GENERA OF ARITHMETIC FUCHSIAN GROUPS

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Introduction

The fundamental invariant of a Riemann surface is its genus. In this paper, using arithmetical means, we calculate the genus of certain Riemann surfaces defined by unit groups in quaternion algebras.

First we recall a well-known general construction of Riemann surfaces. The group $SL_2(\mathbb{R})$ acts on the upper half-plane $\mathcal{H}$ by Möbius transformations. If G is a Fuchsian group, that is, a discrete subgroup of $SL_2(\mathbb{R})$, then it is possible to provide the quotient space $G\backslash\mathcal{H}$ with the structure of a Riemann surface. A distinguished class of Fuchsian groups are the arithmetic ones. These are by definition groups commensurable with unit groups in quaternion orders. The best known example of an arithmetic Fuchsian group is the modular group $SL_2(\mathbb{Z})$.

The genus of the (compactification of the) surfaces corresponding to certain subgroups of $SL_2(\mathbb{Z})$ is well investigated, see for example the first chapter in [16]. A more general investigation can be found in [4], where the authors, among other things, derive a list of all congruence subgroups of $SL_2(\mathbb{Z})$, which give Riemann surfaces with genus equal to 0. Another related result is the determination of all arithmetic triangular groups in [17].

In this paper, we will consider the case of orders in quaternion *division* algebras. This case contains all arithmetic Fuchsian groups, except those commensurable with $SL_2(\mathbb{Z})$. There exists a general formula for the genus in this case. However, the implementation of this was only known explicitly in the simplest case of maximal orders in algebras over $\mathbb{Q}$. The main purpose of the paper is to generalise this to arbitrary orders in rational quaternion algebras and also to maximal orders in algebras over quadratic fields. As an application of these explicit formulas, we give complete lists of all such orders for which the genus is less than or equal to 2.

In Section 1, we give the necessary background and some notations. The following two sections contain general formulas for the area of fundamental domains and the number of elliptic points, which are the ingredients in the genus formula. Section 4 contains auxiliary results on Bass orders, which we will need. In Section 5, we give a completely explicit formula for the genus of $G\backslash\mathcal{H}$, where G corresponds to an arbitrary order in a rational quaternion algebra. We also give a complete list of all orders in rational quaternion algebras for which the genus is less than or equal to 2. This is a complement to the list announced in [4] of all congruence subgroups of $SL_2(\mathbb{Z})$ with genus equal to 0. The last section contains the corresponding result for maximal orders in quaternion algebras over quadratic fields.

I would like to thank J. Brzezinski for helpful discussions and for carefully reading this manuscript and suggesting several improvements.

1. Background and notations

Let $\mathcal{H}$ be the complex upper half-plane model of the hyperbolic plane equipped with the hyperbolic metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

The group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm 1\}$ acts on $\mathcal{H}$ by Möbius transformations

$$g \cdot z = \frac{az+b}{cz+d}, \text{ where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{1.1}$$

Let G be a Fuchsian group, that is, a discrete subgroup of $PSL_2(\mathbb{R})$. It is well-known, that one can give the quotient space $G\backslash\mathcal{H}$ the structure of a Riemann surface. We will from now on assume that $G\backslash\mathcal{H}$ is compact, and that the hyperbolic measure μ of a fundamental domain $\mathcal{F}_G$ of $G\backslash\mathcal{H}$ is finite. Then the genus of the surface $G\backslash\mathcal{H}$ is given by the formula ([10], p. 91)

$$g = g(G) = \frac{1}{4\pi}\mu(\mathcal{F}_G) + 1 - \sum_{m\geq 2} e(m,G)\frac{m-1}{2m}, \tag{1.2}$$

where $e(m,G)$ is the number of elliptic points of order m on $G\backslash\mathcal{H}$.

In this paper, we will investigate the genus of $G\backslash\mathcal{H}$, when G is arithmetic. These groups are related to orders in quaternion algebras. Therefore, we first give the necessary background on quaternion algebras.

Let K be a totally real algebraic number field with ring of integers R and $[K:\mathbb{Q}] = n$. The subgroup of squares in R^* will be denoted by R^{*^2}, and the group of totally positive units by R^{*^+}. If $\mathfrak{p}$ is a prime ideal in R, then $K_\mathfrak{p}$ will be the completion of K corresponding to $\mathfrak{p}$ and $R_\mathfrak{p}$ the ring of integers in $K_\mathfrak{p}$.

