ideal $\mathfrak{a}$ in R, and φ is the natural surjection, then $q_{R'} : L/\mathfrak{a}L \to R/\mathfrak{a}$ will be also denoted by $q_{\mathfrak{a}}$ and called the reduction of q (or (L, q)) modulo $\mathfrak{a}$. For a quadratic module (L, q) define $L_q = \{x \in L \mid b(x, L) = q(x) = 0\}$. It is easy to check that L_q is a submodule of L.

Definition 2.1. We say that that a quadratic module (L, q) over R is étale if $(L/\mathfrak{m}L)_{q_{\mathfrak{m}}} = (0)$ for any maximal ideal $\mathfrak{m}$ in R.

Remark 2.2. If (L, q) is a quadratic module, the rank of q over R can be defined as $\mathrm{rk}(q) = \mathrm{rk}(L) - \mathrm{rk}(L_q)$. Then the definition of étale modules says that all reductions $q_{\mathfrak{m}}$ of

(L, q) modulo all maximal ideals $\mathfrak{m}$ in R have constant ranks at least over the residue fields $R/\mathfrak{m}$ corresponding to the maximal ideals in the connected components of the spectrum of R. In fact, we shall mostly assume that R is a domain, so that the rank of the projective module L is constant over R and the ranks of all reductions of q are equal to it when (L, q) is étale. For the definition of the rank of q, at least in the case when R is a field (what we essentially need), see [Ke], p.10 and [Sw], Prop. 1.1 (see also Proposition 2.4 below). Notice that the extra condition $q(x) = 0$ in the definition of L_q is only essential when the characteristic of R equals 2.

Remark 2.3. The notion of "étale modules" was introduced by Deligne in [D] where they are called "ordinaire". Later Swan in [Sw] studied such modules calling them "nonsingular quadratic modules". Swan comments on the terminological problems related to this notion – the terms nonsingular, regular, non-degenerate are used in different contexts. One can hope that the term "étale" gives a better chance to avoid misunderstandings. Moreover, there is a close relation between étale modules and étale (Clifford) algebras, which motivates using this terminology (see Section 3). Notice that if the rank of L over R is even, then (L, q) is étale if and only if (L, b_q) is a nonsingular bilinear module in the usual sense (L and its dual $L^* = \mathrm{Hom}_R(L, R)$ are isomorphic over R by means of the linear mapping $x \mapsto b_q(x, \cdot)$).

The simplest examples of étale modules are the following: $L = Re_1 + \cdots + Re_n$ a free R-module of rank n over an arbitrary ring R and

$$q(x_1e_1 + \cdots + x_ne_n) = \begin{cases} x_1x_2 + \cdots + x_{2m-1}x_{2m} & \text{if } n = 2m, \\ x_1x_2 + \cdots + x_{2m-1}x_{2m} + \varepsilon x_{2m+1}^2 & \text{if } n = 2m+1, \end{cases} \tag{2.1}$$

where $\varepsilon \in R^*$ (R^* denotes as usual the units in R). This statement follows easily from the definition of étale modules and it is also a direct consequence of our next result.

First, we have to recall some modification of the notion of discriminant for quadratic modules (see [Ke], (2.13) or [Ku], p.208). Let (L, q) be a free quadratic module with $L = Re_1 + \cdots + Re_n$. We have, $b_q(e_i, e_i) = 2q(e_i)$, and if n is odd, then each product of the off-diagonal elements in $\det[b_q(e_i, e_j)]$ appears twice. Therefore for odd n, it is possible to write

$$\det[b_q(e_i, e_j)] = 2P(q(e_i), b_q(e_i, e_j)),$$

where $P(x_{ii}, x_{ij})$ is the polynomial with integer coefficients, which we get by computing over the integers the determinant of the $n \times n-$matrix having $2x_{ii}$ on the $(i, i)-$places and $x_{ij} = x_{ji}$ on the $(i, j)-$places for $i \neq j$ and dividing it by 2. Let $n = 2m$ or $n = 2m + 1$ (note that $m \equiv \frac{n(n-1)}{2} \pmod 2$). We shall call

$$(-1)^m \det[b_q(e_i, e_j)] \text{ for } n \text{ even and } (-1)^m P(q(e_i), b_q(e_i, e_j)) \text{ for } n \text{ odd}$$

the discriminant of the basis $e_1, \ldots, e_n$. We shall write $d(L, q)$ (or sometimes shorter $d(q)$) to denote the R-ideal generated by the discriminant of the basis $e_1, \ldots, e_n$. Of course, $d(L, q)$ does not depend on the choice of a basis for L. When n is odd, then $d(L, q)$ is usually called *half-discriminant* of (L, q) (as the ideal generated by the half-discriminant of a basis), but in this paper we call it discriminant in both even and odd dimensions. If L is an arbitrary R-lattice on V, then by the discriminant $d(L, q)$ (or shorter, $d(q)$) of (L, q), we mean the R-ideal which for every open subset of $\mathrm{Spec}(R)$ over which L is free equals the discriminant defined above.

Now we have the following result, which gives a possibility to give an alternative definition of étale modules (for the equivalence, see [Sw], Prop. 1.1, [Ku], p.208 and [Ke], p.9-10):

Proposition 2.4. *A quadratic module (L, q) is étale if and only if $d(L, q) = R$.*

For the convenience of the Reader, we end this section with some other characterizations of étale modules, which are proved in [Sw] (see also [Kn], Chap. IV, §3). Notice that it is not difficult to prove these properties using the last proposition and suitable localizations. If R is a domain, R_0 denotes its field of quotients.

Proposition 2.5. *A quadratic module (L, q) is étale if and only if each of the following equivalent conditions holds:*

(*a*) *For every prime ideal $\mathfrak{p} \in \mathit{Spec}\, R$ the module $(L/\mathfrak{p}L \otimes_{R/\mathfrak{p}} (R/\mathfrak{p})_0, q_\mathfrak{p} \otimes (R/\mathfrak{p})_0)$ is étale.*

(*b*) *There is an étale covering of the prime spectrum $\mathit{Spec}\, R$ by $\mathit{Spec}\, R_i$ such that the modules $L_i = L \otimes_R R_i$ are free with a basis in which $q_i = q \otimes R_i$ has form* (2.1).

3. ÉTALE LATTICES AND AZUMAYA ALGEBRAS

We recall some facts concerning relations between the étale quadratic spaces and Azumaya algebras. We refer to [Sw] and [Kn] for details.

Let (L, q) be any quadratic module over the ring R. By the Clifford algebra of (L, q) over R, we mean the algebra $C(L, q) = \mathcal{T}(L)/I$, where $\mathcal{T}(L) = \bigoplus_{k=0}^{\infty} L^{\otimes k}$ denotes the tensor algebra of L and I its ideal generated by the elements $x \otimes x - q(x)$, where $x \in L$. The even Clifford algebra $C_0(L, q)$ is the subalgebra of $C(L, q)$ generated by the image of the even part $\mathcal{T}_0(L) = \bigoplus_{k=0}^{\infty} L^{\otimes 2k}$ of the tensor algebra $\mathcal{T}(L)$ in $C(L, q)$.

Let $\Lambda \supseteq R$ be an R-algebra finitely generated and projective as an R-module. The algebra Λ is called separable over R if for each maximal ideal $\mathfrak{m}$ in R, the algebra $\Lambda/\mathfrak{m}\Lambda$ is a (finite) product of simple algebras whose centers are separable field extensions of the field $R/\mathfrak{m}$. If Λ is commutative, then it is called étale over R. If Λ is a central R-algebra and for each maximal ideal $\mathfrak{m}$ in R, $\Lambda/\mathfrak{m}\Lambda$ is central simple over $R/\mathfrak{m}$, then Λ is called Azumaya R-algebra.

The fundamental relation between étale R-lattices (L, q) and separable R-algebras is given by the following result (for a proof see [Sw], Prop. 4.2 and [Kn], Chap. IV, (4.2.1)):

Theorem 3.1. *Let (L, q) be an étale R-lattice.*

(*a*) *If L has odd rank, then $C_0(L, q)$ is an Azumaya R-algebra.*

(*b*) *If L has even rank, then $C(L, q)$ is an Azumaya R-algebra and $C_0(L, q)$ is an Azumaya algebra over its center, which is an étale quadratic algebra over R.*

4. QUADRATIC SPACES

Let R be the ring of integers in a global field K, and let (V, q) be a quadratic space over K. We say that (V, q) is an *étale* (*quadratic*) *space* if there is an R-lattice L on V such that $(L, q_{|L})$ is étale. As we noted before, our main purpose in the paper is to classify and construct all étale quadratic spaces over quadratic number fields K.

In the notations from Section 2, for each quadratic space (V, q) over K, the discriminant is usually defined as the class of the determinant $(-1)^m \det[b_q(e_i, e_j)]$ in K^*/K^{*2}, where $e_1, \dots, e_n$ is a basis for V over K. We need a slight modification of this definition. Let $R_* = R \setminus \{0\}$ denote the multiplicative semi-group of nonzero elements in R and consider the quotient $\mathrm{Q}(R) = R_*/R_*^2$, which evidently is a group. Let $e_1, \dots, e_n$ be a basis of V such that $\det[b_q(e_i, e_j)] \in R$ (of course, such bases exist). Now define the discriminant, $d(V, q)$ or $d(q)$, of (V, q) as the class in $\mathrm{Q}(R)$ of the discriminant of the basis $e_1, \dots, e_n$ (see Section 2). It is easy to check that this class does not depend on the choice of a basis for V with $\det[b_q(e_i, e_j)] \in R$. Notice that the discriminant of a lattice $(L, q_{|L})$ on (V, q) was defined as an R-ideal, while the discriminant of (V, q) is in principle given by an element of R_*.

Let us recall a few facts concerning classification of quadratic spaces (V, q) over algebraic number fields K. We refer mainly to [O'M] for the proofs. Let $\Omega(K) = \Omega^\infty(K) \cup \Omega^f(K)$ denote the set of all primes of K (infinite $\Omega^\infty(K)$ and finite $\Omega^f(K)$). Its elements (the equivalence classes of norms on K) will be denoted by $\mathfrak{p}$.

Every quadratic space (V, q) over K can be characterized by the following set of invariants: the dimension $\dim_K V$, the discriminant $d(V, q)$ (defined above), the Clifford invariants $c_\mathfrak{p}(V, q) = c_\mathfrak{p}(q)$ for each $\mathfrak{p} \in \Omega^f(K)$ and the signature $s_\mathfrak{p}(V, q) = s_\mathfrak{p}(q)$ for each real $\mathfrak{p} \in \Omega^\infty(K)$. (We follow the terminology in [Sc], p.333.) Recall that $c_\mathfrak{p}(V, q)$ (for any $\mathfrak{p}$ in $\Omega(K)$) is defined by the class in the Brauer group $\mathrm{Br}(K_\mathfrak{p})$ of the Clifford algebra $C(V_\mathfrak{p}, q_\mathfrak{p})$ when $\dim_K V$ is even, and the class of the even Clifford algebra $C_0(V_\mathfrak{p}, q_\mathfrak{p})$ when $\dim_K V$ is odd. More precisely, $c_\mathfrak{p}(V, q)$ equals 1, when this class is trivial, and -1, when it is non-trivial. We denote by $s_\mathfrak{p}^+(q)$ the positive and by $s_\mathfrak{p}^-(q)$ the negative index of $(V_\mathfrak{p}, q_\mathfrak{p})$ for real infinite $\mathfrak{p}$, that is, the number of positive and negative coefficients of $q_\mathfrak{p}$ in any orthogonal basis for $V_\mathfrak{p}$. Thus $s_\mathfrak{p}^+(q) + s_\mathfrak{p}^-(q) = n$ and $s_\mathfrak{p}^+(q) - s_\mathfrak{p}^-(q) = s_\mathfrak{p}(q)$.

