A spectral correspondence for Maaß waveforms[1]

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Abstract

Let $\mathcal{O}^1$ be a (cocompact) Fuchsian group, given as the group of units of norm one in a maximal order $\mathcal{O}$ in an indefinite quaternion division algebra over $\mathbb{Q}$. Using the (classical) Selberg trace formula, we show that the eigenvalues of the automorphic Laplacian for $\mathcal{O}^1$ and their multiplicities coincide with the eigenvalues and multiplicities of the Laplacian defined on the Maaß newforms for the Hecke congruence group $\Gamma_0(d)$, when d is the discriminant of the maximal order $\mathcal{O}$. We also show the equality of the traces of certain Hecke operators defined on the Laplace eigenspaces for $\mathcal{O}^1$ and the newforms of level d, respectively.

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1 Introduction

In several situations correspondences between spaces of automorphic forms for cocompact and non-cocompact Fuchsian groups are well-known. From a representation theoretic point of view, the most general such correspondence is covered by the Jacquet-Langlands correspondence [10]. In certain examples it is, however, also possible to study correspondences by classical means only. Our focus will be on non-holomorphic automorphic forms (Maaß waveforms) of weight zero and with trivial multiplier, a situation for which classical investigations can e.g. be found in [7].

To be precise, let $\mathcal{O}$ be an order in an indefinite rational quaternion division algebra so that its group $\mathcal{O}^1$ of units of norm one can be considered as a cocompact Fuchsian group. Maaß waveforms for $\mathcal{O}^1$, i.e. eigenfunctions of the automorphic Laplacian associated with $\mathcal{O}^1$, can then be lifted to Maaß cusp forms for the Hecke congruence group $\Gamma_0(d)$, where d is the (reduced) discriminant of the order $\mathcal{O}$ [2]. These lifts are realised as integral transforms with certain (classical) Siegel theta functions as kernels. As a consequence, theta-lifts preserve eigenvalues of the hyperbolic Laplacian. In [2], we also showed that in the case of so-called Eichler orders the eigenvalues of Hecke operators also remain unchanged. Furthermore in the language of the Atkin-Lehner formalism [1], theta-lifts of a Hecke basis for $L^2(\mathcal{O}^1\backslash\mathcal{H})$ were found to be newforms of a level dividing d. Counting theta-lifts and newforms suggested that in the case of maximal orders a Hecke basis for $L^2(\mathcal{O}^1\backslash\mathcal{H})$ is lifted to a Hecke basis for the newspace of level d.

In this paper, we continue to address the question to what extent theta-lifts are newforms of level d. We concentrate on maximal orders $\mathcal{O}$ in indefinite rational quaternion division algebras and exploit several versions of the (classical) Selberg trace formula [14]. Our first result in this direction is summarised in Theorem 5.1: The Laplace eigenvalues and their multiplicities for the cocompact group $\mathcal{O}^1$ and for the newforms of level d coincide. This, however, does not yet imply that theta-lifts provide isomorphisms between Laplace (and Hecke) eigenspaces in $L^2(\mathcal{O}^1\backslash\mathcal{H})$ and $L^2(\Gamma_0(d)\backslash\mathcal{H})$, respectively, since the theta-lifts are so far neither known to be injective, nor to map into the newspace. Our second result, as summarised in Theorem 5.2, states that the traces of the Hecke operators $\widetilde{T}_p$ defined on the Laplace eigenspace in $L^2(\mathcal{O}^1\backslash\mathcal{H})$ coincide with those of the Hecke operators T_p defined on the corresponding eigenspaces of newforms in $L^2(\Gamma_0(d)\backslash\mathcal{H})$ for the infinitely many primes p not dividing the level d. Since theta-lifts preserve Hecke eigenvalues, one thus concludes that all one dimensional Laplace eigenspaces in $L^2(\mathcal{O}^1\backslash\mathcal{H})$ lift to the newspace of level d.

The outline of this paper is as follows: In Section 2, we fix our notation and recall some basic facts about the Fuchsian groups $\mathcal{O}^1$ and $\Gamma_0(d)$ which are used in the later sections. We then recall the results of our preceding paper [2] that are relevant for the present work. Some arithmetic lemmas that are needed later on are established in Section 3. In Section 4, we first recall several versions of the Selberg trace formula and rewrite them in a manner that is suitable for our later purposes. Then we establish the Selberg trace formula for the groups $\Gamma_0(m)$, m square free, that includes the eigenvalues of Hecke operators T_p, $p\nmid m$. Finally our main results are presented in Section 5.

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2 Background and notations

In this section we fix the notation and also recall some basic facts. Good references for this well-known material are [9] and [12], see also [2].

Throughout the paper $\mathcal{O}$ will denote an order in an indefinite rational quaternion algebra $\mathcal{A}$. Except in Section 3, $\mathcal{A}$ will be a division algebra and $\mathcal{O}$ a maximal order in $\mathcal{A}$. The discriminant of $\mathcal{O}$ will be denoted by $d(\mathcal{O})$. If $\mathcal{O}$ is maximal, then $d(\mathcal{O})$ is equal to the discriminant of $\mathcal{A}$ and hence equal to the product of the *even* number of primes ramified in $\mathcal{A}$. We define

$$\mathcal{O}^n = \{x \in \mathcal{O} : \mathrm{N}(x) = n\}$$

for $n \in \mathbb{Z}$, where $\mathrm{N} : \mathcal{O} \longrightarrow \mathbb{Z}$ is the (reduced) norm. For the following, we choose a fixed embedding $\sigma : \mathcal{O} \longrightarrow M_2(\mathbb{R})$. It is then well-known that $\sigma(\mathcal{O}^1) \subset SL_2(\mathbb{R})$ is a Fuchsian group, which we also denote by $\mathcal{O}^1$. This group acts on the complex upper half-plane $\mathcal{H}$ by Möbius transformations so that the orbit space $X_{\mathcal{O}} = \mathcal{O}^1\backslash\mathcal{H}$ can be considered as a Riemann surface. This surface is compact, since $\mathcal{A}$ is a division algebra. We fix a suitable fundamental domain $\mathcal{F}_{\mathcal{O}} \subset \mathcal{H}$ for $\mathcal{O}^1$ with (hyperbolic) area $A_{\mathcal{O}}$. If $\mathcal{O}$ is a maximal order, then

$$A_{\mathcal{O}} = \frac{\pi}{3} \prod_{p|d(\mathcal{O})} (p-1). \tag{2.1}$$

Let $L^2(X_{\mathcal{O}})$ be the usual Hilbert space of functions which are square integrable on $X_{\mathcal{O}}$ with respect to the hyperbolic volume form. Since $X_{\mathcal{O}}$ is compact, the spectrum of the hyperbolic Laplacian $-\Delta$ on $L^2(X_{\mathcal{O}})$ is discrete, and is comprised of the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\,, \quad \lambda_n \longrightarrow \infty.$$

The counting function for the eigenvalues (with multiplicities) will be denoted by

$$N_{\mathcal{O}}(\lambda) = \#\{n : \lambda_n \leq \lambda\},$$

and $\{\varphi_k : k \in \mathbb{N}_0\}$ will be an orthonormal basis of $L^2(X_{\mathcal{O}})$ with $-\Delta\varphi_k = \lambda_k\varphi_k$. Thus φ_0 is a constant function and spans the eigenspace corresponding to $\lambda_0 = 0$, which we denote by $\mathbb{C}$ for simplicity. We also introduce $L_0^2(X_{\mathcal{O}})$ as the orthogonal complement to $\mathbb{C}$ within $L^2(X_{\mathcal{O}})$.

According to a well-known procedure, one can introduce Hecke operators $\widetilde{T}_n$, $n \in \mathbb{Z}$, on $L^2(X_{\mathcal{O}})$ which are self-adjoint and bounded operators that commute among themselves and also with the hyperbolic Laplacian. It is therefore possible to choose the orthonormal basis $\{\varphi_k : k \in \mathbb{N}_0\}$ for $L^2(X_{\mathcal{O}})$ to consist of joint eigenfunctions of $-\Delta$ and $\widetilde{T}_n$, $n \in \mathbb{Z}$, i.e. the Laplace eigenfunctions φ_k also satisfy $\widetilde{T}_n\varphi_k = \tilde{t}_k(n)\varphi_k$ for constants $\tilde{t}_k(n)$. Such a basis is called a Hecke basis.

The non-compact surfaces we will consider are $X_d = \Gamma_0(d)\backslash\mathcal{H}$, where

$$\Gamma_0(d) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \gamma \equiv 0 \mod d \right\}$$

is the Hecke congruence group of level d with d *square free*. The infinity and all rational numbers are cusps for $\Gamma_0(d)$. If $\omega(d)$ is the number of prime divisors of d, then the number of $\Gamma_0(d)$-equivalence classes of cusps is $2^{\omega(d)}$. As a set of representatives for these we choose

$$F(d) = \left\{ \frac{1}{v} :\ v|d,\ v > 0 \right\}.$$

A suitable fundamental domain $\mathcal{F}_d$ for $\Gamma_0(d)$ adopted to this choice then extends to $\mathbb{R}$ exactly in the points of $F(d)$. The hyperbolic area A_d of $\mathcal{F}_d$ satisfies

$$A_d = \frac{\pi}{3} \prod_{p|d} (p+1). \tag{2.2}$$

Associated to the cusps $\frac{1}{v} \in F(d)$ are their stability groups

$$\Gamma_v = \left\{ \gamma \in \Gamma_0(d) : \ \gamma \frac{1}{v} = \frac{1}{v} \right\}.$$

These are conjugate to $\Gamma_\infty = \{\pm \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) : \ j \in \mathbb{Z}\}$ via some $\sigma_v \in SL_2(\mathbb{R})$, $\sigma_v \infty = \frac{1}{v}$, $\sigma_v^{-1} \Gamma_v \sigma_v = \Gamma_\infty$. For each $\frac{1}{v} \in F(d)$ one also defines a non-holomorphic Eisenstein series $E_{1/v}(z,s)$, $z \in \mathcal{H}$, $s \in \mathbb{C}$, by

$$E_{1/v}(z,s) = \sum_{\gamma \in \Gamma_v \backslash \Gamma_0(d)} \left[\operatorname{Im}(\sigma_v^{-1} \gamma z) \right]^s , \quad \operatorname{Re} s > 1 , \tag{2.3}$$

which can be continued to a meromorphic function in $s \in \mathbb{C}$.

According to the Roelcke-Selberg spectral resolution of the hyperbolic Laplacian on $L^2(X_d)$, one has an orthogonal decomposition

$$L^2(X_d) = \mathcal{E}_d \oplus \mathbb{C} \oplus \mathcal{C}_d,$$

where $\mathcal{E}_d$ is spanned by the Eisenstein series (2.3) analytically continued to $\operatorname{Re} s = \frac{1}{2}$, and $\mathcal{C}_d$ is spanned by the cusp forms. The spectrum of $-\Delta$, when restricted to $\mathbb{C} \oplus \mathcal{C}_d$, is discrete and the eigenvalues satisfy

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots , \quad \mu_n \longrightarrow \infty.$$

In analogy to the compact case, we put $N_d(\mu) = \#\{n : \mu_n \leq \mu\}$ and choose an orthonormal basis $\{g_k : k \in \mathbb{N}_0\}$ of $\mathbb{C} \oplus \mathcal{C}_d$ with $-\Delta g_k = \mu_k g_k$. The Hecke operators on $L^2(X_d)$ will be denoted by T_n, $n \in \mathbb{N}$, and in this case it is possible to choose a Hecke basis $\{g_k\}$ such that $T_n g_k = t_k(n) g_k$ for all n coprime to d.