Let $\mathfrak{A}$ be a quaternion algebra over K. It is always possible to find a basis 1, i, i, ij of $\mathfrak{A}$ over K satisfying

$$i^2 = a,\ j^2 = b \text{ and } ij = -ji, \text{ where } a, b \in K^*.$$

This algebra will be denoted by $(a,b)_K$. There is a natural anti-involution in $\mathfrak{A}$, which in $(a,b)_K$ is given by

$$x = x_0 + x_1 i + x_2 j + x_3 ij \longmapsto \bar{x} = x_0 - x_1 i - x_2 j - x_3 ij.$$

The (reduced) trace and (reduced) norm in $\mathfrak{A}$ are defined by $Tr(x) = x + \bar{x}$ and $N(x) = x\bar{x}$.

We will assume that $\mathfrak{A}$ is a division algebra, that is, $\mathfrak{A} \not\cong M_2(K)$, and that

$$\mathfrak{A} \otimes_\mathbb{Q} \mathbb{R} \cong M_2(\mathbb{R}) \times \mathbb{H}^{n-1}, \tag{1.3}$$

where $\mathbb{H}$ is the Hamiltonian quaternion algebra. Another way to phrase the condition (1.3) is that exactly one of the n real completions of $\mathfrak{A}$ gives the matrices $M_2(\mathbb{R})$. We will throughout the paper identify $\mathfrak{A}$ with an embedding into $M_2(\mathbb{R})$ corresponding to this unique completion.

Let $\mathfrak{p}$ be a prime ideal in R. Then $\mathfrak{A}_\mathfrak{p} := \mathfrak{A} \otimes_K K_\mathfrak{p}$ is a quaternion algebra over $K_\mathfrak{p}$. It is well-known that $\mathfrak{A}_\mathfrak{p}$ is either isomorphic to $M_2(K_\mathfrak{p})$ or to a unique division algebra, which we will denote by $\mathbb{H}_\mathfrak{p}$. The algebra is said to be ramified at $\mathfrak{p}$ if $\mathfrak{A}_\mathfrak{p} \cong \mathbb{H}_\mathfrak{p}$, otherwise it is said to split at $\mathfrak{p}$. Let $(\ , \ )_\mathfrak{p}$ be the Hilbert symbol at $\mathfrak{p}$. Then $(a,b)_K$ is ramified at $\mathfrak{p}$ iff $(a,b)_\mathfrak{p} = -1$. We define the discriminant $d(\mathfrak{A})$ of $\mathfrak{A}$ to be the product of the prime ideals where it is ramified.

An (R-)order $\mathcal{O}$ in a quaternion algebra $\mathfrak{A}$ is a ring containing R, which is a finitely generated R-module, such that $K\mathcal{O} = \mathfrak{A}$. The R-ideal generated by $\det(Tr(x_i\bar{x}_j))$, where $x_1, \dots, x_4 \in \mathcal{O}$, is the square of an ideal $d(\mathcal{O})$. The ideal $d(\mathcal{O})$ is the (reduced) discriminant of $\mathcal{O}$. The dual lattice $\mathcal{O}^\#$ of $\mathcal{O}$ is defined to be

$$\mathcal{O}^\# = \{x \in \mathfrak{A} : Tr(x\mathcal{O}) \subseteq R\}.$$

If $\mathfrak{p}$ is a prime ideal in R, then $\mathcal{O}_\mathfrak{p} := \mathcal{O} \otimes_R R_\mathfrak{p}$.

In [5], Eichler introduced a local invariant of $\mathcal{O}$ depending on the structure of $\mathcal{O}_\mathfrak{p}/J(\mathcal{O}_\mathfrak{p})$, where $J(\mathcal{O}_\mathfrak{p})$ is the Jacobson radical of $\mathcal{O}_\mathfrak{p}$. If $\mathcal{O} \ncong M_2(R_\mathfrak{p})$, then it is defined by

$$(1.4) \quad e(\mathcal{O}_\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathcal{O}_\mathfrak{p}/J(\mathcal{O}_\mathfrak{p}) \cong K_\mathfrak{p} \oplus K_\mathfrak{p}, \\ 0 & \text{if } \mathcal{O}_\mathfrak{p}/J(\mathcal{O}_\mathfrak{p}) \cong K_\mathfrak{p}, \\ -1 & \text{if } \mathcal{O}_\mathfrak{p}/J(\mathcal{O}_\mathfrak{p}) \text{ is a quadratic field extension of } K_\mathfrak{p}. \end{cases}$$

Two orders $\mathcal{O}$ and $\mathcal{O}'$ are said to be in the same genus, if $\mathcal{O}_\mathfrak{p} \cong \mathcal{O}'_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. The number of isomorphism classes in the genus of $\mathcal{O}$ is called the type number of $\mathcal{O}$. It will be denoted by $t(\mathcal{O})$.