We recall the following result, which will be used for computations of the Clifford invariants of quadratic spaces over arbitrary fields K of characteristic different from 2. As usual, if $a, b \in K^*$, we denote by $(a, b)_K$ the Clifford algebra of (K^2, q), where $q(x_1, x_2) = ax_1^2 + bx_2^2$. This is the quaternion algebra over K generated (as an algebra) by i, j such that $i^2 = a$, $j^2 = b$ and $ji = -ij$. The algebra $(a, b)_K$ is either isomorphic to the algebra of 2×2-matrices over K or a division algebra. We define the Hilbert symbol

$$(a, b) = -1 \text{ if } (a, b)_K \text{ is a division algebra, and } (a, b) = 1 \text{ if not.}$$

We denote by $\langle\, a \,\rangle$ the quadratic space (K, q) with $q(x) = ax^2$ for $x \in K$ and write $(V, q) = \langle\, a_1 \,\rangle \perp \cdots \perp \langle\, a_n \,\rangle$ to denote the orthogonal sum of $\langle\, a_i \,\rangle$, where $a_i \in K^*$. With these notations, we have (see [Sc], pp. 81 and 333):

Theorem 4.1. *Let $(V, q) = \langle\, a_1 \,\rangle \perp \cdots \perp \langle\, a_n \,\rangle$, where $a_i \in K^*$ and K is a local field of characteristic different from 2. Then*

$c(V, q) = \prod_{i<j}(a_i, a_j)$ *if* $n \equiv 1, 2 \pmod 8$,

$c(V, q) = \prod_{i<j}(a_i, a_j)(-1, -\prod_i a_i)$ *if* $n \equiv 3, 4 \pmod 8$,

$c(V, q) = \prod_{i<j}(a_i, a_j)(-1, -1)$ *if* $n \equiv 5, 6 \pmod 8$,

$c(V, q) = \prod_{i<j}(a_i, a_j)(-1, \prod_i a_i)$ *if* $n \equiv 7, 8 \pmod 8$.

Corollary 4.2. *Let $(V, q) = \langle\, a_1 \,\rangle \perp \cdots \perp \langle\, a_n \,\rangle$, where $a_i \in \{\pm 1\}$, be a quadratic space over the real numbers. Then $d(q) = (-1)^{\frac{s(q)(s(q)-1)}{2}}$ and the Clifford invariant satisfies*

$$c(q) = 1 \Leftrightarrow \begin{cases} 2 \nmid n \ \textit{and}\ s(q) \equiv d(q) \pmod 8, \\ 2 \mid n \ \textit{and}\ s(q) \equiv 1 - d(q) \pmod 8, \end{cases}$$

and

$$c(q) = -1 \Leftrightarrow \begin{cases} 2 \nmid n \ \textit{and}\ s(q) \equiv 4 + d(q) \pmod 8, \\ 2 \mid n \ \textit{and}\ s(q) \equiv 5 - d(q) \pmod 8. \end{cases}$$

Proof. With $n = 2m$ or $n = 2m+1$ we clearly have, $d(q) = (-1)^m \prod_{i=1}^n a_i = (-1)^{\frac{s(q)(s(q)-1)}{2}}$, since

$$m + s^-(q) \equiv \frac{n(n-1)}{2} + \frac{n - s(q)}{2} = \frac{n^2 - s(q)}{2} \equiv \frac{s(q)^2 - s(q)}{2} \pmod 2.$$

Moreover, Theorem 4.1 gives

$c(q) = (-1)^{\frac{s^-(q)(s^-(q)-1)}{2}} = 1$ iff $s^-(q) \equiv 0, 1 \pmod 4$, when $n \equiv 1, 2 \pmod 8$,

$c(q) = (-1)^{\frac{s^-(q)(s^-(q)-1)}{2} + (s^-(q)+1)} = 1$ iff $s^-(q) \equiv 1, 2 \pmod 4$, when $n \equiv 3, 4 \pmod 8$,

$c(q) = (-1)^{\frac{s^-(q)(s^-(q)-1)}{2} + 1} = 1$ iff $s^-(q) \equiv 2, 3 \pmod 4$, when $n \equiv 5, 6 \pmod 8$,

$c(q) = (-1)^{\frac{s^-(q)(s^-(q)-1)}{2} + s^-(q)} = 1$ iff $s^-(q) \equiv 0, 3 \pmod 4$, when $n \equiv 7, 8 \pmod 8$.

Using the equality $s(q) = n - 2s^-(q)$ in each case above, we easily get the residue of $s(q)$ modulo 8 from the residues of n modulo 8 and $s^-(q)$ modulo 4 and vice versa. Comparing this residue with $d(q)$ gives the claim. Similar arguments give the congruences when $c(q) = -1$. $\square$

We shall repeatedly use the following theorems (see [O'M], 66:4, 63:20 and 72:1):

Theorem 4.3. *Let (V, q) and (V', q') be two non-degenerate quadratic spaces over a local or global field K. Then the two spaces are isometric over K (notation: $(V, q) \cong (V', q')$) if and only if $\dim_K V = \dim_K V'$, $d(q) = d(q')$, $c_\mathfrak{p}(q) = c_\mathfrak{p}(q')$ for each $\mathfrak{p} \in \Omega^f(K)$ and $s_\mathfrak{p}(q) = s_\mathfrak{p}(q')$ for each real $\mathfrak{p} \in \Omega^\infty(K)$.*

Theorem 4.4. *Let $(V'_\mathfrak{p}, q'_\mathfrak{p})$ be a given non-degenerate quadratic space for each prime $\mathfrak{p} \in \Omega(K)$. Then there exists a quadratic space (V, q) over K such that $(V_\mathfrak{p}, q_\mathfrak{p}) \cong (V'_\mathfrak{p}, q'_\mathfrak{p})$ for each $\mathfrak{p} \in \Omega(K)$ if and only if the following three conditions are satisfied:*

(a) there is a $d \in K$ such that $d(q'_\mathfrak{p}) = d$ for each $\mathfrak{p} \in \Omega(K)$,

(b) $c(q'_\mathfrak{p}) = 1$ for almost all $\mathfrak{p} \in \Omega(K)$,

(c) $\prod_{\mathfrak{p} \in \Omega(K)} c(q'_\mathfrak{p}) = 1$.

5. QUADRATIC DEFECTS

In this section, we refer to [O'M], §63A for details and proofs. If $x \in K$ and K is a discrete local field with ring of integers R, then the quadratic defect of x is the intersection of all the ideals bR, where $x = a^2 + b$ for some $a \in K$. We denote the quadratic defect of x by $\mathfrak{d}(x)$ and note that x is a square in K if and only if its quadratic defect is (0). If π is any generator of the maximal ideal $\mathfrak{m}$ in R and the quadratic defect of a non-square x is $\mathfrak{m}^r$, then we shall also say that x, or the class xR^{*2}, has quadratic defect π^r. Similarly, we say that squares in K have quadratic defect 0.

We note some important facts concerning the quadratic defect, which are proved in [O'M]. If 2 is a unit in R (non-dyadic case), then the quadratic defects of the units in R are (0) (squares) and R (non-squares). If 2 is not a unit in R (dyadic case) and $(2) = (\pi^t)$, then (see [O'M], 63:2) the quadratic defects of the units in R are the ideals in the chain:

$$(0) \subset (\pi^{2t}) \subset (\pi^{2t-1}) \subset (\pi^{2t-3}) \subset \cdots \subset (\pi^3) \subset (\pi).$$

We shall repeatedly use the following result (see [O'M], 63:5 and 63:3):

Theorem 5.1. *Let K be a discrete local field with the integers R whose maximal ideal $\mathfrak{m}$ has a generator π. Then:*

(*a*) *If $\varepsilon = \eta^2 + \pi^r\delta$, where $\eta, \delta \in R^*$ and $1 \le r \le 2t-1$ is odd, then $\mathfrak{d}(\varepsilon) = \pi^r R$. In any of these cases, $K[\sqrt{\varepsilon}] = K(\sqrt{\varepsilon})$ is a ramified field extension of K, that is, the integral closure S of R in this field is not separable* (*i.e. not étale*) *over R.*

(*b*) *If $\varepsilon = \eta^2 + 4\alpha$, where $\eta \in R^*$ and $\alpha \in R$, then $\mathfrak{d}(\varepsilon) = 4R$ when α is a unit and* (0) *when α is not a unit. Moreover, in the first case $K[\sqrt{\varepsilon}] = K(\sqrt{\varepsilon})$ is an unramified field extension of K, and in the second one, $K[\sqrt{\varepsilon}] = K \times K$. Hence in both cases the integral closure S of R in $K[\sqrt{\varepsilon}]$ is a quadratic* (*free*) *unramified* (*i.e. étale*) *R-algebra.*

6. LOCAL CASE

Let R be a complete discrete valuation ring and K its quotient field. In this section, we give a description of all étale lattices (L, q) over R. As we know, L is free, so let $L = Re_1 + \cdots + Re_n$, where $e_1, \ldots, e_n$ is a basis of L over R. Let $V = L \otimes_R K$. We shall also use q to denote the natural extension $q \otimes K$ of the quadratic structure from L to V.

Theorem 6.1. *Let R be a complete discrete valuation ring and K its quotient field.*

(*a*) *If (L, q) is an étale lattice over R, then its Clifford invariant $c(L, q) = 1$ and the discriminant $d(V, q) \in R^*/R^{*2}$ has defect* (0) *or $4R$ if the rank of L over R is even.*

(*b*) *For every odd $n \ge 1$ and every $\varepsilon \in R^*/R^{*2}$ there exists an étale R-lattice (L, q) such that $d(V, q) = \varepsilon$. For every even $n \ge 2$ and every $\varepsilon \in R^*/R^{*2}$ of quadratic defect* (0) *or $4R$ there exists an étale R-lattice (L, q) such that $d(V, q) = \varepsilon$.*

(*c*) *Two étale R-lattices (L, q) and (L', q') of the same R-rank are R-isometric if and only if their discriminants are equal. Moreover, (L, q) and (L', q') are R-isometric if and only if $(L \otimes_R K, q \otimes K)$ and $(L' \otimes_R K, q' \otimes K)$ are K-isometric.*

Proof. We start with the cases $n = 1$ and $n = 2$.

If $n = 1$, then $(L = Re_1, q)$ and $q(e_1) = \varepsilon \in R^*$, so the discriminant $d(q)$ is an element of R^*/R^{*2}. Conversely, notice that to each element $\varepsilon \in R^*/R^{*2}$ corresponds an étale R-lattice $(L = Re, q)$ with $q(e) = \varepsilon$ and that L is unique on the quadratic K-space $V = L \otimes_R K$. Of course, $c(L, q) = 1$ and the statement (c) is evident.

If $n = 2$ and $(L = Re_1 + Re_2, q)$ is étale, then according to Theorem 3.1, the algebra $C_0(L, q) = R + Re_1e_2$ is separable and quadratic over R. Let $q(e_i) = a_i$ and $b(e_1, e_2) = b$.

If $x = e_1e_2$, then $x^2 = bx - a_1a_2$, so $b^2 - 4a_1a_2 = \varepsilon \in R^*$ is a unit of quadratic defect (0) or $4R$ (see [Hh], Chap. 3 A and Theorem 5.1). In the first case, $C_0(L, q) \cong R \times R$ and in the second $C_0(L, q)$ is the integral closure of R in the only unramified quadratic field extension of K (see Theorem 5.1). Let $(V = L \otimes_R K, q)$ be the quadratic étale space corresponding to (L, q). Notice that the Clifford invariant of (V, q) is trivial since $C(V, q)$ is a quaternion algebra over K which contains the quaternion order $C(L, q) = R + Re_1 + Re_2 + Re_1e_2$ having discriminant ε^2 (an easy computation). Thus $C(V, q) \cong M_2(K)$, the algebra of two by two matrices over K.

Conversely, let (L, q) be an R-lattice of rank 2 such that the discriminant $d(V, q)$ is a unit of quadratic defect (0) or $4R$. In the first case, $(V = L \otimes K, q)$ is isometric to the hyperbolic plane $(H = Ke_1 + Ke_2, q_0)$, where $q_0(x_1e_1 + x_2e_2) = x_1x_2$. Hence (L, q) is an R-maximal lattice on (V, q), which is R-isometric to the étale lattice $(L_0 = Re_1 + Re_2, q_0)$ (see [O'M], 91:2).