In [2], we constructed bounded linear operators

$$\Theta : \ L_0^2(X_{\mathcal{O}}) \longrightarrow \mathcal{C}_{d(\mathcal{O})} \quad \text{and} \quad \widetilde{\Theta} : \ \mathcal{C}_{d(\mathcal{O})} \longrightarrow L_0^2(X_{\mathcal{O}}) \tag{2.4}$$

preserving both Laplace and Hecke eigenvalues. That is, if $\{\varphi_k\}$ and $\{g_k\}$ are Hecke bases of $L_0^2(X_{\mathcal{O}})$ and $\mathcal{C}_{d(\mathcal{O})}$, respectively, then

$$-\Delta(\Theta(\varphi_k)) = \lambda_k \Theta(\varphi_k) \quad \text{and} \quad -\Delta(\widetilde{\Theta}(g_k)) = \mu_k \widetilde{\Theta}(g_k),$$

and for $(n, d(\mathcal{O})) = 1$

$$T_n(\Theta(\varphi_k)) = \tilde{t}_k(n) \Theta(\varphi_k) \quad \text{and} \quad \widetilde{T}_n(\widetilde{\Theta}(g_k)) = t_k(n) \widetilde{\Theta}(g_k).$$

If g_k is in a Hecke basis of $\mathcal{C}_d$, then it has a Fourier expansion at the cusp at infinity that reads

$$g_k(\tau) = \sum_{n \neq 0} c_k(n) \sqrt{v} \, K_{ir}(2\pi |n| v) \, e^{2\pi i n u} , \tag{2.5}$$

with $\tau = u + iv \in \mathcal{H}$, $\mu_k = r^2 + \frac{1}{4}$ being the Laplace eigenvalue of g_k, and $K_s(z)$ denoting a modified Bessel function. If $(n,d) = 1$, then the n-th Fourier coefficient is related to the n-th Hecke eigenvalue via $c_k(n) = c_k(1)\, t_k(n)$. In [2] we determined the Fourier expansion of $\Theta(\varphi_k)$, when φ_k is part of a Hecke basis of $L^2_0(X_{\mathcal{O}})$, to be

$$\Theta(\varphi_k)(\tau) = 4\, \varphi_k(z_0) \sum_{n=1}^{\infty} \tilde{t}_k(n)\, \sqrt{v}\, K_{ir}(2\pi n v)\ \left[e^{2\pi i n u} + \omega_k\, e^{-2\pi i n u}\right]\ . \tag{2.6}$$

Here $\omega_k \in \{\pm 1\}$, and z_0 is a reference point involved in the construction of Θ which can be chosen such that $\varphi_k(z_0) \neq 0$ for all eigenfunctions in a given Hecke basis.

We now recall the necessary background and results from [2] on newforms. Let $a, m \in \mathbb{N}$, $m < d$, be such that $am|d$, and take some $h \in \mathcal{C}_m$. The inclusion $\Gamma_0(d) \subset \Gamma_0(m)$ implies that $\mathcal{C}_m \subset \mathcal{C}_d$ so that $h \in \mathcal{C}_d$, but also $h^{(a)} \in \mathcal{C}_d$ where $h^{(a)}(\tau) = h(a\tau)$. The linear span of all such forms $h^{(a)} \in \mathcal{C}_d$ that derive from all possible a, m is called the oldspace $\mathcal{C}_d^{old}$. Its orthogonal complement within $\mathcal{C}_d$ is the newspace $\mathcal{C}_d^{new}$, so that $\mathcal{C}_d = \mathcal{C}_d^{old} \oplus \mathcal{C}_d^{new}$. One can introduce a Hecke basis of $\mathcal{C}_d$ such that one part of this basis spans the oldspace, and the remaining part spans the newspace. A Hecke eigenform in the newspace is then called a newform. If h is a newform in $\mathcal{C}_m$, then $h^{(a)}$ is called an oldform in $\mathcal{C}_d$.

It is trivial to check that all $h^{(a)}$ corresponding to a fixed $h \in \mathcal{C}_m$ have the same Laplace eigenvalue. If $h \in \mathcal{C}_m^{new}$, then there are $\tau(\frac{d}{m})$ forms $h^{(a)}$ in $\mathcal{C}_d$ corresponding to h, where $\tau(n)$ is the number of positive divisors of n. Let $\delta(m,\lambda)$ be the dimension of the subspace of $\mathcal{C}_m$ with Laplace eigenvalue $\lambda > 0$, and let $\delta'(m,\lambda)$ be the dimension of the corresponding subspace of $\mathcal{C}_m^{new}$. Since $\mathcal{C}_d = \mathcal{C}_d^{old} \oplus \mathcal{C}_d^{new}$, these satisfy

$$\delta(d,\lambda) = \sum_{m|d} \tau\left(\tfrac{d}{m}\right) \delta'(m,\lambda).$$

Inverting this formula, one gets [1, (6.7)]

$$\delta'(d,\lambda) = \sum_{m|d} \beta\left(\tfrac{d}{m}\right) \delta(m,\lambda), \tag{2.7}$$

with

$$\beta(n) = \sum_{k|n} \mu(k)\mu(\tfrac{n}{k}), \tag{2.8}$$

where $\mu(n)$ is the Möbius function. In particular if n is a product of r distinct primes, then $\beta(n) = (-2)^r$.

If $N_d'(\lambda) = \#\{\mu_k \leq \lambda :\ g_k \in \mathcal{C}_d^{new}\}$, then using (2.7) we observed in [2] that

$$\lim_{\lambda\to\infty} \frac{N_d'(\lambda)}{N_{\mathcal{O}}(\lambda)} \geq 1, \tag{2.9}$$

when $\mathcal{O}$ is a maximal order with discriminant d; equality emerges under the hypothesis that the Laplace spectrum of $L^2(X_{\mathcal{O}})$ is simple. In this paper, we will show the much stronger result that there is a one-to-one correspondence including multiplicities between the Laplace spectra of $L^2_0(X_{\mathcal{O}})$ and $\mathcal{C}_d^{new}$. More generally, we will prove that the traces of the Hecke operators $\widetilde{T}_p$ and T_p on the λ-eigenspaces of $L^2_0(X_{\mathcal{O}})$ and $\mathcal{C}_d^{new}$ coincide for $p \nmid d$.

3 Auxiliary results

Let $\mathcal{O}$ be a so-called Eichler order of square free level m and discriminant $d \cdot m$, where d is a product of an even number of distinct primes. We remark that $(d, m) = 1$ is implicit in the definition. If $d = 1$, then

$$\mathcal{O} \cong M(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \mod m \right\},$$

and if $m = 1$, then $\mathcal{O}$ is a maximal order in an indefinite quaternion algebra $\mathcal{A}$ over $\mathbb{Q}$. Let $\mathcal{O}^1$ denote the group of elements in $\mathcal{O}$ with (reduced) norm equal to 1. In particular, $\mathcal{O}^1 \cong \Gamma_0(m)$ if $d = 1$, and otherwise $\mathcal{O}^1$ is a cocompact arithmetic Fuchsian group.

Let K be a quadratic field extension of $\mathbb{Q}$ and B an order in K. Assume that there is an embedding $\iota : K \longrightarrow \mathcal{A}$. The order B is said to be optimally embedded into $\mathcal{O}$ with respect to ι if $\iota(B) = \mathcal{O} \cap \iota(K)$. Two optimal embeddings, ι_1 and ι_2, are conjugate by γ if $\iota_1(B) = \gamma \cdot \iota_2(B) \cdot \gamma^{-1}$. The number of optimal embeddings of B modulo conjugation by elements in $\mathcal{O}^1$ is given by the formula

$$E(B, \mathcal{O}^1) = \mathrm{cl}(B) \cdot \prod_{p|d} \left(1 - \left(\frac{B}{p}\right)\right) \prod_{p|m} \left(1 + \left(\frac{B}{p}\right)\right), \tag{3.1}$$

where $\mathrm{cl}(B)$ is the class number of B and

$$\left(\frac{B}{p}\right) = \begin{cases} 1 & \text{if } p \text{ is split in } K \text{ or } B \text{ is not a maximal order in } K, \\ -1 & \text{if } p \text{ is unramified in } K \text{ and } B \text{ is maximal}, \\ 0 & \text{if } p \text{ is ramified in } K \text{ and } B \text{ is maximal}. \end{cases}$$

For imaginary extensions K this is a special case of [13, 2.5.], noting that the local embedding number for a prime p dividing m is $\left(1 + \left(\frac{B}{p}\right)\right)$ [17, Th.II.3.2]. However, in this simple case with a rational ground field, it is easy to see that the factor multiplying the product of the local embedding numbers is $\mathrm{cl}(B)$ also for real quadratic extensions. We remark that the fact that all Eichler orders contain elements with norm -1 [11, (5.6)] is important when deriving that the factor is exactly $\mathrm{cl}(B)$.

If $\gamma \in \mathcal{O}$, then the conjugacy class of γ with respect to $\mathcal{O}^1$ is $\{\gamma\}_{\mathcal{O}^1} = \{\alpha\gamma\alpha^{-1} : \alpha \in \mathcal{O}^1\}$. Observe that $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\alpha\gamma\alpha^{-1})$, where $\mathrm{Tr} : \mathcal{O} \longrightarrow \mathbb{Z}$ is the (reduced) trace, and $\mathrm{N}(\gamma) = \mathrm{N}(\alpha\gamma\alpha^{-1})$. For $t, n \in \mathbb{Z}$, we now define $E(t, n, \mathcal{O}^1)$ to be the number of conjugacy classes $\{\gamma\}_{\mathcal{O}^1}$ in $\mathcal{O}$ with $\mathrm{Tr}(\gamma) = t$ and $\mathrm{N}(\gamma) = n$. If γ is an arbitrary element with $\mathrm{Tr}(\gamma) = t$ and $\mathrm{N}(\gamma) = n$, then [17, p.96]

$$E(t, n, \mathcal{O}^1) = \sum_{B \supseteq \mathbb{Z}[\gamma]} E(B, \mathcal{O}^1), \tag{3.2}$$

where the sum is over all orders in $\mathbb{Q}(\gamma)$ containing γ.

We will only need to consider $E(t, n, \mathcal{O}^1)$ for $n = 1$ or $n = p$ a prime, and we may restrict to $t > 0$ since $E(t, n, \mathcal{O}^1) = E(-t, n, \mathcal{O}^1)$. For $n = p$, the case $t = p + 1$ is exceptional, since this is the only situation where $\mathbb{Q}(\gamma)$ is not a field. This implies in particular that $E(p + 1, p, \mathcal{O}^1) = 0$ when $\mathcal{O}$ is an order in a division algebra.