The class number of orders, both in fields and quaternion algebras, will be important in this paper. It will be denoted by $h(M)$, where M is the order. All class numbers will be with respect to locally principal ideals. If $M = \mathcal{O}$ is an order in a quaternion algebra, then $h(\mathcal{O})$ is the one-sided class number, that is, the number of classes of locally principal right $\mathcal{O}$-ideals modulo principal right $\mathcal{O}$-ideals. Instead of right $\mathcal{O}$-ideals, we could equally well take left $\mathcal{O}$-ideals. The type and class number of a quaternion order is related by ([19], p. 88)

$$h(\mathcal{O}) = \sum_{i=1}^{t(\mathcal{O})} H(\mathcal{O}_i),$$

where $\mathcal{O}_1, \dots, \mathcal{O}_{t(\mathcal{O})}$ is a set of representatives of the isomorphism classes in the genus of $\mathcal{O}$, and $H(\mathcal{O}_i)$ is the two-sided class number of $\mathcal{O}_i$, that is, the number of classes of locally principal two-sided $\mathcal{O}$-ideals modulo principal two-sided $\mathcal{O}$-ideals. In particular, $h(\mathcal{O}) = h(\mathcal{O}')$ if $\mathcal{O}$ and $\mathcal{O}'$ are in the same genus.

Let E be any group satisfying $R^{*^2} \subseteq E \subseteq R^{*^+}$. Given an order $\mathcal{O}$ and a group E, we define $\Gamma = \Gamma_E(\mathcal{O}) \subseteq M_2(\mathbb{R})$ to be

$$\Gamma = \{x \in \mathcal{O} : N(x) \in E\}.$$

We can embed Γ in $SL_2(\mathbb{R})$ by

$$x \longmapsto \frac{x}{\sqrt{\det(x)}},$$

and it is well-known that the condition (1.3) is equivalent to Γ being a Fuchsian group with $\mu(\mathcal{F}_\Gamma)$ finite. Moreover, $\mathfrak{A}$ a division algebra implies that $\Gamma\backslash\mathcal{H}$ is compact. We remark that if $E = R^{*^2}$, then $\Gamma_E(\mathcal{O})$ is naturally isomorphic to $\mathcal{O}^1 = \{x \in \mathcal{O} : N(x) = 1\}$. A Fuchsian group is by definition arithmetic iff it is commensurable with any such $\Gamma_E(\mathcal{O})$.

2. Area of fundamental domains

First assume that $\mathcal{O}$ is a maximal order in $\mathfrak{A}$ and $\Gamma = \Gamma_E(\mathcal{O})$. If $\mathcal{F}$ is a fundamental domain of $\Gamma\backslash\mathcal{H}$, then

$$\mu(\mathcal{F}) = (-1)^n \frac{\pi\zeta_K(-1)}{2^{n-3}[E:U^2]} \prod_{\mathfrak{p}\mid d(\mathfrak{A})} (N\mathfrak{p} - 1), \tag{2.1}$$

where ζ_K is the zeta function of K.

A proof of (2.1) can be found in [18]. However, the formula was well-known more than 10 years before – Shimizu states it in [15]. We have specialised the more general statement in [18] to our case. Moreover, we have a different normalization of the hyperbolic measure and have used the functional equation of ζ_K to simplify the formula.

If $\Gamma_1 \subseteq \Gamma_2$ are two Fuchsian groups with finite index $[\Gamma_2 : \Gamma_1]$, then

$$\mu(\Gamma_1) = [\Gamma_2 : \Gamma_1] \cdot \mu(\Gamma_2).$$

So in order to determine the area in general, we need to determine this index. This can be done by local calculations thanks to the following lemma.