If the discriminant $d(V, q)$ is a unit of quadratic defect $4R$, then $(V = L \otimes K, q)$ is isometric to the quadratic space (F, Nr), where F is the unique quadratic unramified extension of K and Nr is the norm from F to K (see [O'M], 63:3). In fact, both spaces (V, q) and (F, Nr) have the same dimension, the same discriminant and the same Clifford invariant, since the Clifford algebras $C(V, q)$ and $C(F, \mathrm{Nr})$ both contain maximal orders whose discriminants are units in R: $C(L, q)$ and $C(S, \mathrm{Nr})$, where S denotes the ring of the integers in F. Thus both Clifford algebras are unramified and as such, isomorphic to the matrix algebra $M_2(K)$. Hence the quadratic spaces (V, q) and (F, Nr) are isometric over K. Since the lattices (L, q) and (S, Nr) are R-maximal on the corresponding quadratic spaces, they are isometric as R-lattices according to [O'M], 91:2. (S, Nr) is an étale lattice. Hence, we have the following result:

Proposition 6.2. (L, q) *is a binary étale quadratic module over a complete discrete valuation ring* R *if and only if* $d(L, q) = R$ *and (any representative of) the discriminant* $d(L \otimes_R K, q)$ *is a unit of quadratic defect* (0) *or* $4R$. *In the first case,* $(L, q) \cong (R \times R, q)$, *where* $q((x_1, x_2)) = x_1x_2$, *and in the second,* $(L, q) \cong (R[\sqrt{\varepsilon}], \mathrm{Nr})$, *where* ε *is any unit in* R *of the quadratic defect* $4R$ *and* Nr *is the norm mapping from* $K(\sqrt{\varepsilon})$ *to* K.

Now let $n \geq 3$ and assume that the statements (a) – (c) are true for étale lattices of ranks less than n. Let (L, q) be an étale R-lattice of rank n over R. Then $L = L_1 \perp \cdots \perp L_t$, where the R-lattices L_i have ranks 1 or 2 and are R-étale (see [O'M], §91C). If n is even, then L is an orthogonal sum of an étale lattice of rank 2 (say, L_1) and an étale lattice L' of even rank. Using the case $n = 2$ and the inductive assumption, we get that the discriminant of (L, q) is a unit of quadratic defect (0) or $4R$ and the Clifford invariant is trivial (see [Kn], Chap. IV, §2 and §3). If n is odd, then L is an orthogonal sum of an étale lattice of rank 1 (say, L_1) and an étale lattice L' of even rank. Similarly, we get that the Clifford invariant $c(L, q) = 1$. This proves (a).

In order to prove (b), notice that if n is odd, then (L, q), where $L = Re_1 + \cdots + Re_n$ and $q(x_1e_1 + \cdots + x_ne_n) = \varepsilon x_1^2 + x_2x_3 + \cdots + x_{n-1}x_n$ is an étale lattice of rank n such that the discriminant $d(L \otimes_R K, q) = \varepsilon$. If n is even and (L_1, q) is an étale lattice of rank 2 such that the discriminant $d(L_1 \otimes_R K, q_1) = \varepsilon$, where ε is a unit of quadratic defect (0) or $4R$, then $L = L_1 + Re_3 + \cdots + Re_n$, with $q(e + x_3e_3 + \cdots + x_ne_n) = q_1(e) + x_3x_4 + \cdots + x_{n-1}x_n$, where $e \in L_1$, is an étale lattice of rank n with the same discriminant as L_1.

As regards (c), notice that an étale R-lattice is R-maximal on the quadratic space ($V = L \otimes_R K, q \otimes K$). Since all R-maximal lattices on V are isometric (see [O'M], 91:2), it follows that if the spaces $(L \otimes_R K, q \otimes K)$ and $(L' \otimes_R K, q' \otimes K)$ are K-isometric, then the corresponding lattices (L, q) and (L', q') are R-isometric. According to general theory of quadratic spaces over local fields (see [O'M], 63:20), the dimension, the discriminant and the Clifford invariant are the three invariants uniquely defining the K-isometry class of a regular quadratic space. This proves (c), since the triviality of the Clifford invariant makes the equality of the dimensions (R-ranks of the lattices) and the discriminants equivalent to the K-isometry of the spaces, and consequently, the maximal R-lattices on them. □

7. QUADRATIC AND DISCRIMINANT GROUPS

Let R be any Dedekind domain with quotient field K. If $d \in R$ is the discriminant of an étale quadratic space (V, q) over K and the dimension of V over K is odd, then according to Theorem 6.1, d is locally a unit modulo squares, that is, for every maximal ideal $\mathfrak{m}$ in R there exists a unit $\varepsilon_{\mathfrak{m}} \in R_{\mathfrak{m}}^*$ and an element $r_{\mathfrak{m}} \in R_{\mathfrak{m}} \setminus \{0\}$ such that $d = \varepsilon_{\mathfrak{m}} r_{\mathfrak{m}}^2$. If the dimension of V is even, then according to Theorem 6.1, such a representation holds with $\varepsilon_{\mathfrak{m}}$ having a quadratic defect (0) or $4R_{\mathfrak{m}}$.

In the study of étale lattices and quadratic spaces in the global case (which is our purpose), these properties create necessity to consider the groups $\mathrm{Dis}(R)$ and $\mathrm{Qu}(R)$. We use to some extent established notations, since both groups appear in other connections (see the comments below).

Definition 7.1. Let $R_* = R \setminus \{0\}$ denote the multiplicative semi-group of nonzero elements in R and (as in Section 4), $\mathrm{Q}(R) = R_*/R_*^2$. We define $\mathrm{Dis}(R)$ as the subgroup of $\mathrm{Q}(R)$ whose elements are the classes of $r \in R_*$ such that for every maximal ideal $\mathfrak{m}$ in R there exists a unit $\varepsilon_{\mathfrak{m}} \in R_{\mathfrak{m}}^*$ and an element $r_{\mathfrak{m}} \in R_{\mathfrak{m}}$ such that $r = \varepsilon_{\mathfrak{m}} r_{\mathfrak{m}}^2$. Finally, $\mathrm{Qu}(R)$ is the subgroup of $\mathrm{Dis}(R)$ whose elements are the classes of $r \in R_*$ for which the quadratic defect of $\varepsilon_{\mathfrak{m}} \in R_{\mathfrak{m}}^*$ in the representations above equals (0) or $4R_{\mathfrak{m}}$.

Remark 7.2. The group $\mathrm{Dis}(R)$ is usually called *the discriminant group of* R (see e.g. [Hh], p.53, [Knu], p.124 and also [C-T], p.5). The group $\mathrm{Qu}(R)$ is sometimes called *the quadratic group of* R (see [Hh], p.175). Essentially this name would be more suitable for the group $\mathrm{Q}(R)$. It is known (see [Hh], (12.4)) that for Dedekind rings R,

$$\mathrm{Dis}(R) = R^*/R^{*^2} \times \mathcal{C}_2(R),$$

where $\mathcal{C}_2(R)$ denotes the subgroup consisting of the elements of order less than or equal to 2 in the class group $\mathcal{C}(R)$ of R. If R is the ring of integers in a global field and $\mathcal{C}^+(R)$ is its narrow ideal class group (that is, the group of all fractional R-ideals modulo the principal R-ideals having a totally positive generator), then (see [Hh], (14.11)):

$$\mathrm{Qu}(R) = \mathcal{C}_2^+(R),$$

the subgroup consisting of the elements of order less than or equal to 2 in $\mathcal{C}^+(R)$. We give a more detailed description of $\mathrm{Dis}(R)$ and $\mathrm{Qu}(R)$ in the case of quadratic fields in Section 9.

Let us note the following important consequence of the discussion at the beginning of this section and the definitions of the groups $\mathrm{Dis}(R)$ and $\mathrm{Qu}(R)$. Part (b) follows directly from Theorem 6.1 (a) and Theorem 4.4:

Proposition 7.3. *Let (V, q) be an étale quadratic space over a global field K.*

(a) If $\dim_K V$ is odd, then $d(V,q) \in \mathrm{Dis}(R)$, and if $\dim_K V$ is even, then $d(V,q) \in \mathrm{Qu}(R)$.

(b) $c_{\mathfrak{m}}(V,q) = 1$ for each finite $\mathfrak{m}$ and $\prod_{\mathfrak{m}} c_{\mathfrak{m}}(V,q) = 1$, where the product is over all real infinite $\mathfrak{m}$.

Observe that for the infinite primes $\mathfrak{m}$, $(n, s_{\mathfrak{m}})$ uniquely determine the isometry classes of the spaces $(V_{\mathfrak{m}}, q_{\mathfrak{m}})$ and consequently, the invariants d (in $K_{\mathfrak{m}}^*/K_{\mathfrak{m}}^{*^2}$) and $c_{\mathfrak{m}}$ (see Corollary 4.2).

8. ÉTALE LATTICES OVER THE INTEGERS

Our aim in this section is to classify all étale quadratic spaces over the rational numbers. First we need some notations. Let A denote the lattice $(\mathbb{Z}, q)$, where $q(x) = x^2$ for $x \in \mathbb{Z}$, H the lattice $(\mathbb{Z}^2, q)$, where $q(x_1, x_2) = x_1x_2$ for $x_1, x_2 \in \mathbb{Z}$, E_7 the lattice $(\mathbb{Z}^7, q)$, where $q(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_5x_6 - x_4x_7$ and E_8 the lattice $(\mathbb{Z}^8, q)$, where $q(\mathbf{x}) = \sum_{1\le i\le 8} x_i^2 + \sum_{1\le i<j\le 8} x_ix_j - x_2(x_1 + x_3)$. To simplify notations, we write $L_1 + L_2$ or $L_1 - L_2$ to denote the orthogonal sum of lattices L_1 and L_2, where the minus sign means that we replace the corresponding quadratic mapping q by $-q$. We denote by $k \cdot L$ the orthogonal sum of k copies of a lattice L. With these notations, we have the following description of étale quadratic spaces over the rational integers:

Theorem 8.1. *Let n, s be integers such that $n \ge 1$, $|s| \le n$, $s \equiv n \pmod 2$, and let $d \in \mathbb{Z}$.*

(a) If n is odd, then there exists an étale quadratic space (V, q) of dimension n, discriminant d and signature s over $\mathbb{Q}$ if and only if $d \in \mathrm{Dis}(\mathbb{Z}) = \{\pm 1\}$ and $s \equiv d \pmod 8$. Moreover, if (n, d, s) satisfies these conditions, then the corresponding space (V, q) is isometric to $V = \mathbb{Q} \otimes \pm(a \cdot H + b \cdot E_8 + dA)$, where $a = \frac{n-|s|-1+d}{2}, b = \frac{|s|-d}{8}$ with the exception of $s = \pm n$, when $V = \mathbb{Q} \otimes \pm(\frac{n-7}{8} \cdot E_8 + E_7)$.

(b) If n is even, then there exists an étale quadratic space (V, q) of dimension n, discriminant d and signature s over $\mathbb{Q}$ if and only if $d \in \mathrm{Qu}(\mathbb{Z}) = \{1\}$ and $s \equiv 0 \pmod 8$. Moreover, if (n, d, s) satisfies these conditions, then the corresponding space (V, q) is isometric to $V = \mathbb{Q} \otimes \pm(a \cdot H + b \cdot E_8)$, where $a = \frac{n-|s|}{2}, b = \frac{|s|}{8}$.

The following result is well known (see [Ch], Corollar zu Satz 2, [CS], Chap. 15, 10.3):

Theorem 8.2. *The definite étale quadratic spaces (V, q) only exist in dimensions n such that $n \equiv 0, \pm 1 \pmod 8$ (thus for $1 < \dim V < 7$ any étale quadratic space (V, q) is indefinite).*

Before we prove Theorem 8.1, let us note the following simple fact:

Lemma 8.3. $\mathrm{Dis}(\mathbb{Z}) = \{\pm 1\}$ *and* $\mathrm{Qu}(\mathbb{Z}) = \{1\}$.