Lemma 3.1. *If p is a prime and m is a product of $\omega(m)$ distinct primes, then*

$$E\big(p + 1, p, \Gamma_0(m)\big) = 2^{\omega(m)}(p - 1).$$

Moreover, if $n_0(v)$ are integers satisfying $n_0(v) \equiv (p-1)v^{-1} \pmod{\frac{m}{v}}$ for positive divisors v of m, then

$$\bigcup_{\substack{v|m \\ v>0}} \left\{ \gamma_{v,n} = \begin{pmatrix} p-nv & n \\ v(p-nv-1) & nv+1 \end{pmatrix} : \; n = n_0(v) + k\frac{m}{v},\; 0 \le k < p-1 \right\}$$

is a set of representatives of the different conjugacy classes with trace $p+1$ and norm p.

Proof. Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(m)$ be an arbitrary element with $\mathrm{Tr}(\alpha) = p+1$ and $\mathrm{N}(\alpha) = p$. Then α has two rational fixed points, namely $\frac{a-1}{c}$ and $\frac{1-d}{c}$. Each of these fixed points is mapped to a unique member in $F(m) = \{\frac{1}{v} : v|m\}$ by some elements β_1 and β_2 in $\Gamma_0(m)$, since all rational numbers are cusps for $\Gamma_0(m)$. Hence we may assume that α has a fixed point in $F(m)$, since we may conjugate by β_1 or β_2.

Fix a positive divisor v of m. The element $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ -v & 1 \end{smallmatrix}\right) \in \Gamma_0(1)$ maps $\frac{1}{v}$ to ∞. Elements with trace $p+1$, norm p and one fixed point in ∞ are either of the form $\left(\begin{smallmatrix} 1 & n \\ 0 & p \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} p & n \\ 0 & 1 \end{smallmatrix}\right)$. Hence α is of one of the forms:

$$\alpha = \gamma^{-1} \begin{pmatrix} 1 & n \\ 0 & p \end{pmatrix} \gamma = \begin{pmatrix} 1-nv & n \\ v(1-nv-p) & nv+p \end{pmatrix}, \tag{3.3}$$

or

$$\alpha = \gamma^{-1} \begin{pmatrix} p & n \\ 0 & 1 \end{pmatrix} \gamma = \begin{pmatrix} p-nv & n \\ v(p-nv-1) & nv+1 \end{pmatrix}. \tag{3.4}$$

A priori, n need not be an integer. However, the condition $\alpha \in M(m)$ forces n to be an integer, and moreover is equivalent to $\frac{m}{v}|(1-nv-p)$ and $\frac{m}{v}|(p-nv-1)$, respectively. Since $(\frac{m}{v}, v) = 1$, this implies that $\alpha \in M(m)$ is equivalent to n being congruent to a unique class modulo $\frac{m}{v}$.

There are only two possibilities for two elements of the form (3.3) or (3.4) to be conjugate by elements in $\Gamma_0(m)$. Either directly by an element in Γ_v for some $v|m$, or otherwise they correspond to the two different choices of fixed point of α combined with an element in Γ_v. First we will consider Γ_v.

One readily checks that $\Gamma_v = \gamma^{-1} \langle \beta \rangle \gamma$, where $\beta = \left(\begin{smallmatrix} 1 & \frac{m}{v} \\ 0 & 1 \end{smallmatrix}\right)$. A direct computation gives

$$(\gamma^{-1}\beta^k\gamma) \begin{pmatrix} 1-nv & n \\ v(1-nv-p) & nv+p \end{pmatrix} (\gamma^{-1}\beta^k\gamma)^{-1} = \begin{pmatrix} 1-n'v & n' \\ v(1-n'v-p) & n'v+p \end{pmatrix},$$

where $n' = n + k(p-1)\frac{m}{v}$. Since n is congruent to a unique class modulo $\frac{m}{v}$, we conclude that we get $p-1$ different conjugacy classes. The form (3.4) is completely analogous. This gives us at most $2(p-1)$ different conjugacy classes associated to each of the different cusps.

A direct computation shows that conjugation with an element taking the fixed point $\frac{a-1}{c}$ to the cusp $\frac{1}{v} \in F(m)$ always gives an element of the form (3.4), and for $\frac{1-d}{c}$ of the form (3.3). Hence we can conclude that the number of different conjugacy classes is $2^{\omega(m)}(p-1)$. Furthermore, they can be chosen to be either of the form (3.3) or (3.4). $\square$

The basis for the definition of the Hecke operator T_p with respect to $\Gamma_0(m)$ for $(p,m) = 1$ is the set $M_p(m) = \{\alpha \in M(m) : \det(\alpha) = p\}$. It has the following decomposition [12, (4.5.25)]

$$M_p(m) = \bigcup_{j=0}^{p-1} \Gamma_0(m) \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} \cup \Gamma_0(m) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

Lemma 3.2. *Assume that p is a prime and m is a square free integer with $(p,m)=1$. Then two cusps which are inequivalent modulo $\Gamma_0(m)$ are also inequivalent modulo $M_p(m)$.*

Proof. We recall that $\{\frac{1}{v} : v|m\}$ is a set of representatives of the different cusps modulo $\Gamma_0(m)$. From (3.5) it follows that it suffices to show that $\gamma\frac{1}{v}$ is $\Gamma_0(m)$-equivalent to $\frac{1}{v}$ for any element γ of the form $\left(\begin{smallmatrix} 1 & j \\ 0 & p \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, that is, $\frac{1+jv}{pv}$ and $\frac{p}{v}$ are $\Gamma_0(m)$-equivalent to $\frac{1}{v}$. More generally, we show that $\frac{1}{v}$ is equivalent to $\frac{x}{yv}$ whenever $(x,yv)=1$ and $(y,m)=1$. Take $\alpha = \left(\begin{smallmatrix} a & b \\ mc & d \end{smallmatrix}\right) \in \Gamma_0(m)$. Then $\alpha\frac{1}{v} = \frac{x}{yv}$ and $\alpha \in \Gamma_0(m)$ is equivalent to

$$\begin{aligned} 1 &= ad - mbc \\ x &= a + bv \\ y &= d + c\tfrac{m}{v}. \end{aligned}$$

Substituting the expressions of a and d into the first equality, we get

$$1 = (y - \tfrac{m}{v}c)(x - bv) - mbc = xy - c\tfrac{m}{v}x - bvy.$$

This has a solution b, c, since $(\frac{m}{v}x, vy) = 1$. $\square$

The next three lemmas contain identities which are the main ingredients when comparing the trace formulae in Section 5.

Lemma 3.3. *Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion algebra with discriminant d. Then*

$$E(B,\mathcal{O}^1) = \sum_{m|d} \beta\left(\tfrac{d}{m}\right) E\big(B,\Gamma_0(m)\big) \tag{3.6}$$

and

$$E(t,n,\mathcal{O}^1) = \sum_{m|d} \beta\left(\tfrac{d}{m}\right) E\big(t,n,\Gamma_0(m)\big), \tag{3.7}$$

where β is given by (2.8).

Proof. The identity (3.7) follows directly from (3.6) by using (3.2). Since $\mathrm{cl}(B)$ only depends on B, it follows from (3.1) that (3.6) is equivalent to

$$\prod_{p|d}\left(1 - \left(\frac{B}{p}\right)\right) = \sum_{m|d} \beta\left(\tfrac{d}{m}\right) \prod_{p|m}\left(1 + \left(\frac{B}{p}\right)\right). \tag{3.8}$$

That $\mathcal{O}$ is a maximal order in an indefinite quaternion algebra implies that d is a product of an even number, $2r$, of different primes. Fix an order B and let $k = \#\{p : p|d,\ (\frac{B}{p}) = 1\}$ and $e = \#\{p : p|d,\ (\frac{B}{p}) = 0\}$. We obviously have

$$\prod_{p|d}\left(1 - \left(\frac{B}{p}\right)\right) = \begin{cases} 0 & \text{if } k > 0, \\ 2^{2r-e} & \text{otherwise.} \end{cases}$$

Moreover, a term of the right-hand side of (3.8) is non-zero only for all possible products of the $k+e$ primes with $(\frac{B}{p}) \neq -1$. This observation yields

$$\begin{aligned} \sum_{m|d} \beta\left(\tfrac{d}{m}\right) \prod_{p|m}\left(1 + \left(\frac{B}{p}\right)\right) &= \sum_{j=0}^{e}\sum_{i=0}^{k} (-2)^{2r-i-j} \cdot 2^i \cdot 1^j \binom{k}{i}\binom{e}{j} \\ &= 2^{2r-e} \sum_{j=0}^{e} (-1)^j 2^{e-j} \binom{e}{j} \sum_{i=0}^{k} (-1)^i \binom{k}{i}. \end{aligned}$$

The desired result follows by observing that

$$\sum_{j=0}^{e}(-1)^j 2^{e-j}\binom{e}{j} = (2-1)^e = 1 \quad \text{and} \quad \sum_{i=0}^{k}(-1)^i\binom{k}{i} = \begin{cases}(1-1)^k = 0 & \text{if } k>0,\\ 1 & \text{if } k=0.\end{cases} \qquad \square$$

A hyperbolic element $\gamma \in \Gamma$, Γ a Fuchsian group, is called primitive in Γ, if there is no element $\alpha \in \Gamma$ such that $\gamma = \alpha^r$, with $r \geq 2$. For $t > 2$, we define $E'(t,1,\mathcal{O}^1)$ to be the number of conjugacy classes in $\mathcal{O}^1$ of primitive elements with trace t. We remark that if γ is hyperbolic, then for any $r \geq 1$ $\{\gamma\}_{\mathcal{O}^1} = \{\beta\}_{\mathcal{O}^1}$ iff $\{\gamma^r\}_{\mathcal{O}^1} = \{\beta^r\}_{\mathcal{O}^1}$. Moreover, $r \longmapsto |\operatorname{Tr}(\gamma^r)|$ is a strictly increasing function for $r \geq 1$ when γ is hyperbolic. Using these two facts it is clear that

$$E(t,1,\mathcal{O}^1) = {\sum_{s\leq t}}' E'(s,1,\mathcal{O}^1), \tag{3.9}$$

where the sum is over all s such that if $\operatorname{Tr}(\gamma) = s$, then there is an $r \geq 1$ such that $\operatorname{Tr}(\gamma^r) = t$. Since $E(3,1,\mathcal{O}^1) = E'(3,1,\mathcal{O}^1)$, the following lemma follows by straightforward induction on t using (3.7) and (3.9).

Lemma 3.4. *Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion algebra with discriminant d. Then*

$$E'(t,1,\mathcal{O}^1) = \sum_{m|d}\beta\big(\tfrac{d}{m}\big)\, E'\big(t,1,\Gamma_0(m)\big).$$

We conclude this section with three simple arithmetic identities which are important in the proof of the main theorems.

Lemma 3.5. *If $d > 1$ is a square free integer and $\omega(m)$ the number of primes dividing m, then*

$$\sum_{m|d}\beta\big(\tfrac{d}{m}\big) = (-1)^{\omega(d)} \quad \text{and} \quad \sum_{m|d}\beta\big(\tfrac{d}{m}\big)\, 2^{\omega(m)} = 0.$$

Moreover, if f is an arbitrary function and d has at least two prime divisors, then

$$\sum_{m|d}\beta\big(\tfrac{d}{m}\big)\, 2^{\omega(m)} \sum_{p|m} f(p) = 0,$$

where the first sum is over all divisors and the second only over prime divisors.