(2.2) Lemma. *Let $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$ be two orders and $R^{*^2} \subseteq E \subseteq \widetilde{E} \subseteq R^{*^+}$. Then*

$$[\widetilde{\mathcal{O}}^1 : \mathcal{O}^1] = \prod_{\mathfrak{p}} [\widetilde{\mathcal{O}}^1_{\mathfrak{p}} : \mathcal{O}^1_{\mathfrak{p}}]$$

and

$$[\Gamma_{\widetilde{E}}(\widetilde{\mathcal{O}}) : \Gamma_E(\mathcal{O})] = [\widetilde{\mathcal{O}}^1 : \mathcal{O}^1] \cdot [N(\Gamma_{\widetilde{E}}(\widetilde{\mathcal{O}})) : N(\Gamma_E(\mathcal{O}))].$$

Proof. A proof of the first equality can be found in [19], p. 107. The second follows from the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}^1 & \longrightarrow & \Gamma_E(\mathcal{O}) & \xrightarrow{N} & N(\Gamma_E(\mathcal{O})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \widetilde{\mathcal{O}}^1 & \longrightarrow & \Gamma_{\widetilde{E}}(\widetilde{\mathcal{O}}) & \xrightarrow{N} & N(\Gamma_{\widetilde{E}}(\widetilde{\mathcal{O}})) & \longrightarrow & 1
\end{array}$$ $\square$

The calculation of the local indices is not complicated, and we have the following result:

(2.3) Lemma. *Let* $\mathcal{O}_\mathfrak{p} \subsetneq \mathcal{M}_\mathfrak{p}$*, where* $\mathcal{M}_\mathfrak{p}$ *is a maximal order in* $\mathfrak{A}_\mathfrak{p}$ *and* $d(\mathcal{O}_\mathfrak{p}) = \mathfrak{p}^n$*. Then*

$$[\mathcal{M}_\mathfrak{p}^1 : \mathcal{O}_\mathfrak{p}^1] = [\mathcal{M}_\mathfrak{p}^* : \mathcal{O}_\mathfrak{p}^*] \cdot [R_\mathfrak{p}^* : N(\mathcal{O}_\mathfrak{p}^*)]^{-1}$$

and

$$[\mathcal{M}_\mathfrak{p}^* : \mathcal{O}_\mathfrak{p}^*] = \begin{cases} q^{n-1}(q^2-1)(q-e(\mathcal{O}_\mathfrak{p}))^{-1} & \text{if } \mathfrak{A}_\mathfrak{p} \cong M_2(K_\mathfrak{p}), \\ q^{n-1}(q+1)(q-e(\mathcal{O}_\mathfrak{p}))^{-1} & \text{if } \mathfrak{A}_\mathfrak{p} \cong \mathbb{H}_\mathfrak{p}, \end{cases}$$

where $q = N\mathfrak{p}$*.*

Proof. It is well-known that the norm $N : \mathcal{M}_\mathfrak{p}^* \longrightarrow R_\mathfrak{p}^*$ is surjective, so the first equality follows from a similar diagram as the one in the proof of (2.2) with $\mathcal{O}_\mathfrak{p}$ and $\mathcal{M}_\mathfrak{p}$ instead of $\mathcal{O}$ and $\mathcal{M}$ and $E = \widetilde{E} = R_\mathfrak{p}^*$. The second equality is contained in [11], Satz 1. $\square$

If we collect the results of this section, we get a formula of the area of the fundamental domain which only contains $\zeta_K(-1)$, the Eichler invariants $e(\mathcal{O}_\mathfrak{p})$ and the image of the norm.

3. Elliptic elements

In order to determine the number of elliptic points, one can make use of a correspondence between some special orders in maximal subfields of $\mathfrak{A}$ and elliptic points. We remark, that determining the number of elliptic points on $\Gamma\backslash\mathcal{H}$ is equivalent to determining the number of conjugacy classes of elliptic elements in Γ. Let $\mathcal{O}$ be an order in $\mathfrak{A}$, and $\Omega = \Omega_\Gamma$ the set of subrings S of $\mathfrak{A}$ satisfying:

1. $L = K(S)$ is a totally imaginary subfield of $\mathfrak{A}$.
2. $S = L \cap \mathcal{O}$, that is, S is optimally embedded in $\mathcal{O}$.
3. $\Gamma \cap S \neq R^*$.