Proof. The first equality is clear. We have $-1 = 1 - 2$, which shows that the quadratic defect of -1 is equal to (2) (see Section 5). This proves the second equality. $\square$

Proof of Theorem 8.1. Let (V, q) be an étale quadratic space of dimension n over $\mathbb{Q}$ and denote by ∞ the infinite prime on $\mathbb{Q}$. By Theorems 4.4 and 6.1, we have $c_\infty(V, q) = 1$.

If n is even, then since $d(q) \in \mathrm{Qu}(\mathbb{Z})$, we have $d(q) = 1$. Thus Corollary 4.2, implies that $s = s(q) \equiv 0 \pmod 8$.

If n is odd, then $d(q) \in \mathrm{Dis}(\mathbb{Z})$, so according to Lemma 8.3, we have two possible discriminants $d(q) = \pm 1$ and Corollary 4.2 gives $s = s(q) \equiv d(q) \pmod 8$.

Thus the necessary conditions for the existence of an étale quadratic space (V, q) of dimension n over $\mathbb{Q}$ are: n even, $d(q) = 1$ and $s(q) \equiv 0 \pmod 8$, or n odd, $d(q) = \pm 1$, $s(q) \equiv d(q) \pmod 8$. It follows from Theorem 4.4 that these conditions are sufficient. However, we want to give an explicit construction of the corresponding spaces (V, q).

Let (n, d, s) be given where n is even, $d = 1$ and $s \equiv 0 \pmod 8$. Let $a = \frac{n-s}{2}$ and $b = \frac{s}{8}$, where $s \geq 0$. Since $n - s \geq 0$, we have $a \geq 0$ and the space $V = \mathbb{Q} \otimes L$, where $L = aH + bE_8$ is étale and has the required invariants (n, d, s). If $s < 0$, we define V using $-L$ instead of L.

Assume now that (n, d, s) is given with n odd, $d = \pm 1$ and $s \equiv d \pmod 8$. If $s > 0$, we choose $a = \frac{n-s-1+d}{2}$ and $b = \frac{s-d}{8}$. Then $L = aH + bE_8 + dA$ is a lattice such that $\mathbb{Q} \otimes L$ has the invariants (n, d, s) unless $a < 0$ (notice that $b \geq 0$). We have $a < 0$ if and only if $n = s$ and $d = -1$. In this case, $L = \frac{n-7}{8} \cdot E_8 + E_7$ has the required invariants. If $s < 0$, we apply the same construction to $(n, -d, -s)$ and change the sign of the lattice. $\square$

Observe that $E_7 + H \cong E_8 - A$ over the integers. In fact, both sides span quadratic spaces over $\mathbb{Q}$, which have the same invariants $(9, -1, 7)$. Hence the spaces are isometric, and since the lattices are maximal on the corresponding spaces, they are isometric over the integers thanks to the strong approximation property of the indefinite quadratic spaces (see [O'M], 102:10 and 104:10).

9. QUADRATIC NUMBER FIELDS

In this section, we describe the étale quadratic spaces (V, q) over quadratic number fields $K = \mathbb{Q}(\sqrt{\Delta})$, where $\Delta \neq 1$ is a square-free integer. If $\Delta < 0$, then the situation is rather simple, since there are no infinite real primes. If $\Delta > 0$, we have two infinite real primes, which we denote by ∞_1 and ∞_2 (that is, $\Omega^\infty(K) = \{\infty_1, \infty_2\}$). Let R denote the ring of all integers in K. The groups $\mathrm{Dis}(R)$ and $\mathrm{Qu}(R)$ have the following description:

Theorem 9.1. *If R is the ring of all integers in the quadratic field $K = \mathbb{Q}(\sqrt{\Delta})$, where Δ is square-free, then*

(a) $\mathrm{Qu}(R)$ is the subgroup of $\mathrm{Q}(R)$ generated by 1 and the classes of all numbers $p^* = (-1)^{\frac{p-1}{2}} p$, where p are all odd prime numbers dividing Δ.

(b) $\mathrm{Dis}(R)$ is the subgroup of $\mathrm{Q}(R)$ generated by the classes of all units in R, $\mathrm{Qu}(R)$ and, if Δ is not divisible by a prime congruent to 3 modulo 4, by the class of the number $s + \sqrt{\Delta}$, where $\Delta = r^2 + s^2$ for suitable integers r, s with r odd.

Proof. (a) By Theorem 5.1 (b), the class of $r \in R_*$ belongs to $\mathrm{Qu}(R)$ if and only if $R_\mathfrak{m}[\sqrt{\varepsilon_\mathfrak{m}}]$ is an unramified (that is, étale) quadratic extension of $R_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ in R, where $r = \varepsilon_\mathfrak{m} r_\mathfrak{m}^2$ according to the definition of $\mathrm{Qu}(R)$. Thus $r \in R_*$ belongs to $\mathrm{Qu}(R)$ if and only if $K[\sqrt{r}]$ is an unramified quadratic algebra over K. The classes of all such r are 1 (corresponding to $K \times K$) and, according to [Hs], p.48, the products of the classes of p^* (see also [GL], p.108).

(b) If the class of $r \in R_*$ belongs to $\mathrm{Dis}(R)$, then for every maximal ideal $\mathfrak{m}$ in R there exists a unit $\varepsilon_\mathfrak{m} \in R_\mathfrak{m}^*$ and an element $r_\mathfrak{m} \in R_\mathfrak{m}$ such that $r = \varepsilon_\mathfrak{m} r_\mathfrak{m}^2$. It is clear that $(r_\mathfrak{m})$ is an idelé for which the corresponding ideal in R has order at most 2 in the class group of R. Moreover, it is easy to see that $\mathrm{Dis}(R)$ consists exactly of the classes in this group of the generators of squares of such ideals. Now the group of elements of order at most 2 in the class group of R (the "ambiguous" ideals) is well known and its description can be found in [Hi], Satz 107 and [CH], §18. The generators of the squares of such ideals are obtained by multiplying the generators described above in (b) by the classes of units in R. $\square$

In order to give a description of all étale quadratic spaces over quadratic fields, we need some notations. As before, we denote by H the lattice (R^2, q), where $q(x_1, x_2) = x_1x_2$ for $x_1, x_2 \in R$. According to Proposition 7.3, the elements of $\mathrm{Dis}(R)$ correspond to the isometry classes of the étale quadratic lattices of rank one over R. Let $d \in R$ represent an element of $\mathrm{Dis}(R)$. We denote by $L_d^{(1)}$ (the isometry class of) the R-lattice $(\mathfrak{I}_d, \frac{1}{d}q)$, where $\mathfrak{I}_d$ is an R-ideal whose norm equals (d) and $q(x) = x^2$. Finally, if $d \in R$ represents an element in $\mathrm{Qu}(R)$, then $L_d^{(2)}$ will stand for (the isometry class of) the R-lattice (S_d, Nr), where S_d denotes the R-integers in the algebra $K[\sqrt{d}\,]$ and Nr is the norm map from $K[\sqrt{d}\,]$ to K (we follow the convention that $K[\sqrt{1}\,]$ denotes $K \times K$). Notice that $L_1^{(2)} = H$. With these notations, we have:

Theorem 9.2. *Let (V, q) be an étale quadratic space of dimension n over $K = \mathbb{Q}(\sqrt{\Delta})$, where $\Delta < 0$.*

(a) If n is odd, then there exists an element $d \in \mathrm{Dis}(R)$ such that $V = K \otimes (\frac{n-1}{2} \cdot H + L_d^{(1)})$.

(b) If n is even, then there exists an element $d \in \mathrm{Qu}(R)$ such that $V = K \otimes (\frac{n-2}{2} \cdot H + L_d^{(2)})$.

Moreover, every étale quadratic space (V, q) over K is isometric to a space given by (a) or (b) and its isometry class is uniquely determined by (n, d).

Proof. Since the set $\Omega^\infty(K)$ of infinite primes is empty, Theorems 4.3 and 6.1 say that every pair (n, d) uniquely defines the isometry class of (V, q). If $n = \dim_K V$ is odd, then we easily check that the lattice $L = \frac{n-1}{2} \cdot H + L_d^{(1)}$, is étale, has rank n and the discriminant d. Similarly, if $\dim_K V$ is even, then the lattice $\frac{n-2}{2} \cdot H + L_d^{(2)}$ is étale, has rank n and the discriminant d (recall that $L_1^{(2)} = H$). $\square$

Assume now that K is a real quadratic field and let (V, q) be an étale space over K. The pair $s_\infty = s_\infty(V, q) = (s_{\infty_1}(V, q), s_{\infty_2}(V, q))$ will be called the signature of (V, q). We shall also write $c_\infty = c_\infty(V, q) = (c_{\infty_1}(V, q), c_{\infty_2}(V, q))$.

In order to describe all étale spaces over K, we need some more types of "standard lattices" $L_d^{(3)}$ for $d \in \mathrm{Dis}(R)$, and $L_d^{(4)}$, $L_{d,\chi}^{(4)}$ for $d \in \mathrm{Qu}(R)$, $d > 0$, which we define next. First, we have to recall some facts concerning quaternion algebras.

Let $\mathbb{H}_*$ be the quaternion algebra over K, which is ramified only at ∞_1 and ∞_2. It follows from the general theory of central simple algebras over global fields that such an algebra exists and is unique up to isomorphism over K (see [V], Thm. 3.1, p.74). Thus $\mathbb{H}_*$ is a division algebra over K, which is isomorphic to the algebra of the (2×2)-matrices over all completions $K_\mathfrak{m}$ for all $\mathfrak{m} \in \Omega^f(K)$ and to the Hamiltonian quaternions over the two infinite completions $K_{\infty_i} = \mathbb{R}$ for $i = 1, 2$. Let Γ be a maximal order in $\mathbb{H}_*$ over R and let $L_1^{(4)} = \Gamma$ with the quadratic structure given by the reduced norm Nr from $\mathbb{H}_*$ to K restricted to Γ (of course, $L_1^{(4)}$ is not defined uniquely). Let $L_1^{(3)} = \{x \in \Gamma | \operatorname{Tr}(x) = 0\}$, where Tr is the reduced trace from $\mathbb{H}_*$ to K. It is clear that $(L_1^{(4)}, \mathrm{Nr})$ has rank 4, discriminant R and the signature $s_\infty = (4, 4)$, while $L_1^{(3)}$ has rank 3, discriminant R and the signature $s_\infty = (3, 3)$. Let $L_d^{(3)} = L_d^{(1)} \otimes L_1^{(3)}$ for $d \in \mathrm{Dis}(R)$. Notice that $K \otimes_R L_d^{(3)}$ has discriminant $-d$. Its signature depends on the signs of d in both infinite embeddings of K.

Finally, we want to define $L_d^{(4)}$ for $d \in \mathrm{Qu}(R)$ such that the corresponding spaces (V, q) are positive definite at both infinite places of K:

Lemma 9.3. *Let (V, q) be an étale space of dimension 4, which is positive definite at both infinite places of K. Then $d(V, q) > 0$. Conversely, if $d \in \mathrm{Qu}(R)$ and $d > 0$, then there is an étale space (V, q) for which $d(V, q) = d$. Moreover, if $d(V, q) = 1$, then (V, q) is similar to $(\mathbb{H}_*, \mathrm{Nr})$.*

Proof. Since (V, q) has signature 4 at both infinite places of K, Corollary 4.2 says that the discriminant $d(V, q) > 0$ and the Clifford invariant $c_\infty(V, q) = -1$.

We already know that if $d = 1$, then $(\mathbb{H}_*, \mathrm{Nr})$ is an étale quaternary space with discriminant 1 and the signature $(4, 4)$.