Proof. The first identity follows from the binomial theorem, since

$$\sum_{m|d}\beta\big(\tfrac{d}{m}\big) = \sum_{i=0}^{\omega(d)}(-2)^{\omega(d)-i}\binom{\omega(d)}{i} = (1-2)^{\omega(d)} = (-1)^{\omega(d)}.$$

The second identity follows analogously,

$$\sum_{m|d}\beta\big(\tfrac{d}{m}\big)\, 2^{\omega(m)} = \sum_{i=0}^{\omega(d)}(-2)^{\omega(d)-i}\cdot 2^i\binom{\omega(d)}{i} = (2-2)^{\omega(d)} = 0.$$

From this identity we derive the third one by

$$\begin{aligned}\sum_{m|d}\beta\big(\tfrac{d}{m}\big)\, 2^{\omega(m)}\sum_{p|m} f(p) &= \sum_{p|d} f(p)\sum_{m|\frac{d}{p}}\beta\Big(\frac{d}{mp}\Big)\, 2^{\omega(mp)}\\ &= 2\cdot\sum_{p|d} f(p)\sum_{m|\frac{d}{p}}\beta\Big(\frac{d/p}{m}\Big)\, 2^{\omega(m)} = 0.\end{aligned}$$

$\square$

4 Trace formulae

The basic tool we are going to apply in order to prove our main result will be the Selberg trace formula. We will need several versions of it; with and without inclusion of Hecke-eigenvalues, for the cocompact groups $\mathcal{O}^1$ and the Hecke congruence groups $\Gamma_0(m)$. Without inclusion of Hecke-eigenvalues the trace formula for all Fuchsian groups occurring are well-known. When considering Hecke operators $\widetilde{T}_p$, T_p, for primes $p \nmid d(\mathcal{O})$, the relevant trace formula for the groups $\Gamma_0(m)$, $m > 1$, still has to be evaluated explicitly. The result for $\mathcal{O}^1$ can be found in [5, ch.V,Thm.8.1] and for $\Gamma_0(1)$ in [4, (11.10)]. In this section, we will first recall the known results, and thereby take the opportunity to introduce some notation that will be useful. Then we will derive the necessary trace formula not yet to be found in the literature.

In the sequel $h : \mathbb{C} \to \mathbb{C}$ always denotes a function satisfying

- $h(r) = h(-r)$,
- $h(r)$ is holomorphic in the strip $|\operatorname{Im} r| \leq \frac{1}{2} + \varepsilon$, for some $\varepsilon > 0$,
- $|h(r)| \leq C\,(1 + \operatorname{Re} r)^{-2-\delta}$ for some $C > 0$ and $\delta > 0$.

The Fourier transform of h will then be written as

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\, e^{-iru}\, dr \ .$$

Since the unit group $\mathcal{O}^1$ is a cocompact Fuchsian group, the Selberg trace formula without Hecke-eigenvalues reads as follows:

Proposition 4.1. *Let $\lambda_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(X_{\mathcal{O}})$, counted with multiplicities. Then*

$$\begin{aligned}
\sum_{k=0}^{\infty} h(r_k) =& \frac{A_{\mathcal{O}}}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\, dr \\
&+ \sum_{t \in \{0,1\}} \frac{E'(t,1,\mathcal{O}^1)}{2m_t} \sum_{k=1}^{m_t - 1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1 + e^{-2\pi r}}\, dr \\
&+ \sum_{t=3}^{\infty} E'(t,1,\mathcal{O}^1)\, \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)} \ .
\end{aligned}$$

Proof. Recall the trace formula for cocompact Fuchsian groups [14, (3.2)], [5, ch.II,Thm.5.1]. The sums over representatives of the primitive elliptic and hyperbolic conjugacy classes of elements in $\mathcal{O}^1$ are rewritten as sums over the traces $t \in \mathbb{N}_0$ of the representatives. The number of primitive conjugacy classes with trace t is $E'(t,1,\mathcal{O}^1)$. In the elliptic case, where $t \in \{0,1\}$, m_t denotes the order of the primitive element with trace t. □

We now consider the Hecke operators $\widetilde{T}_p$ when p is a prime not dividing $d(\mathcal{O})$. In this case the relevant trace formula is given by

Proposition 4.2. *Let $\lambda_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(X_{\mathcal{O}})$, counted with multiplicities. Fix a prime $p \nmid d(\mathcal{O})$ and denote by $\tilde{t}_k(p)$ the eigenvalues* *of $\widetilde{T}_p$ on a Hecke basis $\{\varphi_k : \; k \in \mathbb{N}_0\}$ of $L^2(X_\mathcal{O})$. Then*

$$\sum_{k=0}^{\infty} \tilde{t}_k(p)\, h(r_k) = \sum_{\substack{\{\gamma\}_{\mathcal{O}^1} \\ \gamma \; ellip.}} \frac{1}{m_\gamma \sqrt{4p-t^2}} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-2r \arcsin \sqrt{1-\frac{t^2}{4p}}}}{1+e^{-2\pi r}}\; dr$$

$$+ \frac{1}{\sqrt{p}} \sum_{\substack{\{\gamma\}_{\mathcal{O}^1} \\ \gamma \; hyperb.}} \frac{\operatorname{arccosh}(|\log \varepsilon_\gamma|)\, \hat{h}\left(2 \operatorname{arccosh}(\frac{t}{2\sqrt{p}})\right)}{2 \sinh\left(\operatorname{arccosh}(\frac{t}{2\sqrt{p}})\right)} \;,$$

where $t = \mathrm{Tr}(\gamma)$ and $p = \mathrm{N}(\gamma)$. Moreover, m_γ is the order of the centraliser $Z_{\mathcal{O}^1}(\gamma)$ when $\gamma \in \mathcal{O}^p$ is elliptic, and ε_γ is a generator of $Z_{\mathcal{O}^1}(\gamma)$ when $\gamma \in \mathcal{O}^p$ is hyperbolic.

Proof. In order to define the Hecke operator $\widetilde{T}_p$,

$$\widetilde{T}_p \varphi(z) = \frac{1}{\sqrt{p}} \sum_{j=1}^{d(p)} \varphi(\gamma_j z) \;, \qquad \varphi \in L^2(X_\mathcal{O}) \;,$$

one needs the decomposition

$$\mathcal{O}^p = \{u \in \mathcal{O} : \; \mathrm{N}(u) = p\} = \bigcup_{j=1}^{d(p)} \mathcal{O}^1 \gamma_j \;,$$

which derives from the decomposition

$$\mathcal{O}^1 u \mathcal{O}^1 = \bigcup_{j=1}^{d(u)} \mathcal{O}^1 \gamma_j$$

of the distinct double cosets $\mathcal{O}^1 u \mathcal{O}^1$ with $u \in \mathcal{O}^p$.

Clearly, $\widetilde{T}_p$ commutes with $-\Delta$, and also with any integral operator

$$L\varphi(z) = \int_{\mathcal{F}_\mathcal{O}} K(z,w)\, \varphi(w)\; d\mu(w) \;, \qquad \varphi \in L^2(X_\mathcal{O}) \;,$$

which has a kernel of the form

$$K(z,w) = \sum_{\gamma \in \mathcal{O}^1/\{\pm E_2\}} k(\gamma z, w) \;, \qquad k(z,w) = \phi\left(\frac{|z-w|^2}{\operatorname{Im} z \operatorname{Im} w}\right) \;,$$

where $d\mu(z) = \frac{dx\,dy}{y^2}$ is the hyperbolic volume form and $\phi \in C_0^2(\mathbb{R})$ is arbitrary. The fact that $\widetilde{T}_p L = L \widetilde{T}_p$ follows from [5, ch.V,Prop.2.19] and from the observation that $\widetilde{T}_p$ is $\frac{1}{\sqrt{p}}$ times a finite sum of operators as in [5, ch.V,Def.2.9]. According to [5, ch.V,Prop.2.22], $\widetilde{T}_p L$ has an integral kernel

$$K_p(z,w) = \frac{1}{\sqrt{p}} \sum_{j=1}^{d(p)} \sum_{\gamma \in \mathcal{O}^1/\{\pm E_2\}} k(\gamma \gamma_j z, w) = \frac{1}{\sqrt{p}} \sum_{\gamma \in \mathcal{O}^p/\{\pm E_2\}} k(\gamma z, w) \;. \tag{4.1}$$

Following [5, ch.V] further, one calculates $\operatorname{Tr} L\widetilde{T}_p$ on the one hand from (4.1), and on the other hand from the spectral expansion

$$K_p(z,w) = \sum_{k=0}^{\infty} \tilde{t}_k(p)\, h(r_k)\, \varphi_k(z)\, \overline{\varphi_k}(w) \ .$$

This yields

$$\begin{aligned}\sum_{k=0}^{\infty} \tilde{t}_k(p)\, h(r_k) &= \frac{1}{\sqrt{p}} \sum_{\gamma\in\mathcal{O}^p/\{\pm E_2\}} \int_{\mathcal{F}_{\mathcal{O}}} k(\gamma z, z)\ d\mu(z) \\ &= \frac{1}{\sqrt{p}} \sum_{\{\gamma\}_{\mathcal{O}^1}} \sum_{\sigma\in\mathcal{O}^1/Z_{\mathcal{O}^1}(\gamma)} \int_{\sigma\mathcal{F}_{\mathcal{O}}} k(\sigma z, z)\ d\mu(z) \ . \end{aligned} \tag{4.2}$$

In the second line the first sum extends over all conjugacy classes $\{\gamma\}_{\mathcal{O}^1}$ of elements $\gamma \in \mathcal{O}^p/\{\pm E_2\}$. For the second sum one needs to know the centraliser

$$Z_{\mathcal{O}^1}(\gamma) = \big\{\sigma \in \mathcal{O}^1 :\ \sigma\gamma = \gamma\sigma\big\} = \mathbb{Q}(\gamma) \cap \mathcal{O}^1$$

of $\gamma \in \mathcal{O}^p$. Since $\mathcal{O}$ is a maximal order in a division algebra, any quadratic extension $\mathbb{Q}(\rho)$, $\rho \in \mathcal{O}$, is a field. Moreover, $\mathbb{Q}(\rho) \cap \mathcal{O}$ is the maximal order in $\mathbb{Q}(\rho)$. It follows that the centraliser $\mathbb{Q}(\gamma) \cap \mathcal{O}^1$ is the group of units of norm one in the maximal order of $\mathbb{Q}(\sqrt{D})$, where D is square free such that $t^2 - 4p = n^2 D$. Elliptic elements of $\mathcal{O}^p$ are characterised by $t^2 - 4p < 0$, which implies that D is negative. In this case $Z_{\mathcal{O}^1}(\gamma)$ is a finite cyclic group of order m_γ. In the hyperbolic case, where $t^2 - 4p > 0$ and hence D is positive, the centraliser is infinite cyclic, generated by ε_γ. Going on as in [5, ch.V], we arrive at the desired result; compare [5, ch.V,Thm.8.1]. □