Two orders $\mathcal{O}_1$ and $\mathcal{O}_2$ in Ω_Γ are Γ-conjugated if there exists $x \in \Gamma$, such that $\mathcal{O}_1 = x\mathcal{O}_2 x^{-1}$. Now there is a one-to-one correspondence between elliptic points on $\Gamma\backslash\mathcal{H}$ and Γ-conjugacy classes in Ω given as follows: Let $x \in \Gamma$ be a generator of the fix group of an elliptic point z of $\Gamma\backslash\mathcal{H}$. Then the order corresponding to z is $S = K(x) \cap \mathcal{O}$. Conversely given $S \in \Omega$, then $\Gamma \cap S$ is the non-trivial fix group of some $z \in \Gamma\backslash\mathcal{H}$.

The problem to determine the number of Γ-conjugacy classes in Ω is naturally divided into two parts. First one divides Ω into isomorphism classes and calculates the number of Γ-conjugacy classes in each isomorphism class. This was first done by Shimizu in [15]. In [14], Schneider filled in some details in Shimizu's work and also considered the problem to determine the different isomorphism classes.

Both these papers only consider maximal orders. However, it is easy to deduce from [14], that the formulas remain true if only the order $\mathcal{O}$ satisfies the following conditions:

(3.1) Condition. 1. If $\mathcal{O} \cong \mathcal{O}'$ and both orders contain S, then there is a locally principal S-ideal $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}\mathfrak{a}^{-1} = \mathcal{O}'$.

2. If $\mathcal{M}$ is a maximal order containing $\mathcal{O}$, then $h(\mathcal{O}) = h(\mathcal{M})$. □

The second condition of (3.1) may seem to be very restrictive. However at least for orders in rational algebras, this is satisfied by all orders containing elliptic elements, see (5.8).

Fix $S = K(x) \cap \mathcal{O} \in \Omega$ and let $S_1 \subseteq S \subseteq S_0$, where $S_1 = R[x]$ and S_0 is the maximal order in $L = K(S)$. Here x is a generator of the fix group of the elliptic point corresponding to S. It is well-known that there is an R-ideal $\mathfrak{f}$, such that

$$S = R + \mathfrak{f} \cdot S_0.$$

Let $\mathfrak{p}$ be a prime in R and define $\chi(\mathfrak{p}) = \chi_L(\mathfrak{p})$ to be the Kronecker symbol, that is,

$$\chi(\mathfrak{p}) = \begin{cases} -1 & \text{if } \mathfrak{p} \text{ unramified in } L, \\ 0 & \text{if } \mathfrak{p} \text{ ramified in } L, \\ 1 & \text{if } \mathfrak{p} \text{ split in } L. \end{cases}$$

We need the following relative class number formula for non-maximal orders in quadratic extensions. For a proof, see [13], (12.12), and [14].

(3.2) Lemma. *Let $S = R + \mathfrak{f} \cdot S_0$ be as above. Then*

$$h(S) = h(S_0)\frac{\prod_{\mathfrak{p}|\mathfrak{f}}[(S_0)_{\mathfrak{p}}^* : S_{\mathfrak{p}}^*]}{[S_0^* : S^*]} = \frac{h(S_0)}{[S_0^* : S^*]} \cdot N\mathfrak{f} \prod_{\mathfrak{p}|\mathfrak{f}} (1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}})$$

Let $l(S)$ denote the number of Γ-conjugacy classes in Ω in the R-isomorphism class of S.

(3.3) Proposition. *Let $\mathcal{O}$ be an order satisfying* (3.1). *If $S = R + \mathfrak{f} \cdot S_0$ belongs to Ω, then*

$$l(S) = \frac{[N(\mathcal{O}^*) : N(\Gamma)] \cdot h(S_0)}{2[S_0^* : \Gamma \cap S] \cdot h(\mathcal{O})} N\mathfrak{f} \prod_{\mathfrak{p}|\mathfrak{f}} (1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}}) \prod_{\mathfrak{p}|d(\mathfrak{A})} e^*(S_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}}),$$

where $e^*(S_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}})$ is the number of optimal embeddings of $S_{\mathfrak{p}}$ into $\mathcal{O}_{\mathfrak{p}}$ modulo conjugation by the elements of $\mathcal{O}_{\mathfrak{p}}^*$.

Proof. This is Satz 2.5. in [14], except that we have rewritten it using the identity

$$\frac{h(S)}{[S^* : \Gamma \cap S]} = \frac{h(S_0)}{[S_0^* : \Gamma \cap S]} N\mathfrak{f} \prod_{\mathfrak{p}|\mathfrak{f}} (1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}}),$$

which follows from (3.2). □

We remark that the embedding numbers $e^*(S_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}})$ are known in great generality, see [2].