Let $d \in \mathrm{Qu}(R)$, $d \neq 1$ and $d > 0$. Denote $M = K(\sqrt{d})$. Let $x \mapsto x^*$ be the natural involution of $\mathbb{H}_*$ over K and $x \mapsto \bar{x}$ the non-trivial automorphism of M over K. Let $A = \mathbb{H}_* \otimes_K M$ and let $x \mapsto x^*$ and $x \mapsto \bar{x}$ be the natural extensions to A of the two previously defined maps on $\mathbb{H}_*$ and M. Define $W_d = \{x \in A \,|\, x^* = \bar{x}\}$. According to [P], Prop. 2, p.5 (see also [Sh], Thm. 7.4c, p.55), (W_d, Nr), where $\mathrm{Nr}(x) = xx^*$, is a quaternary space over K whose discriminant equals d, the Clifford invariants are defined by the class of $\mathbb{H}_*$ in the Brauer groups corresponding to the completions of K and the signature is $(4, 4)$.

We want to show that (W_d, Nr) is étale constructing a suitable étale lattice $L_d^{(4)}$ on W_d. Let Γ be any maximal R-order on $\mathbb{H}_*$ and S the maximal R-order in M. Define

$$L_d^{(4)} = W_d \cap (\Gamma \otimes_R S). \tag{9.1}$$

It is not difficult to check that $L_d^{(4)} \subset W_d$ is a lattice on (W_d, Nr) whose discriminant equals R. In fact, for each maximal ideal $\mathfrak{m}$ of R, the completion $(\mathbb{H}_*)_\mathfrak{m}$ is the algebra $M_2(K_\mathfrak{m})$ of the (2×2)–matrices over the completion $K_\mathfrak{m}$. We can choose a basis of $(\mathbb{H}_*)_\mathfrak{m}$ over $K_\mathfrak{m}$ such that $\Gamma_\mathfrak{m} = M_2(R_\mathfrak{m})$. Since S is unramified over K, we have two possibilities for $S_\mathfrak{m}$ – either $S_\mathfrak{m} = R_\mathfrak{m} \times R_\mathfrak{m}$ or $S_\mathfrak{m} = R_\mathfrak{m} + R_\mathfrak{m}\omega$, where $\omega = \frac{1+\sqrt{\varepsilon}}{2}$, and ε is a unit in $R_\mathfrak{m}$ of quadratic defect $4R_\mathfrak{m}$.

Let e_{ij}, $i, j = 1, 2$, be the matrices with 1 at $ij-$position and 0 elsewhere. We have a basis $e_{ij} \otimes (1, 0)$, $e_{ij} \otimes (0, 1)$ for $\Gamma_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} S_{\mathfrak{m}}$ if $S_{\mathfrak{m}} = R_{\mathfrak{m}} \times R_{\mathfrak{m}}$, and $e_{ij} \otimes 1$, $e_{ij} \otimes \omega$ when $S_{\mathfrak{m}} = R_{\mathfrak{m}} + R_{\mathfrak{m}}\omega$. It is easy to find a basis of $(L_d^{(4)})_{\mathfrak{m}}$ noting that $x \mapsto x^*$ is the adjoint matrix for $x \in \Gamma_{\mathfrak{m}}$ and $x \mapsto \bar{x}$ is the nontrivial automorphism of $S_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$ ($(a, b) \mapsto (b, a)$ in the first case, and $\sqrt{\varepsilon} \mapsto -\sqrt{\varepsilon}$ in the second). Then it is a matter of a simple computation to check that the discriminant of $(L_d^{(4)})_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$. □

Remark 9.4. (a) It is possible to construct an étale space (V, q) with $d(V, q) = 1$ using a similar procedure as for the spaces with $d(V, q) = d$ and $d \neq 1$ starting from $M = K \times K$ and replacing A by $\mathbb{H}_* \times \mathbb{H}_*$, where $\mathbb{H}_*$ is the quaternion algebra ramified only at the two infinite places of K.

(b) If $\mathbb{H}_* = K + Ki + Kj + Kij$, where $1, i, j, ij$ is a basis of $\mathbb{H}_*$ over K such that $i^2, j^2 \in K$ and $ji = -ij$ (see Section 4), then a trivial computation shows that $W_d = K + Ki\sqrt{d} + Kj\sqrt{d} + Kij\sqrt{d}$, where $M = K(\sqrt{d})$ with d as in the proof of Lemma 9.3.

The lattices $L_{d,\chi}^{(4)}$ will be constructed in a similar way on étale spaces (V, q) with $d(V, q) = d$ for $d \in \mathrm{Qu}(R)$, $d > 0$, having signature $(4, -4)$.

In order to prove the existence of such spaces, we need one more auxiliary result:

Lemma 9.5. *Let K be a real quadratic field and $d \in \mathrm{Qu}(R)$, $d > 0$. Then there exists $\alpha \in K$ such that $\mathrm{Nr}(\alpha) < 0$ and the Hilbert symbol $(\alpha, d) = 1$.*

Proof. If $d = 1$, then the statement is clear for any $\alpha \in K$ whose norm is negative. Assume that $d \neq 1$ and let $M = K(\sqrt{d})$. Choose $\beta \in M$ such that $\mathrm{Nr}_{M/\mathbb{Q}}(\beta) < 0$ and define $\alpha = \mathrm{Nr}_{M/K}(\beta)$. Then $\mathrm{Nr}_{K/\mathbb{Q}}(\alpha) < 0$ and if $\beta = x + y\sqrt{d}$, where $x, y \in K$, then $\mathrm{Nr}_{M/K}(\beta) = x^2 - dy^2 = \alpha$, so the Hilbert symbol $(\alpha, d) = 1$. □

The last Lemma implies the following result:

Proposition 9.6. *Let (V, q) be an étale space of signature (s_1, s_2) and even dimension over a real quadratic field K and let $\alpha \in K$ be such that $(\alpha, d(V, q)) = 1$ and $\mathrm{Nr}_{K/\mathbb{Q}}(\alpha) < 0$. Assume that the $\alpha_{\infty_1} > 0$. Then $(V, \alpha q)$ is an étale space of signature $(s_1, -s_2)$.*

Proof. Of course, $d(V, q) = d(V, \alpha q)$ and $(s_{\infty_1}(V, \alpha q), s_{\infty_2}(V, \alpha q)) = (s_1, -s_2)$. According to Theorem 4.1, $c(V, \alpha q) = (\alpha, d(V, q))c(V, q)$, so according to Lemma 9.5, both spaces $(V, \alpha q)$ and (V, q) have the same Clifford invariants. Thus $(V, \alpha q)$ is an étale space. In fact, if L is an étale lattice on (V, q), then using the local-global principle (see e.g.[V], p.83), it is possible to construct an étale lattice L' on $(V, \alpha q)$ such that $L'_{\mathfrak{p}} = L_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$ of R for which α is a unit in $R_{\mathfrak{p}}$, and $L'_{\mathfrak{p}}$ equal to an étale lattice on $(V_{\mathfrak{p}}, \alpha q_{\mathfrak{p}}) \cong (V_{\mathfrak{p}}, q_{\mathfrak{p}})$ for the remaining finite number of prime ideals $\mathfrak{p}$ of R. □

Now we define $L_{d,\chi}^{(4)}$ as any integral maximal lattice on the étale space $(W_d, \alpha\mathrm{Nr})$ with $d(W_d, \alpha\mathrm{Nr}) = d(W_d, \mathrm{Nr}) = d$, where W_d is defined in Lemma 9.3 and α satisfies the condition in Lemma 9.5.

In order to formulate the final results for real quadratic fields, we need some more notations. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, where x_i, y_i are integers, then we write $x \equiv y \pmod{m}$ when $x_i \equiv y_i \pmod{m}$ for $i = 1, 2$. Let us note that the discriminant $d = d(V, q)$ defines a pair $\mathrm{sgn}(d) = (\mathrm{sgn}(d_1), \mathrm{sgn}(d_2))$, where d_i is the embedding (of the class) of $d = d(V, q)$ into the corresponding copy of the real numbers and $\mathrm{sgn}(d_i) = \pm 1$ its sign.

According to Corollary 4.2, $\mathrm{sgn}(d)$ puts some necessary congruence conditions on $s_\infty(V,q)$. If $\mathrm{sgn}(d_1)=\mathrm{sgn}(d_2)$, then sometimes we write $\mathrm{sgn}(d)$ meaning the common value of both coordinates. This will be clear from the context. Now we have:

Theorem 9.7. *Let $(n,d,s=(s_1,s_2))$ be such that n is even, $d\in\mathrm{Qu}(R)$ and $\mid s_i\mid\le n$, $s\equiv(n,n) \pmod 2$.*

(a) An étale space (V,q) of dimension n with discriminant d and signature s exists if and only if $s_1\equiv s_2 \pmod 8$ and $s\equiv(1,1)-sgn(d) \pmod 4$.

(b) If (n,d,s) satisfies the conditions in (a) and $s_1\ge|s_2|$, then the corresponding space (V,q) is isometric to $V=K\otimes L(s,d)$, where $L(s,d)=a\cdot H+b\cdot L_1^{(4)}+c\cdot L_{1,\chi}^{(4)}+L_d^{(2)}$ for $a=\frac{n-s_1-1-\mathrm{sgn}(d)}{2}$, $b=\frac{s_1+s_2-2+2\mathrm{sgn}(d)}{8}$ and $c=\frac{s_1-s_2}{8}$ with the exception of $s_1=n$ and $\mathrm{sgn}(d)=1$, when $L(s,d)=b\cdot L_1^{(4)}+c\cdot L_{1,\chi}^{(4)}+L_d^{(4)}$ for $b=\frac{s_1+s_2-8}{8}$ and $c=\frac{s_1-s_2}{8}$ except when $s_1=-s_2=n$ and $\mathrm{sgn}(d)=1$, when $L(s,d)=c\cdot L_{1,\chi}^{(4)}+L_{d,\chi}^{(4)}$ for $c=\frac{n-4}{4}$. Moreover, we have $c_\infty(V,q)=(1,1)$ iff $2\mid b+c$ in the first case, $2\nmid b+c$ in the second and $2\nmid c$ in the last case.

Proof. Let (V,q) be given. We first note that by Theorem 9.1, the sign of the discriminant d is the same in both infinite places. Corollary 4.2 then implies that $s_1\equiv s_2\equiv 1-\mathrm{sgn}(d) \pmod 8$ if $c_\infty(V,q)=(1,1)$ and $s_1\equiv s_2\equiv 5-\mathrm{sgn}(d) \pmod 8$ if $c_\infty(V,q)=(-1,-1)$. This proves the necessity of the conditions in (a).

Conversely assume that s and d satisfy the conditions in (a) and $s_1\ge|s_2|$. It's now straightforward to check that the constants a, b and c are non-negative integers and that the form $L(s,d)$ has the desired rank, signature and discriminant. Hence the given conditions are also sufficient.

Finally $c_\infty(V,q)=(1,1)$ iff $s_1\equiv 1-\mathrm{sgn}(d) \pmod 8$. Observe that

$$b+c=\frac{s_1-1+\mathrm{sgn}(d)}{4}$$

in the first case, $b+c=\frac{s_1}{4}-1$ in the second and $c=\frac{s_1}{4}-1$ in the last case. This implies the final statement of the theorem. □

Remark 9.8. Notice that the assumption $s_1\ge|s_2|$ in Theorem 9.7 as well as in Theorem 9.9 has only a technical character. In fact, if we replace $L_d^{(4)}$ and $L_{d,\chi}^{(4)}$ by $-L_d^{(4)}$ and $-L_{d,\chi}^{(4)}$, then we shift s_1 and s_2, while the change of the sign of the whole lattice $L(s,d)$ replaces (s_1,s_2) by $(-s_1,-s_2)$.