Next we consider the Hecke congruence groups $\Gamma_0(m)$ with $m|d(\mathcal{O})$. Thus m is square free and consist of $\omega(m)$ prime divisors. For the rest of this section the discriminant $d(\mathcal{O})$ is not important so that m denotes an arbitrary square free positive integer. Without the inclusion of Hecke-eigenvalues the trace formula for $\Gamma_0(m)$ is well-known. We recall [4, Thm.9.9] together with [4, (10.2),(10.4)] and use the same notation as in Proposition 4.1:

Proposition 4.3. *Let $\mu_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(X_m)$, counted with multiplicities. Then*

$$\begin{aligned}\sum_{k=0}^{\infty} h(r_k) =& \frac{A_m}{4\pi} \int_{-\infty}^{+\infty} h(r)\, r\, \tanh(\pi r)\ dr \\ &+ \sum_{t\in\{0,1\}} \frac{E'(t,1,\Gamma_0(m))}{2m_t} \sum_{k=1}^{m_t-1} \frac{1}{\sin(\frac{k\pi}{m_t})} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-\frac{2k\pi r}{m_t}}}{1+e^{-2\pi r}}\ dr \\ &+ \sum_{t=3}^{\infty} E'(t,1,\Gamma_0(m))\ \operatorname{arccosh}(\tfrac{t}{2}) \sum_{k=1}^{\infty} \frac{\hat{h}\left(2k \operatorname{arccosh}(\frac{t}{2})\right)}{\sinh\left(k \operatorname{arccosh}(\frac{t}{2})\right)} \\ &+ 2^{\omega(m)} \Bigg\{ \hat{h}(0)\log(\tfrac{\pi}{2}) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\ \left[\frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir) + \frac{\Gamma'}{\Gamma}(1+ir)\right]\ dr \\ &\qquad + 2\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n}\, \hat{h}(2\log n) - \sum_{\substack{p|m \\ p\ \text{prime}}} \sum_{k=0}^{\infty} \frac{\log p}{p^k}\, \hat{h}(2k\log p) \Bigg\} \ .\end{aligned}$$

Our main concern will now be to set up the trace formula for $\Gamma_0(m)$ that includes a Hecke operator T_p, where $p \nmid m$. On the spectral side, a proof of the trace formula requires to know the eigenvalues of T_p not only on a basis for $\mathcal{C}_m$, but also on Eisenstein series. We therefore first establish

Lemma 4.4. *Let $E_{1/v}(z,s)$, $v|m$, be the non-holomorphic Eisenstein series (2.3) associated with the representatives $\frac{1}{v} \in F(m)$ of inequivalent cusps for $\Gamma_0(m)$. Then these are eigenfunctions of T_p, $p \nmid m$, with eigenvalues $p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}$, i.e.,*

$$T_p E_{1/v}(z,s) = \left(p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}\right) E_{1/v}(z,s) \ .$$

Proof. We first recall [16, Thm.6.3.3], which states that

$$T_p E_{1/v}(z,s) = \sum_{u|m} H_{vu}(s,p)\, E_{1/u}(z,s) \ ,$$

where the unknowns $H_{vu}(s,p)$ can be obtained as coefficients of y^s in a Fourier expansion of $T_p E_{1/v}(\sigma_u z, s)$. According to the decomposition (3.5) one obtains for $\mathrm{Re}\, s > 1$

$$\sqrt{p}\, T_p E_{1/v}(\sigma_u z, s) = \sum_{\gamma \in \Gamma_v \backslash M_p(m)} \left[\mathrm{Im}\left(\sigma_v^{-1} \gamma \sigma_u z\right)\right]^s = \sum_{\tau \in \Gamma_\infty \backslash \sigma_v^{-1} M_p(m) \sigma_u} \left[\mathrm{Im}\,(\tau z)\right]^s \ . \tag{4.3}$$

As in the case of the Fourier expansion of $E_{1/v}(\sigma_u z, s)$, see [9, sec.3.4], the coefficient of y^s in (4.3) derives from the elements $\tau_\infty = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \sigma_v^{-1} M_p(m) \sigma_u$. Now consider $\gamma = \sigma_v \tau_\infty \sigma_u^{-1} \in M_p(m)$. Since τ_∞ fixes ∞, one obtains that $\gamma \frac{1}{u} = \frac{1}{v}$. In Lemma 3.2, we found that the $\Gamma_0(m)$-inequivalent cusps $\frac{1}{v}$, $v|m$, are also inequivalent with respect to $M_p(m)$. Hence $\sigma_v^{-1} M_p(m) \sigma_u \neq \emptyset$ iff $u = v$. This implies that $H_{vu}(s,p)$ are the entries of a diagonal matrix, and hence the Eisenstein series are eigenfunctions of T_p.

We now determine the eigenvalues $H_{vv}(s,p)$. A direct computation shows that $\tau_\infty \in \sigma_v^{-1} M_p(m) \sigma_v$ is of the form

$$\tau_\infty = \begin{pmatrix} a+bv & b\frac{v}{m} \\ 0 & d-bv \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} a & b \\ mc & d \end{pmatrix} \in M_p(m) \ .$$

Thus, one either has (i) $a+bv = p$ and $d-bv = 1$, or (ii) $a+bv = 1$ and $d-bv = p$. These conditions, together with the relation $ad - mbc = p$, imply that $b(\pm v(p-1) - bv^2 - mc) = 0$. Apart from $b = 0$, one therefore gets the equations $bv + c\frac{m}{v} = \pm(p-1)$. Since v and $\frac{m}{v}$ are coprime, the solutions b, c are such that b is in a unique class mod $\frac{m}{v}$. Furthermore,

$$\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & b\frac{v}{m} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & b\frac{v}{m} + N \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b\frac{v}{m} \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & b\frac{v}{m} + Np \\ 0 & p \end{pmatrix} \ .$$

Therefore, there is one equivalence class mod Γ_∞ of elements $\tau_\infty \in \sigma_v^{-1} M_p(m) \sigma_v$ in the case (i), whereas there are p classes in the case (ii). Moreover, since

$$\left[\mathrm{Im}\,(\tau_\infty z)\right]^s = \begin{cases} p^s\, y^s & \text{in case (i),} \\ p^{-s}\, y^s & \text{in case (ii),} \end{cases}$$

we conclude that the coefficient of y^s on the right-hand side of (4.3) is given by $p^s + p^{1-s}$. Division by $\sqrt{p}$ then yields the eigenvalue of $E_{1/v}(z,s)$. ∎

Theorem 4.5. *Let $\mu_k = r_k^2 + \frac{1}{4}$ run through all eigenvalues of the hyperbolic Laplacian on $L^2(X_m)$, counted with multiplicities. Fix a prime $p \not| m$ and denote by $t_k(p)$ the eigenvalues of T_p on a Hecke basis $\{g_k : k \in \mathbb{N}_0\}$ of $\mathbb{C} \oplus \mathcal{C}_m$. Then*

$$\begin{aligned}
\sum_{k=0}^{\infty} t_k(p)\, h(r_k) = & \sum_{\substack{\{\gamma\}_{\Gamma_0(m)} \\ \gamma \text{ ellip.}}} \frac{1}{m_\gamma \sqrt{4p - t^2}} \int_{-\infty}^{+\infty} h(r)\, \frac{e^{-2r \arcsin \sqrt{1 - \frac{t^2}{4p}}}}{1 + e^{-2\pi r}}\, dr \\
& + \frac{1}{\sqrt{p}} \sum_{\substack{\{\gamma\}_{\Gamma_0(m)},\, t \neq p+1 \\ \gamma \text{ hyperb.}}} \frac{\operatorname{arccosh}(|\log \varepsilon_\gamma|)\, \hat{h}\left(2 \operatorname{arccosh}(\frac{t}{2\sqrt{p}})\right)}{2 \sinh\left(\operatorname{arccosh}(\frac{t}{2\sqrt{p}})\right)} \\
& + 2^{\omega(m)} \Bigg\{ 2\, \hat{h}(\log p) \left[\log \pi + \log(p-1) - \frac{\log X(p-1)}{p-1} \right] \\
& \quad - \tfrac{1}{2} h(0) + \int_{\log p}^{\infty} \hat{h}(u)\, \frac{e^{\frac{u}{2}} + e^{-\frac{u}{2}}}{e^{\frac{u}{2}} - e^{-\frac{u}{2}} + p^{\frac{1}{2}} - p^{-\frac{1}{2}}}\, du \\
& \quad - \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(r)\, \left[p^{ir} + p^{-ir}\right]\, \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + ir)\, dr \\
& \quad + 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \left[\hat{h}(2 \log n - \log p) + \hat{h}(2 \log n + \log p) \right] \\
& \quad - \sum_{\substack{q | m \\ q \text{ prime}}} \sum_{k=0}^{\infty} \frac{\log q}{q^k} \left[\hat{h}(2k \log q - \log p) + \hat{h}(2k \log q + \log p) \right] \Bigg\} .
\end{aligned}$$

Here $X(n) = \prod_{k \bmod n} (k, n)$. In the elliptic case, m_γ denotes the order of the centraliser $Z_{\Gamma_0(m)}(\gamma) = \mathbb{Q}(\gamma) \cap \Gamma_0(m)$, and in the hyperbolic case ε_γ is a generator of $Z_{\Gamma_0(m)}(\gamma)$.

Proof. To start with, we recall some well-known facts about the trace formula for the non-cocompact Fuchsian group $\Gamma_0(m)$, compare [6, 9, 16]. One first defines the integral operator

$$Lg(z) = \int_{\mathcal{F}_m} K(z, w)\, g(w)\, d\mu(w) \ , \qquad g \in \mathcal{C}_m \ ,$$

whose kernel is constructed from a point-pair invariant $k(z, w) = \phi(\frac{|z-w|^2}{\operatorname{Im} z \operatorname{Im} w})$, where ϕ is an arbitrary function in $C_0^2(\mathbb{R})$, and

$$K(z, w) = \sum_{\gamma \in \Gamma_0(m)/\{\pm E_2\}} k(\gamma z, w) \ .$$

Since ϕ has compact support, $K(\cdot, w) \in L^2(X_m)$ for any fixed $w \in \mathcal{H}$. According to the spectral resolution of the Laplacian, one then obtains the spectral expansion

$$K(z, w) = \sum_{k=0}^{\infty} h(r_k)\, g_k(z)\, \overline{g_k}(w) + \frac{1}{4\pi} \sum_{v | m} \int_{-\infty}^{+\infty} h(r)\, E_{1/v}(z, \tfrac{1}{2} + ir)\, E_{1/v}(w, \tfrac{1}{2} - ir)\, dr \ , \tag{4.4}$$

see e.g. [9, Thm.7.4]. Here $\{g_k : k \in \mathbb{N}_0\}$ is a Hecke basis for $\mathbb{C} \oplus \mathcal{C}_m$, and $E_{1/v}(z,s)$ are the Eisenstein series (2.3) associated with $\frac{1}{v} \in F(m)$. The function $h(r)$ is related to the point-pair invariant via the Selberg transforms [14, (3.1)]

$$\begin{aligned} Q(t) &= \int_t^\infty \frac{\phi(u)}{\sqrt{u-t}}\, du \ , \\ \phi(u) &= -\frac{1}{\pi}\int_u^\infty \frac{Q'(t)}{\sqrt{t-u}}\, dt \ , \\ \hat{h}(u) &= Q\left(e^u + e^{-u} - 2\right) \ . \end{aligned} \tag{4.5}$$