Finally, let $S_1 = R[x] = R + \mathfrak{a} \cdot S_0$ be a minimal order containing an elliptic element x of order m. In order to get the number of elliptic points of order m, one has to sum over all $S = R + \mathfrak{f} \cdot S_0$, such that $\mathfrak{f}|\mathfrak{a}$ and S does not contain elliptic elements of higher order than m.

4. Bass orders

In this section, we will derive some auxiliary results mainly on so called Bass orders. An order $\mathcal{O}$ is called a Gorenstein order, if the dual $\mathcal{O}^{\#}$ is projective as an $\mathcal{O}$-module. It is called a Bass order, if all orders containing it are Gorenstein orders. If $\mathcal{O}$ contains the maximal order of a quadratic extension of the ground field, then $\mathcal{O}$ is a Bass order. The converse is true in the local case ([2], (1.11)), but in the global case this is an open question. We also remark, that if $\mathcal{O}$ is an order in an algebra over an algebraic number field, then $\mathcal{O}$ is a Bass order iff $\mathcal{O}_{\mathfrak{p}}$ is a Bass order for all prime ideals $\mathfrak{p}$.

First we recall a useful description of Bass orders with Eichler invariant not equal to 1 ([2], (1.12)).

(4.1) Lemma. *Suppose that $\mathcal{O}$ is a Bass order in a quaternion algebra over a $\mathfrak{p}$-adic field K, with $e(\mathcal{O}) = 0$ or $e(\mathcal{O}) = -1$. Then there is a unique minimal hereditary order $H(\mathcal{O})$ containing $\mathcal{O}$. Furthermore, suppose that L is a quadratic extension with maximal order S. If $S \subset \mathcal{O}$, then*

$$\mathcal{O} \cong S + J(H(\mathcal{O}))^m,$$

where $J(H(\mathcal{O}))$ is the Jacobson radical of $H(\mathcal{O})$ and $m \geq 0$.

Using this one can prove the following generalization due to J. Brzezinski of the Eichler-Hasse-Noether-Chevalley theorem:

(4.2) Theorem. *Let $\mathcal{O}$ and $\mathcal{O}'$ be two isomorphic* (*Bass*) *orders in a quaternion algebra over a $\mathfrak{p}$-adic field K. Let L be a quadratic extension of K with maximal order S. Suppose that S is contained in both $\mathcal{O}$ and $\mathcal{O}'$. Then there is an $\alpha \in S$ such that*

$$\alpha\mathcal{O}\alpha^{-1} = \mathcal{O}'.$$

Proof. For a proof when $\mathcal{O}$ and $\mathcal{O}'$ are hereditary orders, see [7], Satz 7. Now suppose that $e(\mathcal{O}) = 0$ or $e(\mathcal{O}) = -1$. We can find $\alpha \in S$, such that $\alpha H(\mathcal{O})\alpha^{-1} = H(\mathcal{O})'$, since the theorem is true for hereditary orders and $\mathcal{O} \cong \mathcal{O}'$ implies $H(\mathcal{O}) \cong H(\mathcal{O}')$. Now from (4.1), we conclude that

$$\alpha\mathcal{O}\alpha^{-1} = \alpha S\alpha^{-1} + \alpha J(H(\mathcal{O}))^m\alpha^{-1} = S + J(\alpha H(\mathcal{O})\alpha^{-1})^m = \mathcal{O}'.$$

The remaining case $e(\mathcal{O}) = 1$ was proved in [3]. □

Another way to phrase (4.2) is to say that there is an S-automorphism between $\mathcal{O}$ and $\mathcal{O}'$, that is, an automorphism that is the identity on S. From this theorem, we immediately derive the following upper bound on the type number of Bass orders.