Theorem 9.9. *Let $(n,d,s=(s_1,s_2))$ be such that n is odd, $d\in\mathrm{Dis}(R)$ and $\mid s_i\mid\le n$, $s\equiv(n,n) \pmod 2$.*

(a) An étale space (V,q) of dimension n with discriminant d and signature s exists if and only if $8\mid s_1+s_2$ or $8\mid s_1-s_2$ and $s\equiv\mathrm{sgn}(d) \pmod 8$ or $s\equiv(4,4)+\mathrm{sgn}(d) \pmod 8$.

(b) If (n,d,s) satisfies the conditions in (a) and $s_1\ge|s_2|$, then the corresponding space (V,q) is isometric to one of the following:

(b_1) If $8\mid s_1-s_2$ and $s_1\equiv r \pmod 4$, $r=1$ or -1, then $V=K\otimes(a\cdot H+b\cdot L_1^{(4)}+c\cdot L_{1,\chi}^{(4)}+rL_d^{(2-r)})$, where $a=\frac{n-s_1}{2}$, $b=\frac{s_1+s_2+2r-4}{8}$, $c=\frac{s_1-s_2}{8}$. Moreover, we have $c_\infty(V,q)=(1,1)$ iff $2\mid b+c$, when $r=1$ and $2\nmid b+c$, when $r=-1$.

(b_2) *If* $8 \mid s_1+s_2$ *and* $s_1 \equiv r \pmod 4$, $r = 1$ *or* -1, *then* $V = K\otimes(a\cdot H+b\cdot L_1^{(4)}+c\cdot L_{1,\chi}^{(4)}+rL_d^{(2-r)})$, *where* $a = \frac{n-s_1}{2}$, $b = \frac{s_1+s_2}{8}$, $c = \frac{s_1-s_2+2r-4}{8}$. *Moreover, we have* $c_\infty(V, q) = (1, 1)$ *iff* $2 \mid b + c$, *when* $r = 1$ *and* $2 \nmid b + c$, *when* $r = -1$.

Proof. Corollary 4.2 implies that $s_1 \equiv s_2 \pmod 8$ if the signs of the discriminant d are the same in both embeddings and $s_1 \equiv -s_2 \pmod 8$ if not. Again it is straightforward to check that the explicit forms have the correct rank, signature and discriminant and hence that the conditions are sufficient. Finally from $b+c = \frac{s_1+r-2}{4}$ and $c_\infty(V, q) = (1, 1)$ iff $s_1 \equiv \pm 1 \pmod 8$, we deduce the last claims in (b_1) and (b_2). □

Remark 9.10. Theorems 9.2, 9.7 and 9.9 in their parts concerning the invariants of étale lattices (parts (a) in the last two Theorems) can be deduced from Satz 1 in [Ch]. However, the statement in Satz 3 in [Ch] concerning the signatures of the lattices whose discriminant is a unit in R (in the notations of the present paper) is not correct, but the problem is simply in translation of the conditions on the Clifford invariants $c_\infty(V, q)$ to the conditions on the signatures $s_\infty(V, q)$ (the modulo 8 relations between s_1 and s_2 in [Ch] became too week).

10. CONSTRUCTIONS OF LATTICES $L_d^{(4)}$

As in the previous section, we assume that $K = \mathbb{Q}(\sqrt{\Delta})$ with ring of integers R. We also assume that $\Delta > 0$. As before, we denote by $\mathbb{H}_*$ the quaternion K-algebra, which is only ramified at the two infinities of K

In this section, we describe how to effectively construct the lattices $L_d^{(4)}$, where $d \in \mathrm{Qu}(R)$ and $d > 0$. Notice that Lemma 9.5 and Proposition 9.6 give a possibility to effectively construct the lattices $L_{d,\chi}^{(4)}$ when $L_d^{(4)}$ are known. The lattices $L_d^{(3)}$, where $d \in \mathrm{Dis}(R)$, are explicitly given by $L_1^{(4)}$.

Let, as before, Γ be any maximal R-order in $\mathbb{H}_*$ and S the maximal R-order in $M = K[\sqrt{d}]$ (as usual, $M = K \times K$, when $d = 1$). Define $\Gamma_0 = \{x \in \Gamma | \operatorname{Tr}(x) = 0\}$ and $S_0 = \{x \in S | \operatorname{Tr}(x) = 0\}$, where the trace functions are from $\mathbb{H}_*$ and from S to R, respectively. Notice that $\Gamma = R\iota + \Gamma_0$ and $S = R\iota + S_0$, where ι is an element (in Γ and S, respectively) whose trace equals 1, are direct sums of R-modules, since the trace functions $\operatorname{Tr} : \Gamma \to R$ and $\operatorname{Tr} : S \to R$ are surjective with the corresponding kernels Γ_0 and S_0. In Corollary 10.6, we prove that

$$L_d^{(4)} = R\iota + \Gamma_0 \otimes_R S_0, \tag{10.1}$$

where $\iota \in \Gamma\otimes_R S$ is a suitable element. Thus, it is sufficient to give an effective construction of Γ_0 and S_0 for maximal R-orders Γ and S in the quaternion algebra $\mathbb{H}_*$ and in $M = K[\sqrt{d}]$. This will be done below.

First of all notice that the integers in M, and consequently the integers whose trace equals 0, can be easily described, since $M = \mathbb{Q}(\sqrt{\Delta}, \sqrt{d})$ is a biquadratic extension of $\mathbb{Q}$:

Proposition 10.1. *The generators of* S_0 *as* R*-module are:*

(a) $\sqrt{d}, \sqrt{\frac{\Delta}{d}}$, *when* $\Delta \equiv 2$ *or* $3 \pmod 4$, (*b*) $\sqrt{d}, \frac{1+\sqrt{\Delta}}{2}\sqrt{\frac{\Delta}{d}}$, *when* $\Delta \equiv 1 \pmod 4$.

Proof. Notice that by Theorem 9.1, $d \in \mathrm{Qu}(R)$ and $d > 0$ imply that $d \equiv 1 \pmod 4$. Then the given description of S_0 follows immediately from e.g. [Mr], p.51–2. $\square$

The construction method of Γ relies on "resolution of quaternion orders" and can be also applied for similar constructions of étale lattices of ranks 3 and 4 over rings of integers in arbitrary number fields or even over arbitrary Dedekind rings. It is described in detail in the Appendix. The method gives a possibility to construct the minimal over-orders of any Gorenstein quaternion order Λ (see the definition in the Appendix), which, in a finite number of steps, leads to the maximal orders containing Λ. The point is that it is easy to choose a Gorenstein R-order in the algebra $\mathbb{H}_*$ over K, but it is not evident how to describe the maximal orders containing it. According to (10.2), we get explicit generators of $L_d^{(4)}$ as tensor products of generators of S_0 given in Proposition 10.1 and the generators of Γ_0 constructed below.

The first step is to construct the algebra $\mathbb{H}_*$ over K. If $(a,b)_K$ is a quaternion algebra over K and $a, b \in R$, then $(a,b)_R$ will denote the R-order $\Lambda = R \oplus Ri \oplus Rj \oplus Rij$, where $i^2 = a, j^2 = b$ and $ji = -ij$. Its discriminant is $4ab$, so the only finite ramification points of $(a,b)_K$ are amongst the primes in R dividing $4ab$.

Proposition 10.2. *Let* $\mathbb{H}_*$ *be the quaternion algebra over* $K = \mathbb{Q}(\sqrt{\Delta})$ *ramified only at the two infinite places. Then*

$$\mathbb{H}_* \cong (-1, -p)_K,$$

where p *is any rational prime satisfying* $p \equiv 3 \pmod 4$ *and* $\left(\frac{\Delta}{p}\right) = -1$.

Proof. The algebra $(a,b)_K$ with $a, b \in \mathbb{Q}$ and $a, b < 0$ is obviously ramified at both infinite places of K. Hence we only need to check that the algebra given above is not ramified at any finite primes of K. We do this by showing that there is at most one finite prime at which it could be ramified. This combined with the fact that the number of ramification points is always even, proves the result.

The algebra $(-1,-p)_{\mathbb{Q}}$ over $\mathbb{Q}$ is obviously ramified at infinity. Since $\left(\frac{-1}{p}\right) = -1$, it is ramified at p. Of course, $(-1,-p)_{\mathbb{Q}}$ is unramified at all finite odd primes different from p, so it must be unramified at 2. Now $(-1,-p)_K$ is ramified at the two infinities, so it can not be ramified at (p), which remains prime in R depending on the assumption $\left(\frac{\Delta}{p}\right) = -1$. $\square$

Remark 10.3. If $\Delta \not\equiv 1 \pmod 8$, then a similar argument shows that $\mathbb{H}_* \cong (-1,-1)_K$, which gives the familiar Hamilton quaternions better suited for computations. Unfortunately, this choice does not work when $\Delta \equiv 1 \pmod 8$.

Next we will use results and ideas from the Appendix to find maximal orders in $\mathbb{H}_*$. We remark that we only construct one maximal order and that in general there will be several non-isomorphic ones.

In order to follow the details of the proofs, one might want to first consult the Appendix, but the final results here can be checked without the formal constructions presented there. First we need a description of a maximal order in the algebra $\mathbb{H}_* \cong (-1,-p)_K$.

Lemma 10.4. *Let $p \equiv 3 \pmod 4$ be a prime and $q : \mathbb{Z}^3 \to \mathbb{Z}$ the quadratic map $q(x_1,x_2,x_3) = mx_1^2 + x_2^2 + x_1x_2 + x_3^2$, where $p = 4m-1$. A maximal order in $(-1,-p)_{\mathbb{Q}} \cong C_0(\mathbb{Q}^3, q)$ is $C_0(\mathbb{Z}^3,q) = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij$, where $i^2 = -1, j^2 = -m, ij + ji = 1$.*

Proof. It is a matter of easy computation to check that $d(\mathbb{Z}^3, q) = p$, $(-1,-p)_{\mathbb{Q}} \cong C_0(\mathbb{Q}^3, q)$ and that the even Clifford algebra of $(\mathbb{Z}^3, q)$ has the given basis $1, i, j, ij$. Notice that $(i+2j)^2 = -p$. □

Using the algorithm described in the Appendix, we can now construct a maximal order in the algebra $\mathbb{H}_* = (-1,-p)_K$:

Theorem 10.5. *Let $\mathbb{H}_*$ be the quaternion algebra over $K = \mathbb{Q}(\sqrt{\Delta})$, $\Delta > 0$, ramified only at the two infinite places. Let p be any rational prime satisfying $p \equiv 3 \pmod 4$, $p = 4m-1$, $\left(\frac{\Delta}{p}\right) = -1$. Then there exists $a \in R$ such that $a^2 = -1 \pmod p$ and*

$$\Gamma = R + Rj + Rij + R\alpha,$$

where $i^2 = -1, j^2 = -m, ij + ji = 1$ and

$$\alpha = \frac{a}{p} + \frac{1}{p}i + \frac{2}{p}j - \frac{2a}{p}ij,$$

is a maximal order in $\mathbb{H}_* = (-1,-p)_K$ containing $\Lambda = R + Ri + Rj + Rij$. Moreover, Γ_0 is a free R-module with a basis $j, 1-2ij, \alpha$.