One furthermore defines

$$\begin{aligned} H(z,w) &:= \frac{1}{4\pi}\sum_{v|m}\int_{-\infty}^{+\infty} h(r)\, E_{1/v}(z, \tfrac{1}{2}+ir)\, E_{1/v}(w, \tfrac{1}{2}-ir)\, dr \\ K_0(z,w) &:= K(z,w) - H(z,w) \ , \end{aligned}$$

and then proceeds to calculate $\operatorname{Tr} K_0$. This finally yields Proposition 4.3. Instead of this, we are here going to consider the operator T_pL, $p \not| m$, with kernel

$$K_p(z,w) = \frac{1}{\sqrt{p}}\sum_{j=0}^{p-1} K\left(\frac{z+j}{p}, w\right) + \frac{1}{\sqrt{p}} K\left(pz, w\right) = \frac{1}{\sqrt{p}} \sum_{\gamma \in M_p(m)/\{\pm E_2\}} k(\gamma z, w) \ , \tag{4.6}$$

where the last equality follows from (3.5). The spectral expansion of $K_p(z,w)$ can be derived from (4.4) and Lemma 4.4, together with the choice of $\{g_k : k \in \mathbb{N}_0\}$ as a Hecke basis,

$$\begin{aligned} K_p(z,w) = &\sum_{k=0}^\infty h(r_k)\, t_k(p)\, g_k(z)\, \overline{g_k}(w) \\ &+ \frac{1}{4\pi}\sum_{v|m}\int_{-\infty}^{+\infty} h(r)\ \left[p^{ir} + p^{-ir}\right]\ E_{1/v}(z, \tfrac{1}{2}+ir)\, E_{1/v}(w, \tfrac{1}{2}-ir)\, dr \ . \end{aligned} \tag{4.7}$$

In order to compute $\operatorname{Tr} K_0$, we follow the usual procedure and first express $K_p(z,z) - T_pH(z,z)$ in terms of the spectral expansion (4.7). An integration over the fundamental domain $\mathcal{F}_m$ yields the left-hand side of the trace formula in Theorem 4.5. Then the expansion (4.6) will be used to integrate $K_p(z,z) - T_pH(z,z)$ over the truncated fundamental domain $\mathcal{F}_m^Y$, and finally the limit $Y \to \infty$ will be taken. Here $\mathcal{F}_m^Y$ is the fundamental domain $\mathcal{F}_m$ with cuspidal regions $\sigma_v P^Y$ removed for $v|m$, where $P^Y := \{z = x + iy \in \mathcal{H} : -\frac{1}{2} \le x \le \frac{1}{2},\ y \ge Y\}$. We thus have to compute the right-hand side of

$$\begin{aligned} \sum_{k=0}^\infty t_k(p)\, h(r_k) = \lim_{Y\to\infty} \Bigg\{ &\int_{\mathcal{F}_m^Y} K_p(z,z)\ d\mu(z) - \frac{1}{4\pi}\sum_{v|m}\int_{-\infty}^{+\infty} h(r)\ \left[p^{ir} + p^{-ir}\right] \\ &\int_{\mathcal{F}_m^Y} E_{1/v}(z, \tfrac{1}{2}+ir)\, E_{1/v}(z, \tfrac{1}{2}-ir)\ d\mu(z)\, dr \Bigg\} \ . \end{aligned} \tag{4.8}$$

As a first step, we compute the contribution of $T_pH(z,z)$. To this end we recall the

Maaß-Selberg relation

$$\begin{aligned}\int_{\mathcal{F}_m^Y} E_{1/v}(z,\tfrac{1}{2}+ir)\, E_{1/v}(z,\tfrac{1}{2}-ir)\; d\mu(z) =& 2\log Y - \sum_{u|m} \Phi'_{vu}\left(\tfrac{1}{2}+ir\right)\Phi^{-1}_{uv}\left(\tfrac{1}{2}+ir\right)\\ &+ \frac{Y^{2ir}}{2ir}\,\Phi_{vv}\left(\tfrac{1}{2}-ir\right) - \frac{Y^{-2ir}}{2ir}\,\Phi_{vv}\left(\tfrac{1}{2}+ir\right) + o(1)\ ,\end{aligned} \tag{4.9}$$

as $Y\to\infty$, compare [9, Prop.6.8.]. Here $\Phi(s) = (\Phi_{uv}(s))$ denotes the scattering matrix for $\Gamma_0(m)$, whose explicit form can be found in [6, ch.11,Lem.4.6]:

$$\begin{aligned}\Phi_{vu}(s) &= \varphi(s) \prod_{p|(v,u)(\frac{m}{v},\frac{m}{u})} \frac{p-1}{p^{2s}-1} \prod_{p|(u,\frac{m}{v})(v,\frac{m}{u})} \frac{p^s-p^{1-s}}{p^{2s-1}}\ ,\\ \varphi(s) &= \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,\frac{\zeta(2s-1)}{\zeta(s)}\ .\end{aligned}$$

In order to perform the integration of $T_pH(z,z)$ over $\mathcal{F}_m^Y$, see (4.8), we now multiply (4.9) with $\frac{h(r)}{4\pi}\,[p^{ir}+p^{-ir}]$, integrate over $r\in\mathbb{R}$ and sum over the $2^{\omega(m)}$ divisors $v|m$. The first term on the right-hand side of (4.9) then contributes

$$\frac{2^{\omega(m)}}{2\pi}\log Y \int_{-\infty}^{+\infty} h(r)\ \left[p^{ir}+p^{-ir}\right]\ dr = 2^{\omega(m)+1}\,\hat{h}(\log p)\,\log Y\ . \tag{4.10}$$

For the second term, we imitate the computation in [6, pp.537] and use the known result for $m=1$ as found in [4, (11.10)]. This yields

$$\begin{aligned}-\frac{1}{4\pi}\int_{-\infty}^{+\infty} h(r)\ &\left[p^{ir}+p^{-ir}\right]\ \mathrm{Tr}\,\Phi'\Phi^{-1}\left(\tfrac{1}{2}+ir\right)\ dr =\\ -\,2^{\omega(m)}\Bigg\{ & 2\,\hat{h}(\log p)\,\log\pi - \frac{1}{2\pi}\int_{-\infty}^{+\infty} h(r)\ \left[p^{ir}+p^{-ir}\right]\ \frac{\Gamma'}{\Gamma}(\tfrac{1}{2}+ir)\ dr\\ &+2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\ \Big[\hat{h}(2\log n-\log p)+\hat{h}(2\log n+\log p)\Big]\\ &-\sum_{\substack{q|m\\ q\ \text{prime}}}\ \sum_{k=0}^{\infty}\frac{\log q}{q^k}\ \Big[\hat{h}(2k\log q-\log p)+\hat{h}(2k\log q+\log p)\Big]\Bigg\}\ .\end{aligned} \tag{4.11}$$

In order to treat the contributions from (4.9) still remaining, we follow [9, p.155] and find

$$\begin{aligned}\frac{1}{8\pi i}\int_{-\infty}^{+\infty}\frac{h(r)}{r}\ &\left[p^{ir}+p^{-ir}\right]\ \left[Y^{2ir}\ \mathrm{Tr}\,\Phi(\tfrac{1}{2}-ir) - Y^{-2ir}\ \mathrm{Tr}\,\Phi(\tfrac{1}{2}+ir)\right]\ dr\\ &= \tfrac{1}{2}\ \mathrm{Tr}\,\Phi(\tfrac{1}{2})\,h(0)+o(1)\\ &= 2^{\omega(m)-1}\,h(0)+o(1)\ .\end{aligned} \tag{4.12}$$

This completes the spectral side of the calculation.

Next we integrate $K_p(z,z)$, expressed as in (4.6), over $\mathcal{F}_m^Y$. First notice that $M_p(m)$ contains only elliptic and hyperbolic elements. One can therefore proceed as in the case of the cocompact Fuchsian group $\mathcal{O}^1$, see the right-hand side of (4.2). Again, the centraliser

$Z_{\Gamma_0(m)}(\gamma)$ of an element $\gamma \in M_p(m)$ is given by $\mathbb{Q}(\gamma) \cap \Gamma_0(m)$. In case $\mathbb{Q}(\gamma)$ is a field, this is the group of units of norm one in an order of the quadratic field $\mathbb{Q}(\gamma) \cong \mathbb{Q}(\sqrt{D})$. The only exception occurs for $\pm\operatorname{Tr}(\gamma) = p+1$, where $D=1$ and $\mathbb{Q}(\gamma)$ is not a field. In this case $Z_{\Gamma_0(m)}(\gamma)$ only consists of the identity, since $\pm\operatorname{Tr}(\gamma) = p+1$ and $\mathrm{N}(\gamma) = p$ implies that γ has two distinct fixed points on $\mathbb{Q} \cup \{\infty\}$. If there existed a non-trivial $\sigma \in Z_{\Gamma_0(m)}(\gamma)$, its fixed points would have to be $\Gamma_0(m)$-equivalent to the ones of γ, i.e. they would be two distinct cusps of $\Gamma_0(m)$, which is impossible. In the non-exceptional cases, $\pm\operatorname{Tr}(\gamma) \neq p+1$, one can proceed as after (4.2), thereby already performing the limit $Y \to \infty$. This yields the first two terms on the right-hand side of the trace formula in Theorem 4.5.

The exceptional cases require a separate treatment. Their contribution to the trace formula, before taking the limit $Y \to \infty$, reads

$$I(Y) := \frac{1}{\sqrt{p}} \sum_{\substack{\gamma \in M_p(m) \\ \operatorname{Tr}(\gamma)=p+1}} \int_{\mathcal{F}_m^Y} k(\gamma z, z)\ d\mu(z) = \frac{1}{\sqrt{p}} \sum_{\gamma_{v,n}} \int_{\widetilde{\mathcal{H}}^Y} k(\gamma_{v,n} z, z)\ d\mu(z)\ , \tag{4.13}$$

where the sum over $\gamma_{v,n}$ extends over the representatives of the exceptional conjugacy classes as described in Lemma 3.1. The domain of integration is given by

$$\begin{aligned} \widetilde{\mathcal{H}}^Y := \bigcup_{\alpha \in \Gamma_0(m)} \alpha \mathcal{F}_m^Y &= \mathcal{H} \setminus \bigcup_{u|m} \bigcup_{\alpha \in \Gamma_0(m)} \alpha \sigma_u P^Y \\ &= \mathcal{H} \setminus \bigcup_{u|m} \bigcup_{\tau \in \Gamma_\infty \backslash \sigma_u^{-1} \Gamma_0(m) \sigma_u} \sigma_u \tau P_\infty^Y\ , \end{aligned} \tag{4.14}$$

where $P_\infty^Y := \{z \in \mathcal{H} :\ \operatorname{Im} z \geq Y\} = \Gamma_\infty P^Y$. The limit $Y \to \infty$ cannot be taken before evaluating the integral on the right-hand side of (4.13), since the domain $\widetilde{\mathcal{H}}^Y$ approaches $\mathcal{H}$ in that limit and the integral over $\mathcal{H}$ diverges. One is therefore forced to evaluate the integral with the truncation of the domain present.