(4.3) Proposition. *Let K be an algebraic number field and suppose that L is a quadratic extension of K with maximal order S. If $\mathcal{O}$ is an order in a quaternion algebra over K such that $S \subset \mathcal{O}$, then*

$$t(\mathcal{O}) \leq h(S).$$

Proof. Suppose that $\mathcal{O}$ and $\mathcal{O}'$ are in the same genus, that is, $\mathcal{O}_\mathfrak{p} \cong \mathcal{O}'_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. From (4.2), we get $\alpha_\mathfrak{p} \in L_\mathfrak{p}$ such that $\alpha_\mathfrak{p}\mathcal{O}_\mathfrak{p}\alpha_\mathfrak{p}^{-1} = \mathcal{O}'_\mathfrak{p}$, or equivalently an S-ideal $\mathfrak{a}$ such that $\mathcal{O}_\mathfrak{a} := \mathfrak{a}\mathcal{O}\mathfrak{a}^{-1} = \mathcal{O}'$. In this way, we can associate an S-ideal to every order in the genus of $\mathcal{O}$. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be equal in the class group of S, that is, $\mathfrak{a}_1 = \mathfrak{a}_2 x$ with $x \in L$. Then

$$\mathcal{O}_{\mathfrak{a}_1} = \mathfrak{a}_1\mathcal{O}\mathfrak{a}_1^{-1} = \mathfrak{a}_2 x\mathcal{O}(\mathfrak{a}_2 x)^{-1} = x(\mathfrak{a}_2\mathcal{O}\mathfrak{a}_2^{-1})x^{-1} = x\mathcal{O}_{\mathfrak{a}_2}x^{-1},$$

from which we immediately derive the desired inequality. □

We conclude this section with a class number formula for non-maximal orders, which is analogous to the one in (3.2) for algebraic number fields.

(4.4) Proposition. *Let $\mathcal{O}$ be an order in a quaternion algebra which is not totally definite, and let $\mathcal{M}$ be a maximal order containing $\mathcal{O}$. If we denote the (one sided) class numbers of $\mathcal{O}$ and $\mathcal{M}$ by $h(\mathcal{O})$ and $h(\mathcal{M})$, then*

$$h(\mathcal{O}) = h(\mathcal{M})\frac{\prod_\mathfrak{p}[\mathcal{M}_\mathfrak{p}^* : \mathcal{O}_\mathfrak{p}^*]}{[\mathcal{M}^* : \mathcal{O}^*]} = h(\mathcal{M})\frac{\prod_\mathfrak{p}[R_\mathfrak{p}^* : N(\mathcal{O}_\mathfrak{p}^*)]}{[N(\mathcal{M}^*) : N(\mathcal{O}^*)]}.$$

Proof. The proof of the first equality is the same as in the commutative case, see [13], (12.12) except for the complication that we need a group structure on the ideals. However, if $\mathfrak{A}$ is not totally definite, then the norm provides us with such a structure.

The second equality follows directly from (2.2) and (2.3). □

5. Rational case

Let $\mathcal{O}$ be an arbitrary order in an indefinite rational quaternion algebra. In this section, we will use the general formulas in Sections 2 and 3 to give an explicitly computable formula for the genus g of $\mathcal{O}^1\backslash\mathcal{H}$. We also give a complete list of all such orders for which $g \leq 2$.

First we recall a well-known correspondence between quaternion orders and ternary quadratic forms. Assume that R is a PID with field of fractions K. Let f be a ternary quadratic form integral over R. Then the even Clifford algebra, $C_0(f)$, is an order in a quaternion algebra $\mathfrak{A}$ over K. Conversely, let $\mathcal{O}$ be an order in $\mathfrak{A}$ with $d(\mathcal{O}) = (d)$ and f_1, f_2, f_3 an R-basis of $\Lambda = \mathcal{O}^\# \cap \mathfrak{A}_0$, where $\mathfrak{A}_0 = \{x \in \mathfrak{A} : Tr(x) = 0\}$. Then

$$f_\mathcal{O} = d \cdot N(X_1 f_1 + X_2 f_2 + X_3 f_3)$$

is a ternary quadratic form integral over R. The maps $f \longmapsto C_0(f)$ and $\mathcal{O} \longmapsto f_\mathcal{O}$ are inverses to each other. Furthermore, this gives a one-to-one correspondence between isomorphism classes of quaternion orders and similarity classes of ternary quadratic forms. (Two forms f and g are similar, $f \sim g$, if there exists $\epsilon \in R^*$, such that f is isometric to ϵg.) For details, see [1].