Proof. Let $\Lambda = C_0(\mathbb{Z}^3, q) \otimes_{\mathbb{Z}} R = R + Ri + Rj + Rij$, where the notations are as in Lemma 10.4. Λ is a hereditary order in $\mathbb{H}_* = (-1,-p)_K$ whose reduced discriminant is (p). Thus its Eichler invariant (see the Appendix) equals 1 and we can apply Proposition A.4. Denoting

$$f_1 = \frac{2m}{p}i + \frac{1}{p}j, \quad f_2 = \frac{1}{p}i + \frac{2}{p}j, \quad f_3 = \frac{1}{p} - \frac{2}{p}ij,$$

we have $L = \Lambda^{\#} \cap \mathbb{H}_{*0} = Rf_1 + Rf_2 + Rf_3$. The quadratic structure on L is given by $p\mathrm{Nr}$ and we easily find the corresponding quadratic form $p\mathrm{Nr}(x_1f_1 + x_2f_2 + x_3f_3) = mx_1^2 + x_2^2 + x_1x_2 + x_3^2$. $R/(p)$ is a quadratic field extension of $\mathbb{Z}/(p)$ in which there exist both $\sqrt{\Delta}$ and $\sqrt{-\Delta} \in \mathbb{Z}/(p)$ (since $\left(\frac{-\Delta}{p}\right) = 1$). Thus, there is $a \in R$ such that $a^2 = -1 \pmod p$. Noting that $4m = 1 \pmod p$, we get

$$mx_1^2 + x_2^2 + x_1x_2 + x_3^2 \equiv \left(\frac{1}{2}x_1 + x_2 + ax_3\right)\left(\frac{1}{2}x_1 + x_2 - ax_3\right) \pmod p.$$

According to Proposition A.4, we may choose $e_1 = 2f_1 - f_2 = i$ and $e_2 = af_2 - f_3 = -\frac{1}{p} + \frac{a}{p}i + \frac{2a}{p}j + \frac{2}{p}ij$ as two vectors in L whose images in L/pL generate the kernel of the linear map to $R/(p)$ defined by $\frac{1}{2}x_1 + x_2 + ax_3$. Denoting $\alpha = e_1^* e_2 = \frac{a}{p} + \frac{1}{p}i + \frac{2}{p}j - \frac{2a}{p}ij$, we get a maximal order containing Λ:

$$\Gamma = \Lambda + Re_1^*e_2 = R + Ri + Rj + Rij + R\alpha = R + Rj + Rij + R\alpha.$$

□

A lattice $L_d^{(4)}$ for $d \in \mathrm{Qu}(R)$, $d \neq 1$ can be now constructed in the following way:

Corollary 10.6. *With the notations in Theorem* 10.5*, let S be the maximal order in $M = K(\sqrt{d})$ and S_0 the set of elements of S whose trace equals* 0*. Then*

$$L_d^{(4)} = R\iota + \Gamma_0 \otimes_R S_0, \tag{10.2}$$

where $\iota = 1 \otimes \frac{1+\sqrt{d}}{2} - ij \otimes \sqrt{d}$. Moreover, $L_d^{(4)}$ is free if and only if S is free.

Proof. First recall that according to Theorem 9.1 (a), the condition $d \in \mathrm{Qu}(R)$ implies that $d \equiv 1 \pmod 4$. We have $\Gamma = Rij + \Gamma_0$ and $S = R\frac{1+\sqrt{d}}{2} + S_0$. We get immediately that $\iota \in W_d$ and $\mathrm{Tr}(\iota) = 1$. Now it is a matter of easy computation to show that $W_d \cap (\Gamma \otimes_R S) = R\iota + \Gamma_0 \otimes_R S_0$.

According to Proposition 10.1, $\mathfrak{q} = \sqrt{d}S_0$ is an ideal of R and $\mathfrak{q}^2 = (d)$. Thus the Steinitz invariant of $L_d^{(4)}$ is isomorphic to $\mathfrak{q}^3 = d\mathfrak{q}$. Hence $L_d^{(4)}$ is free if and only if $\mathfrak{q}$ is principal, which occurs exactly when S is free. □

Using the basis of Γ_0 constructed in Theorem 10.5 and the generators of S_0 given in Proposition 10.1, we get the lattices $L_d^{(4)}$ for $d \neq 1$ according to (10.2). Recall that $L_1^{(4)} = \Gamma$ and the quadratic structure is in all cases given by the norm Nr on $\mathbb{H}_*$.

Remark 10.7. In [Ma], Maass gives some necessary and sufficient conditions for the existence of totally positive, even dimensional free étale lattices over real quadratic integers. He also constructs such lattices, which in his case are given by concrete quadratic forms. The result of Theorem 9.7 (a) concerning the invariants of the even dimensional totally positive étale spaces imply of course the corresponding result in [Ma]. This is noted in [Ch] (see pp. 64–65). But Theorem 9.7 (b) also gives examples of étale quadratic forms when the dimension is even and the discriminant $d(V,q) \in \mathrm{Qu}(R)$ of the étale quadratic space (V,q) is (represented by) a unit in R. Such examples can be easily given in all cases, even when $\Delta \equiv 1 \pmod 8$, which could not be constructed by the methods used in [Ma]. In fact, we have the following result:

Proposition 10.8. *Let (V,q) be an étale quadratic space. A free étale lattice L on V exists if and only if the class of $d(V,q)$ in $\mathrm{Qu}(R)$, when $\dim_K V$ is even, and the class of $d(V,q)$ in $\mathrm{Dis}(R)$, when $\dim_K V$ is odd, is represented by a unit in R.*

Proof. It is clear that if there exists a free étale lattice L on V, then the corresponding element $d(V,q) \in \mathrm{Qu}(R)$, when $\dim_K V$ is even, and $d(V,q) \in \mathrm{Dis}(R)$, when $\dim_K V$ is odd, can be represented by a unit, since the determinant of a basis of L is a unit in R. Conversely, let (V,q) be an étale quadratic space such that $d(V,q)$ is represented by a unit $d = \varepsilon \in R$.

If the dimension of V is even, then by Theorem 3.1 (b), the ring of integers S in $M = K[\sqrt{\varepsilon}]$ is separable and, of course, free over R ($S = R \times R$ if ε represents the identity of $\mathrm{Qu}(R)$ and $S = R + R\frac{1+\sqrt{\varepsilon}}{2}$ otherwise). The lattices $L_d^{(r)}$ for $r = 2, 4$ are free – for $r = 2$ by the definition ($L_d^{(2)}$ is equal to S) and for $r = 4$ by Corollary 10.6. Thus the lattices constructed in Theorem 9.7 (b) are also free, since the lattice H is free by definition, and the lattices $L_1^{(4)} = \Gamma$, $L_{1,\chi}^{(4)}$ are free by Theorem 10.5 ($L_{1,\chi}^{(4)}$ is isomorphic to $L_1^{(4)} = \Gamma$ as $R-$module).

If the dimension of V is odd, then the lattices constructed in Theorem 9.9 (b) are free, since the lattices $L_d^{(1)}$ and $L_d^{(3)}$ (defined by means of Γ in Theorem 10.5) are free when d is (represented by) a unit in R. □

Thus the results of Theorems 9.7 (b), 9.9 (b) and the explicit constructions of this section give concrete quadratic forms when such forms exist. In [BB], there is a description of all separable extensions $S \supset R$ such that S is free over R (see Theorems 5–7 in [BB]). Thus combining the results of [BB] with the results of the present paper, it is possible to effectively construct all étale quadratic spaces over real quadratic integers covering free étale lattices.

We conclude this section with some examples of free lattices $L_d^{(4)}$. In general, there will be several different classes in the genus of $L_d^{(4)}$. We construct a particular lattice applying Theorem 10.5 and its Corollary 10.6. We adopt the notations of these two and also define $w \in R$ by $1 + a^2 = pw$.

Examples. (a) As noted above, $L_1^{(4)}$ is the maximal order Γ with the quadratic structure given by the norm form. A direct computation using the basis given in Theorem 10.5, gives the following form:

$$x_1^2 + x_1x_3 + mx_2^2 + x_2x_4 + mx_3^2 - ax_3x_4 + wx_4^2.$$

(b) Now assume $d \neq 1$. We want to give a description of all free lattices $L_d^{(4)}$. According to Corollary 10.6, $L_d^{(4)}$ is free if and only if the maximal order S in $M = K(\sqrt{d})$ is a free separable $R-$algebra. According to [BB], $M = K(\sqrt{\varepsilon})$, where ε is the fundamental unit in R (see Theorems 5–7 in [BB] and notice that ε must be positive, since M is a real field by Lemma 9.3). The equality $K(\sqrt{d}) = K(\sqrt{\varepsilon})$ implies that $d\varepsilon$ is a square in R. Denote $\sqrt{d\varepsilon} = r$. Moreover, $S = R + R\frac{1+\sqrt{\varepsilon}}{2}$, so $S_0 = R\sqrt{\varepsilon}$. Then using (10.2), we get that $L_d^{(4)}$ is given by

$$\frac{1+pd}{4}x_1^2 + rpx_1x_3 + rax_1x_4 + \varepsilon mx_2^2 + \varepsilon x_2x_4 + \varepsilon px_3^2 + 2\varepsilon ax_3x_4 + \varepsilon wx_4^2.$$

(Remember that $p \equiv 3 \pmod 4$ and $d \equiv 1 \pmod 4$.) The discriminant of this form is ε^3. Notice that the corresponding quadratic space $K \otimes_R L_d^{(4)}$ over K is unique up to isometry as d is uniquely determined by ε.

Using the results of [BB], it is possible to give explicit conditions characterizing the fields $K = \mathbb{Q}(\sqrt{\Delta})$ over which there exists a free étale lattice $L_d^{(4)}$ with $d > 1$. For $\Delta \not\equiv 1 \pmod 8$ such spaces were constructed in [Ma] by a different method. The above comments together with Theorems 5–7 of [BB] and Lemma 9.3 imply the following result:

Theorem 10.9. *Let $K = \mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field with Δ square-free and the fundamental unit $\varepsilon = u + v\omega$, where $\omega = \sqrt{\Delta}$ if $\Delta \equiv 2, 3 \pmod 4$ and $\omega = \frac{1+\sqrt{\Delta}}{2}$ when $\Delta \equiv 1 \pmod 4$. Then a free lattice $L_d^{(4)}$ with $d \neq 1$ exists if and only if $\mathrm{Nr}(\varepsilon) = 1$ and*

(a) $\Delta \equiv 3 \pmod 4$ and $2 \mid v$,

(b) $\Delta \equiv 2 \pmod 4$ and $(u, v) \equiv (1, 0)$ or $(3, 2) \pmod 4$,

(c) $\Delta \equiv 1 \pmod 4$ and $(u, v) \equiv (1, 0) \pmod 4$, or $\Delta \equiv 5 \pmod{16}$ and $(u, v) \equiv (1, 1)$ or $(2, 3) \pmod 4$, or $\Delta \equiv 13 \pmod{16}$ and $(u, v) \equiv (0, 3)$ or $(1, 3) \pmod 4$. □

The least $\Delta \equiv 1 \pmod 8$ when $L_d^{(4)}$ exists is $\Delta = 105$ (a case which is not covered by the methods in [Ma]). Here $\varepsilon = 37 + 8\omega$ in $K = \mathbb{Q}(\sqrt{105})$ and $K(\sqrt{\varepsilon}) = K(\sqrt{5})$, that is, $d = 5$. In this case, we may take $p = 11$, $a = 3\sqrt{105}$. Plugging this into the general formula above gives the form:

$$\begin{gathered}14x_1^2 + 11(10 + \sqrt{105})x_1x_3 + 15(21 + 2\sqrt{105})x_1x_4 + 3(41 + 4\sqrt{105})x_2^2 \\ +(41 + 4\sqrt{105})x_2x_4 + (451 + 44\sqrt{105})x_3^2 \\ +6(420 + 41\sqrt{105})x_3x_4 + 86(41 + 4\sqrt{105})x_4^2.\end{gathered}$$

For the values $\Delta < 100$, $L_d^{(4)}$ exists only for $\Delta = 30$ ($d = 5$), 39 ($d = 13$), 34 ($d = 17$), 55 ($d = 5$), 66 ($d = 3 \cdot 11$), 70, ($d = 5$), 95 ($d = 5$).

Appendix A. Resolution of Gorenstein orders

In this Appendix, we give an explicit method of constructing minimal over-orders of Gorenstein quaternion orders over Dedekind rings. The construction is used in Section 10 to find maximal orders in the quaternion algebra $\mathbb{H}_*$ over $\mathbb{Q}(\sqrt{\Delta})$ ($\Delta > 0$) ramified exactly at the two infinite primes. A particular case of the construction described here was considered in [E], p. 632.