In order to proceed further we recall the double coset decomposition

$$\sigma_u^{-1} \Gamma_0(m) \sigma_u = \Omega_\infty \cup \bigcup_{c>0} \bigcup_{d \,(\mathrm{mod}\ mc)} \Omega_{d/mc}\ ,$$

see [9, Thm.2.7]. Here the set Ω_∞ is such that $\Gamma_\infty \backslash \Omega_\infty$ consists of a single class with representative $\omega_\infty = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in \sigma_u^{-1}\Gamma_0(m)\sigma_u$. Moreover, c and d run over numbers such that $\sigma_u^{-1}\Gamma_0(m)\sigma_u$ contains $\omega_{d/mc} = \left(\begin{smallmatrix} a & b \\ mc & d \end{smallmatrix}\right)$, and $\Gamma_\infty \backslash \Omega_{d/mc} = \omega_{d/mc}\Gamma_\infty$. Thus

$$\bigcup_{\tau \in \Gamma_\infty \backslash \sigma_u^{-1}\Gamma_0(m)\sigma_u} \tau P_\infty^Y = P_\infty^Y \cup \bigcup_{c>0} \bigcup_{d \,(\mathrm{mod}\ mc)} \omega_{d/mc} P_\infty^Y\ . \tag{4.15}$$

We now exploit the fact that $\alpha \partial P_\infty^Y$, $\alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{R})$, $C \neq 0$, is a horocycle of radius $\frac{1}{2C^2Y}$ touching $\partial\mathcal{H}$ at $\frac{A}{C}$, and that α^{-1} maps $\frac{A}{C}$ to ∞. In our situation, $\omega_{d/mc}P_\infty^Y$ hence is a disc of radius $\frac{1}{2m^2c^2Y}$ touching the real axis in the point $\frac{a}{mc}$, $(a, mc) = 1$, which is mapped to ∞ by $\omega_{d/mc}^{-1}$. This implies that, letting $\omega_{d/mc}$ run over the set as specified in (4.15), the points $\sigma_u(\frac{a}{mc})$ are comprised of all rational cusps which are $\Gamma_0(m)$-equivalent to $\frac{1}{u}$. According to (4.14), the domain $\widetilde{\mathcal{H}}^Y$ therefore consists of the hyperbolic plane with horocyclic neighbourhoods of all cusps removed.

Special attention has to be devoted to the horocycles touching the real axis in the fixed points of $\gamma_{v,n}$. The latter are given by $-\frac{n}{p-1-nv}$ and $\frac{1}{v}$. In the integral on the right-hand side of (4.13), we change variables from z to $w = \sigma_v^{-1} z$. Then $\gamma_{v,n}$ gets conjugated to the element $\sigma_v^{-1}\gamma_{v,n}\sigma_v$, which has fixed points $-\frac{v}{m}\frac{n}{p-1}$ and ∞. We thus notice that the horocycle at the fixed point $-\frac{v}{m}\frac{n}{p-1}$ corresponds to $\omega_{d/mc}P_\infty^Y$ with $a=\frac{vn}{(vn,mp-m)}$ and $mc=\frac{m(p-1)}{(vn,mp-m)}$. To simplify the integral further we now shift the variable w by $\frac{v}{m}\frac{n}{p-1}$, so that the fixed points of the transformation $\tilde{\gamma}_{v,n}$ defined by a conjugation of $\sigma_v^{-1}\gamma_{v,n}\sigma_v$ with the shift turn out to be 0 and ∞. The matrix representing $\tilde{\gamma}_{v,n}$ is hence diagonal. From the knowledge of $\mathrm{Tr}(\tilde{\gamma}_{v,n})=p+1$ and $\mathrm{N}(\tilde{\gamma}_{v,n})=p$ one furthermore concludes that one can choose $\tilde{\gamma}_{v,n}=\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$. We then denote the shifted domain of integration by $\widehat{\mathcal{H}}^Y$ and introduce the abbreviation $T:=p^{\frac{1}{2}}-p^{-\frac{1}{2}}$, so that

$$I(Y)=\frac{1}{\sqrt{p}}\sum_{\gamma_{v,n}}\int_{\widehat{\mathcal{H}}^Y}\phi\left(\frac{|\tilde{\gamma}_{v,n}z-z|^2}{\mathrm{Im}\,\tilde{\gamma}_{v,n}z\ \mathrm{Im}\,z}\right)d\mu(z)=\frac{1}{\sqrt{p}}\sum_{\gamma_{v,n}}\int_{\widehat{\mathcal{H}}^Y}\phi\left(T^2\left(1+\frac{x^2}{y^2}\right)\right)\frac{dx\,dy}{y^2}\ . \tag{4.16}$$

By construction, the domain $\widehat{\mathcal{H}}^Y$ consists of the hyperbolic plane $\mathcal{H}$ where discs, $U_{a/mc}^Y:=\omega_{d/mc}P_\infty^Y+\frac{v}{m}\frac{n}{p-1}$ of specified radii and touching the boundary $\partial\mathcal{H}$ at the cusps $\frac{a}{mc}$, are removed.

It now turns out that any divergence of $I(Y)$ in the limit $Y\to\infty$ comes from the contributions of the cusps that are fixed points of the transformation appearing under the integral. In (4.16) these cusps are 0 and ∞. To see the finiteness in the other cusps, we consider the integral over any of the discs U_ξ^Y for $\xi\in\mathbb{Q}\setminus\{0\}$,

$$\left|\int_{U_\xi^Y}\phi\left(T^2\left(1+\frac{x^2}{y^2}\right)\right)\frac{dx\,dy}{y^2}\right|\leq\mathrm{meas}\left(U_\xi^Y\right)\cdot\sup_{z\in U_\xi^Y}\left|y^{-2}\phi\left(T^2\left(1+\frac{x^2}{y^2}\right)\right)\right|\ , \tag{4.17}$$

where $\mathrm{meas}\,(U_\xi^Y)$ denotes Lebesgue measure. Since $\phi\in C_0^2(\mathbb{R})$ is supposed to be compactly supported, $\phi(u)=0$ for $|u|>K$ with some sufficiently large constant $K>0$. In particular, we choose $K>T^2$. Then $\phi(T^2(1+\frac{x^2}{y^2}))=0$ if $y<|x|(\frac{K}{T^2}-1)^{-\frac{1}{2}}$. In other words, we obtain the bound $y^{-2}\leq x^{-2}(\frac{K}{T^2}-1)$ in the support of ϕ. If Y is large enough, then $x\neq 0$ for any $x+iy\in U_\xi^Y$. Thus the right-hand side of (4.17) is finite, and vanishes as $Y\to\infty$. Consequently,

$$I(Y)=\frac{1}{\sqrt{p}}\sum_{\gamma_{v,n}}\int_{\mathcal{H}\setminus(U_0^Y\cup U_\infty^Y)}\phi\left(T^2\left(1+\frac{x^2}{y^2}\right)\right)\frac{dx\,dy}{y^2}+o(1)\ .$$

We remark that the integrals depend on $\gamma_{v,n}$ through the radius of the disc U_0^Y, since this is a shift of the disc at the fixed point $\frac{v}{m}\frac{n}{p-1}$ of $\gamma_{v,n}$. Therefore, the radius of U_0^Y is

$$R_{v,n}=\frac{1}{2m^2c^2Y}=\frac{(vn,mp-m)^2}{2m^2(p-1)^2Y}\ . \tag{4.18}$$

We now introduce polar coordinates $r\in\mathbb{R}_+$ and $\theta\in[0,\pi]$, $z=x+iy=r\,e^{i\theta}$, and obtain

$$\begin{aligned}\int_{\mathcal{H}\setminus(U_0^Y\cup U_\infty^Y)}\phi\left(T^2\left(1+\frac{x^2}{y^2}\right)\right)\frac{dx\,dy}{y^2}&=\int_0^\pi\int_{2R_{v,n}\sin\theta}^{\frac{Y}{\sin\theta}}\phi\left(\frac{T^2}{\sin^2\theta}\right)\frac{1}{r\,\sin^2\theta}\,dr\,d\theta\\&=\int_0^\pi\log\left(\frac{Y}{2R_{v,n}\sin^2\theta}\right)\phi\left(\frac{T^2}{\sin^2\theta}\right)\frac{1}{\sin^2\theta}\,d\theta\\&=\frac{1}{T}\int_{T^2}^\infty\log\left(\frac{Yu}{2R_{v,n}T^2}\right)\frac{\phi(u)}{\sqrt{u-T^2}}\,du\ .\end{aligned}$$

In the last line the variable $u := \frac{T^2}{\sin^2\theta}$ has been introduced. In order to proceed further, we recall (4.5) and find

$$\int_{T^2}^{\infty} \frac{\phi(u)}{\sqrt{u-T^2}}\, du = Q\left(p+p^{-1}-2\right) = \hat{h}(\log p) \ .$$

Moreover, using (4.5) and changing the order of the integrations one observes

$$\begin{aligned}\int_{T^2}^{\infty} \frac{\log u}{\sqrt{u-T^2}}\, \phi(u)\ du &= -\frac{1}{\pi}\int_{T^2}^{\infty} \frac{\log u}{\sqrt{u-T^2}} \int_u^{\infty} \frac{Q'(t)}{\sqrt{t-u}}\, dt\, du \\ &= -\frac{1}{\pi}\int_{T^2}^{\infty} Q'(t) \int_{T^2}^{t} \frac{\log u}{\sqrt{u-T^2}\sqrt{t-u}}\, du\, dt \ .\end{aligned}$$

The integral over u can be calculated to yield $2\pi\left[\log(T+\sqrt{t})-\log 2\right]$, so that an integration by parts yields

$$\begin{aligned}\int_{T^2}^{\infty} \frac{\log u}{\sqrt{u-T^2}}\, \phi(u)\ du &= 2\log T\, \hat{h}(\log p) + \int_{T^2}^{\infty} \frac{Q(t)}{T+\sqrt{t}}\ \frac{dt}{\sqrt{t}} \\ &= 2\log T\, \hat{h}(\log p) + \int_{\log p}^{\infty} \hat{h}(u)\, \frac{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}{e^{\frac{u}{2}}-e^{-\frac{u}{2}}+p^{\frac{1}{2}}-p^{-\frac{1}{2}}}\ du \ ,\end{aligned}$$

where the last line was obtained through the change of variables $t = e^u + e^{-u} - 2$. Collecting the above results one obtains

$$\begin{aligned}\frac{1}{\sqrt{p}} \int_{\mathcal{H}\setminus(U_0^Y\cup U_\infty^Y)} \phi\left(T^2\left(1+\frac{x^2}{y^2}\right)\right)\ \frac{dx\,dy}{y^2} &= \frac{\hat{h}(\log p)}{p-1}\ \log\left(\frac{Y}{2R_{v,n}}\right) \\ &\quad + \frac{1}{p-1}\int_{\log p}^{\infty} \hat{h}(u)\, \frac{e^{\frac{u}{2}}+e^{-\frac{u}{2}}}{e^{\frac{u}{2}}-e^{-\frac{u}{2}}+p^{\frac{1}{2}}-p^{-\frac{1}{2}}}\ du \ .\end{aligned}$$

In order to compute $I(Y)$ we still have to perform the sum over the $2^{\omega(m)}(p-1)$ representatives $\gamma_{v,n}$, which is non-trivial only for the term containing $R_{v,n}$. According to (4.18) and Lemma 3.1 this sum reads

$$\begin{aligned}\sum_{\gamma_{v,n}} \log\left(\frac{Y}{2R_{v,n}}\right) &= 2^{\omega(m)+1}(p-1)\ \log Y + 2\sum_{\gamma_{v,n}} \log\left(\frac{m(p-1)}{(vn, mp-m)}\right) \\ &= 2^{\omega(m)+1}(p-1)\ \log Y + 2\sum_{v|m}\sum_{k=0}^{p-2} \log\left(\frac{p-1}{(k,p-1)}\right) \\ &= 2^{\omega(m)+1}(p-1)\ \log Y + 2^{\omega(m)+1}\left[(p-1)\ \log(p-1) - \log X(p-1)\right] \ ,\end{aligned}$$

where the function $X(n) = \prod_{k \bmod n} (k,n)$ has been introduced. Collecting all contributions to $I(Y)$, one first of all observes that the only term that diverges in the limit $Y\to\infty$ is identical to (4.10).