A.1. **Introduction.** Let R be a Dedekind domain with quotient field K. We assume that $\operatorname{char} K = 0$. If R is a complete discrete valuation ring, then we denote by π a generator of the maximal ideal of R and by $\widehat{R}$ the residue field $R/(\pi)$. A quaternion algebra A over K is a central simple four dimensional K-algebra. There is a unique anti-involution $x \mapsto x^*$ on A, called the canonical involution, such that $\mathrm{Tr}(x) = x + x^* \in K$ and $\mathrm{Nr}(x) = xx^* \in K$ for all $x \in A$. The map $\mathrm{Tr} : A \to K$ is called the reduced trace of A, and $\mathrm{Nr} : A \to K$ the reduced norm.

We recall a few facts concerning orders in quaternion algebras over Dedekind rings. For proofs and references see [B]. An R-order Λ in A is a subring of A containing R, which is an R-lattice such that $K\Lambda = A$. If Λ is an order in A, define $\Lambda^\# = \{x \in A \mid \mathrm{Tr}(x\Lambda) \subseteq R\}$. The R-ideal $[\Lambda^\# : \Lambda]$ is a square, so $[\Lambda^\# : \Lambda] = d(\Lambda)^2$ for some R-ideal $d(\Lambda) \subseteq R$ which is called the reduced discriminant of Λ. An order Λ is called hereditary, if every (left) Λ-module is projective. An order is hereditary if and only if the discriminant $d(\Lambda)$ is square free. The order Λ is a Gorenstein order if $\Lambda^\#$ is projective as left (or equivalently right) Λ-module. The order Λ is a Bass order if every order containing Λ is a Gorenstein order.

From now on, we assume that R be a complete discrete valuation ring. Let $J(\Lambda)$ denote the Jacobson radical of the R-order Λ. If Λ is a Gorenstein order which is not Azumaya over R (see Section 3), then the Eichler invariant $e(\Lambda)$ is defined by

$$e(\Lambda) = \begin{cases} -1 & \text{if } \Lambda/J(\Lambda) \text{ is a quadratic field extension of } \widehat{R}, \\ 1 & \text{if } \Lambda/J(\Lambda) \cong \widehat{R} \times \widehat{R}, \\ 0 & \text{if } \Lambda/J(\Lambda) \cong \widehat{R}. \end{cases}$$

For every order Λ, we can in a natural way construct a ternary quadratic R-lattice (L, q) and an isomorphism $\varphi : C_0(L, q) \to \Lambda$, where $C_0(L, q)$ is the even Clifford algebra of (L, q) (see Section 3). Let $A_0 = \{x \in A \mid \mathrm{Tr}(x) = 0\}$ and let d_0 be a generator of $d(\Lambda)$. Define a ternary lattice $L = \Lambda^{\#} \cap A_0$ and a quadratic form $q : L \to R$ by $q(l) = d_0 \,\mathrm{Nr}(l)$. Define an R-linear mapping $L \otimes L \to A$, by $l_1 \otimes l_2 \mapsto d_0 l_1^* l_2$ for $l_1, l_2 \in L$. This map can be uniquely extended to a ring homomorphism $\varphi : \mathcal{T}_0(L) \to A$. It is clear that if $l \in L$, then $\varphi(l \otimes l - q(l)) = 0$. Hence φ factors through the Clifford algebra $C_0(L, q)$. In fact, we have:

Proposition A.1. *The map $\varphi : C_0(L, q) \to \Lambda$ is an isomorphism.*

We have, $d(C_0(L, q)) = d(q)$. Furthermore, the order $C_0(L, q)$ is Gorenstein if and only if the form q is primitive. Let $\widehat{L} = L/\pi L$ and consider the reduction $\widehat{q} : \widehat{L} \to \widehat{R}$ of the quadratic form q modulo (π).

Proposition A.2. *The reduction $\widehat{q}$ of q gives the following information about the order Λ:*

(*a*) *if* $\mathrm{rk}\,\widehat{q} = 3$, *then Λ is an Azumaya R-algebra,*

(*b*) *if* $\mathrm{rk}\,\widehat{q} = 2$ *and $\widehat{q}$ is irreducible, then* $e(\Lambda) = -1$,

(*c*) *if* $\mathrm{rk}\,\widehat{q} = 2$ *and $\widehat{q}$ is reducible, then* $e(\Lambda) = 1$,

(*d*) *if* $\mathrm{rk}\,\widehat{q} = 1$, *then* $e(\Lambda) = 0$,

(*e*) *if* $\widehat{q} = 0$, *then Λ is not Gorenstein.*

Proof. If $\widehat{q} = 0$, then q is not a primitive form, and hence $C_0(L, q)$ is not a Gorenstein order. Assume now that $\widehat{q} \neq 0$. Let M be the null space of $\widehat{q}$, so the quadratic form $\widehat{L}/M \to \widehat{R}$ is non-degenerate. We clearly have a surjective ring homomorphism $\Lambda \to C_0(\widehat{L}/M, \widehat{q})$. Since $C_0(\widehat{L}/M, \widehat{q})$ is a simple algebra, this gives a surjection

$$\Lambda/J(\Lambda) \to C_0(\widehat{L}/M, \widehat{q}). \tag{A.1}$$

Considering the possible isomorphism classes of the two rings $\Lambda/J(\Lambda)$ and $C_0(\widehat{L}/M, \widehat{q})$, the only possible case when the map (A.1) could fail to be an isomorphism is if $\Lambda/J(\Lambda) \cong \widehat{R} \times \widehat{R}$ and $C_0(\widehat{L}/M, \widehat{q}) \cong \widehat{R}$, i.e. if $e(\Lambda) = 1$ and $\mathrm{rk}(\widehat{q}) = 1$. But from Proposition 2.1 in [B], it follows that if $e(\Lambda) = 1$, then $\mathrm{rk}(\widehat{q}) = 2$. Hence we have that the map (A.1) is an isomorphism and the claim follows. □

A.2. **Resolution.** We now consider the problem of finding minimal over-orders of a given Bass order. A chain $\Lambda = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n$, where Λ_{i+1} is a minimal over-order of Λ_i for all i and Λ_n is maximal, is called a resolution of the order Λ. The reason behind this terminology, is that such chains are related to the problem of resolving the singularities of the corresponding Brauer-Severi scheme (see e.g. references in [B]).

The proof of the following result can be found in [B]:

Proposition A.3. *If $e(\Lambda) = 1$, then Λ is a Bass order. If $d(\Lambda) = (\pi^n)$, then $\Lambda \cong \left(\begin{smallmatrix} R & R \\ \pi^n R & R \end{smallmatrix}\right)$. There exists exactly two minimal over-orders Λ' of Λ. They satisfy $[\Lambda' : \Lambda] = (\pi)$ and $e(\Lambda') = 1$ if Λ' is not maximal.*

If $e(\Lambda) = -1$, then Λ is a Bass order. If Λ is not hereditary, then there is a unique minimal order Λ' containing Λ. It satisfies $[\Lambda' : \Lambda] = (\pi^2)$, and if Λ' is not maximal, then $e(\Lambda') = -1$.

If $e(\Lambda) = 0$, then Λ has a uniquely determined over-order Λ' with $[\Lambda' : \Lambda] = (\pi)$. If Λ is not hereditary and Λ' is Gorenstein, then $e(\Lambda') = 0$.

We now prove two results which show how the minimal over-orders of a Gorenstein order can be computed. We state the results locally for simplicity, but the point is that they can immediately be applied in the global case and hence give an algorithm to compute the maximal orders which contain a given order. See Section 10 for examples in the global case.

Proposition A.4. *Let Λ be an order with $e(\Lambda) = 0$ or $e(\Lambda) = 1$. Let $l : \widehat{L} \to \widehat{R}$ be a linear factor of the reducible quadratic form $\widehat{q}$. Take 2 generators of $\ker(l)$, and let e_1 and e_2 be liftings of these elements to L. Then*

$$\Lambda' = \Lambda + \frac{1}{\pi} d(\Lambda) e_1^* e_2$$

is an over-order of Λ with $[\Lambda' : \Lambda] = (\pi)$. This order only depends on the choice of l. If $e(\Lambda) = 1$ and $\widehat{q} = l_1 l_2$, then the two over-orders constructed using l_1 and l_2 respectively are different.

Proof. There is an element e_3 such that e_1, e_2, e_3 is a basis of L. The quadratic form q on L is given by $q(x) = d_0 \operatorname{Nr}(x)$. We have $q(ae_1 + be_2) \in (\pi)$ for all $a, b \in R$. Define $L' = \langle e_1, e_2, \pi e_3 \rangle$ (the R-module generated by $e_1, e_2, \pi e_3$), with quadratic form $q'(x) = \pi^{-1} d_0 \operatorname{Nr}(x)$. It is clear that q' is integral. The natural images, i.e. the images of the corresponding maps φ given by Proposition A.1, of the orders $C_0(L, q)$ and $C_0(L', q')$ are $\Lambda = \langle 1, d_0 e_1^* e_2, d_0 e_1^* e_3, d_0 e_2^* e_3 \rangle$ and $\Lambda' = \langle 1, \pi^{-1} d_0 e_1^* e_2, d_0 e_1^* e_3, d_0 e_2^* e_3 \rangle$ respectively. The claim follows.

Consider the case $e(\Lambda) = 1$ and assume that $\widehat{q} = l_1 l_2$. We can choose a basis e_1, e_2, e_3 for L such that $l_1(e_1) = l_1(e_2) = l_2(e_2) = l_2(e_3) = 0$. The orders we get are $\Lambda + \frac{1}{\pi} d(\Lambda) e_1^* e_2$ and $\Lambda + \frac{1}{\pi} d(\Lambda) e_2^* e_3$ respectively, and these are clearly different. □

Proposition A.5. *Let Λ be a non-hereditary order with $e(\Lambda) = -1$. If e_1 is a primitive element of L such that $q(e_1) \in (\pi)$, then*

$$\Lambda' = \Lambda + \frac{1}{\pi} d(\Lambda) e_1 L$$

is the unique over-order of Λ with $[\Lambda' : \Lambda] = (\pi^2)$.

Proof. Since e_1 is a primitive element of L, we can choose elements e_2 and e_3 such that e_1, e_2, e_3 is a R-basis of L. Let $b(x, y) = q(x + y) - q(x) - q(y)$ be the bilinear form on

L associated to q. Using that the form $\widehat{q}$ is irreducible of rank 2, it is straightforward to verify that the matrix $M = (b(e_i, e_j))$ must be of the form

$$M = \begin{pmatrix} 2q(e_1) & \pi a_{12} & \pi a_{13} \\ \pi a_{12} & 2a_{22} & a_{23} \\ \pi a_{13} & a_{23} & 2a_{33} \end{pmatrix},$$

where $a_{ij} \in R$. Since the rank of $\widehat{q}$ is 2, we get that $4a_{22}a_{33} - a_{23}^2 \in R^*$. Furthermore, we have that $2\pi^2 \mid \det(M)$, since Λ is not hereditary. Now since, $\det M \equiv 2q(e_1)(4a_{22}a_{33} - a_{23}^2) \pmod{2\pi^2}$, we get $\pi^2 \mid q(e_1)$. Hence, if we define the lattice $L' = \langle e_1, \pi e_2, \pi e_3 \rangle$ with quadratic form $q'(x) = \pi^{-2} d_0 \operatorname{Nr}(x)$, then q' is integral. The natural images of the orders $C_0(L, q)$ and $C_0(L', q')$ are $\Lambda = \langle 1, d_0 e_1^* e_2, d_0 e_1^* e_3, d_0 e_2^* e_3 \rangle$ and $\Lambda' = \langle 1, \pi^{-1} d_0 e_1^* e_2, \pi^{-1} d_0 e_1^* e_3, d_0 e_2^* e_3 \rangle$ respectively. Thus

$$\Lambda' = \Lambda + \frac{1}{\pi} d(\Lambda) e_1^* e_2 + \frac{1}{\pi} d(\Lambda) e_1^* e_3 = \Lambda + \frac{1}{\pi} d(\Lambda) e_1^* L = \Lambda + \frac{1}{\pi} d(\Lambda) e_1 L. \qquad \square$$

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