According to (4.8) the right-hand side of the trace formula follows, if we add the contributions of the elliptic and of the non-exceptional hyperbolic conjugacy classes, add $I(Y)$, and subtract the contributions (4.10), (4.11) and (4.12) of the Eisenstein series. The divergent parts therefore cancel, and the limit $Y\to\infty$ can be performed. One thus obtains the right-hand side of the trace formula in Theorem 4.5, however, for a restricted class of test functions

h. Since up to now we required the point-pair invariant ϕ to be twice differentiable and compactly supported, the same properties hold for $\hat{h}$. Then h is smooth and $h(r) = O(r^{-2})$ as $|r| \to \infty$, see [5, ch.I,Prop.4.1]. An extension to the class of test functions introduced at the beginning of this section is possible since all sums and integrals appearing in the trace formula converge absolutely with these test functions. The extension goes via an approximation argument as in [5, pp.32] and [6, Thm.13.8]. This is a standard step and will not be reproduced here, see also [9, Thm.10.2]. $\square$

5 Main theorems

We will need to consider trace formulae for several groups, and to distinguish them we let

$$\sum_{g_k \in S} h(r_k)$$

denote the sum over a Hecke basis $\{g_k\}$ of S, where $\lambda_k = r_k^2 + \frac{1}{4}$ is the Laplace eigenvalue of g_k and h is a test function as specified at the beginning of Section 4.

Theorem 5.1. *Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion division algebra with discriminant d. Then the positive Laplace eigenvalues, including multiplicities, on $X_{\mathcal{O}}$ coincide with the Laplace spectrum on Maaß-newforms for the Hecke congruence group $\Gamma_0(d)$.*

Proof. We already showed in [2, Lem.5.4] that all eigenvalues of $-\Delta$ on $L^2(X_{\mathcal{O}})$ also occur as eigenvalues of $-\Delta$ on $L^2(X_d)$. It thus suffices to show that

$$\sum_{g_k \in L^2(X_{\mathcal{O}})} h(r_k) = \sum_{g_k \in \mathbb{C} \oplus \mathcal{C}_d^{new}} h(r_k) \tag{5.1}$$

for an arbitrary test function h.

In order to prove (5.1), we will use the right-hand sides of the trace formulae in Propositions 4.1 and 4.3. To be able to apply this procedure to the right-hand side of (5.1), we observe that (2.7) implies

$$\sum_{g_k \in \mathcal{C}_d^{new}} h(r_k) = \sum_{m|d} \beta\big(\tfrac{d}{m}\big) \sum_{g_k \in \mathcal{C}_m} h(r_k). \tag{5.2}$$

We now apply the first identity of Lemma 3.5 to obtain

$$\sum_{m|d} \beta\big(\tfrac{d}{m}\big) \sum_{g_k \in \mathbb{C} \oplus \mathcal{C}_m} h(r_k) = h(r_0) + \sum_{m|d} \beta\big(\tfrac{d}{m}\big) \sum_{g_k \in \mathcal{C}_m} h(r_k) = \sum_{g_k \in \mathbb{C} \oplus \mathcal{C}_d^{new}} h(r_k)\ .$$

Using Propositions 4.1 and 4.3, we therefore have to show that

$$\sum_{g_k \in L^2(X_{\mathcal{O}})} h(r_k) = \sum_{m|d} \beta\big(\tfrac{d}{m}\big) \sum_{g_k \in \mathbb{C} \oplus \mathcal{C}_m} h(r_k)\ . \tag{5.3}$$

For the terms corresponding to the identity on the right-hand sides of the trace formulae, one has to show that

$$A_{\mathcal{O}} = \sum_{m|d} \beta\big(\tfrac{d}{m}\big)\, A_m.$$

This, however, was already proved in [2] using (2.1) and (2.2).

In the elliptic and hyperbolic terms, only the numbers $E'(t,1,\Gamma)$ depend on the group Γ. Hence, that these terms are identical is equivalent to

$$E'(t,1,\mathcal{O}^1) = \sum_{m|d} \beta\big(\tfrac{d}{m}\big)\, E'\big(t,1,\Gamma_0(m)\big)$$

for all traces t. But this is exactly Lemma 3.4.

Now it only remains to prove that the parabolic terms vanish; but this follows from Lemma 3.5 since these are of the form

$$\sum_{m|d} \beta\big(\tfrac{d}{m}\big)\, 2^{\omega(m)} \Big(C + \sum_{p|m} f(p)\Big),$$

where C is a constant (i.e. independent of m) and f only depends on p. □

We define V_λ^{new} to be the subspace of $\mathcal{C}_d^{new}$ with Laplace eigenvalue λ, and W_λ to be the corresponding subspace of $L^2(X_\mathcal{O})$. The statement of Theorem 5.1 is exactly $\dim V_\lambda^{new} = \dim W_\lambda$ for all λ. But more generally, we will now prove that the traces of the Hecke operators T_p and $\widetilde{T}_p$, restricted to V_λ^{new} and W_λ respectively, coincide when $p \nmid d$.

Theorem 5.2. *Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion division algebra with discriminant d. If $p \nmid d$, then the traces of T_p on V_λ^{new} and $\widetilde{T}_p$ on W_λ coincide for all λ.*

Proof. Let $\lambda = r^2 + \frac{1}{4} > 0$ be a Laplace eigenvalue that occurs in the newform spectrum of $\Gamma_0(d)$. Choose a Hecke basis for $\mathcal{C}_d^{new}$ such that $\{g_1, \dots, g_N\}$ span V_λ^{new}. According to Theorem 5.1, then one can choose a Hecke basis $\{\varphi_1, \dots, \varphi_N\}$ of W_λ. Let $t_k(p)$ be the p-th Hecke eigenvalue of g_k and $\tilde{t}_k(p)$ the p-th Hecke eigenvalue of φ_k. We have to show that

$$\sum_{k=1}^{N} \tilde{t}_k(p) = \sum_{k=1}^{N} t_k(p) \tag{5.4}$$

for any prime p with $p \nmid d$. In order to prove this identity, we will make use of the trace formulae for Hecke operators contained in Proposition 4.2 and Theorem 4.5, respectively, because it suffices to prove that

$$\sum_{g_k \in L^2(X_\mathcal{O})} \tilde{t}_k(p)\, h(r_k) = \sum_{g_k \in \mathbb{C} \oplus \mathcal{C}_d^{new}} t_k(p)\, h(r_k) \tag{5.5}$$

for arbitrary test functions h. Since $p \nmid d$, the Hecke operators T_p on $\mathcal{C}_m$ are defined the same way for all $m|d$. We can therefore employ the same strategy as in the proof of Theorem 5.1, see (5.3), according to which (5.5) is equivalent to

$$\sum_{g_k \in L^2(X_\mathcal{O})} \tilde{t}_k(p)\, h(r_k) = \sum_{m|d} \beta\big(\tfrac{d}{m}\big) \sum_{g_k \in \mathbb{C} \oplus \mathcal{C}_m} t_k(p)\, h(r_k) \ .$$

There are no terms with trace $t = p+1$ occurring in the trace formula for the left-hand side, since $\mathcal{O}$ is an order in a division algebra. The fact that the terms with $t = p+1$ as well as the contributions from the Eisenstein series to the trace formula for the right-hand side vanish follows from Lemma 3.5, since these are of the form

$$\sum_{m|d} \beta\big(\tfrac{d}{m}\big)\, 2^{\omega(m)} \Big(C + \sum_{\substack{q|m \\ q \text{ prime}}} f(q)\Big) \ ,$$

where C is a constant (i.e. independent of m) and f only depends on q.

When comparing the elliptic and the remaining hyperbolic terms, one has to be a little more careful than in the proof of Theorem 5.1. First we divide the terms for a fixed trace into separate sums corresponding to the different orders B optimally embedded into $\mathcal{O}$ and $\Gamma_0(m)$, respectively. Then we observe that the centraliser, and hence m_γ and ε_γ, only depends on B, since it is precisely the elements in B with norm equal to 1. Hence the identity of the elliptic and the remaining hyperbolic terms follows from (3.6), and thus (5.4) is established. $\square$

Corollary 5.3. *If $\lambda > 0$ is a Laplace eigenvalue on $L^2(X_\mathcal{O})$ with $\dim W_\lambda = 1$, then $\Theta(W_\lambda) = V_\lambda^{new}$.*

Proof. Let φ be the element of a Hecke basis for $L^2(X_\mathcal{O})$ that spans W_λ. Then $\Theta(\varphi) \in \mathcal{C}_d$ with Laplace eigenvalue λ, see [2, Prop.5.1]. Moreover, according to [2, Prop.6.1] $\widetilde{T}_p\varphi = \tilde{t}(p)\varphi$ implies that $T_p\Theta(\varphi) = \tilde{t}(p)\Theta(\varphi)$. If g is the element of a Hecke basis for $\mathcal{C}_d$ that spans V_λ^{new}, Theorem 5.2 says that $\Theta(\varphi)$ and g have the same Hecke eigenvalues for all $p \nmid d$. The corollary then follows from the non-holomorphic analogue of [1, Lem.20]. $\square$

As a final remark let us mention that for maximal orders $\mathcal{O}$ the Laplace spectrum of $L^2(X_\mathcal{O})$ is conjectured to be simple, see e.g. [2]. According to Corollary 5.3, this would imply that $\Theta(L_0^2(X_\mathcal{O})) = \mathcal{C}_d^{new}$. Following Theorem 5.1, the simplicity of the Laplace spectrum of $L^2(X_\mathcal{O})$ is equivalent to a simple Laplace spectrum on $\mathcal{C}_d^{new}$. The latter has long been conjectured, see e.g. [3], and found confirmation in extensive numerical calculations of eigenvalues for some groups, see e.g. [3, 8, 15].

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