We remark, that the only possible elliptic elements in $\mathcal{O}$ are of orders 2 and 3 corresponding to $\mathbb{Z}[i]$ and $\mathbb{Z}[\epsilon]$ respectively. This is clear, since the minimal polynomial of an elliptic element has to be of the form

$$x^2 - tx + 1 = 0,\ t \in \mathbb{Z} \text{ and } |t| < 2.$$

Both $\mathbb{Z}[i]$ and $\mathbb{Z}[\epsilon]$ are maximal. Hence, only Bass orders can contain elliptic elements. For proofs of the following two results on orders containing $\mathbb{Z}[i]$ and $\mathbb{Z}[\epsilon]$, see [9].

(5.1) Proposition. *Let $\mathcal{O}$ be an order in a rational quaternion algebra, such that $\mathcal{O} \supset \mathbb{Z}[i]$. Then $\mathcal{O} \cong C_0(f_i)$, where f_i is uniquely chosen among the three families:*

$$\begin{cases} f_1(a) = X_1^2 + X_2^2 + aX_3^2 \\ f_2(a) = X_1^2 + X_2^2 + aX_3^2 + X_1X_3 \\ f_3(a) = X_1^2 + X_2^2 + aX_3^2 + X_1X_3 + X_2X_3. \end{cases}$$

(5.2) Proposition. *Let $\mathcal{O}$ be an order in a rational quaternion algebra, such that $\mathcal{O} \supset \mathbb{Z}[\epsilon]$. Then $\mathcal{O} \cong C_0(f_i)$, where f_i is uniquely chosen from the two families:*

$$\begin{cases} f_1(a) = X_1^2 + X_2^2 + aX_3^2 + X_1X_2 \\ f_2(a) = X_1^2 + X_2^2 + aX_3^2 + X_1X_2 + X_2X_3 \end{cases}$$

In order to use the results in Section 3, we will have to show that all Bass orders satisfy the two conditions in (3.1). The first was proved in (4.2) and the second will be proved in (5.8) below. But first we prove the following two lemmas, which we also need for our calculations of the area of the fundamental domain.

(5.3) Lemma. *Let $\mathcal{O}$ be an order in a rational quaternion algebra and p a prime.*

If p is odd, then

$$[\mathbb{Z}_p^* : N(\mathcal{O}_p^*)] = \begin{cases} 2 & \text{if } e(\mathcal{O}_p) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $p = 2$, then

$$[\mathbb{Z}_2^* : N(\mathcal{O}_2^*)] = \begin{cases} 4 & \text{if } r_1 + r_2 \geq 3 \text{ or } s_1 \geq 3, \\ 2 & \text{if } r_1 + r_2 = 2 \text{ or } (r_2 \leq 1 \text{ and } r_3 \geq 2 + r_2) \text{ or } s_1 = 2, \\ 1 & \text{otherwise,} \end{cases}$$

where $f_{\mathcal{O}_2}$ is similar to one of the following forms:

$$f_1 = 2^{r_1}X^2 + \delta 2^{r_2}Y^2 + \epsilon 2^{r_3}Z^2,\ 0 \leq r_1 \leq r_2 \leq r_3,\ \delta \text{ and } \epsilon \text{ odd},$$

$$f_2 = 2^{s_1}XY + 2^{s_2}Z^2,\ s_1, s_2 \geq 0,$$

$$f_3 = 2^{s_1}(X^2 + XY + Y^2) + 2^{s_2}Z^2,\ s_1, s_2 \geq 0.$$

Proof. First assume that

$$f_{\mathcal{O}_p} \sim p^{r_1}X^2 + \delta p^{r_2}Y^2 + \epsilon p^{r_3}Z^2, \tag{5.4}$$

where $0 \leq r_1 \leq r_2 \leq r_3$ and $(\delta, p) = (\epsilon, p) = 1$. This is always true if p is odd. In this case, $e(\mathcal{O}_p) \neq 0$ iff p is odd and $r_2 = 0$. The norm form of $\mathcal{O}_p$ is then

$$N = X_0^2 + \delta p^{r_1+r_2} X_1^2 + \epsilon p^{r_1+r_3}(X_2^2 + \delta p^{r_2-r_1} X_3^2), \tag{5.5}$$

from which we derive the desired result by inspection.

If $p = 2$ and $f_{\mathcal{O}_2}$ is not similar to a diagonal form, then $f_{\mathcal{O}_2} \sim f_2$ or $f_{\mathcal{O}_2} \sim f_3$. Moreover, $e(\mathcal{O}_2) \neq 0$ iff $s_1 = 0$. The norm forms corresponding to f_2 and f_3 are

$$\begin{aligned} N_2 &= X_0^2 + 2^{s_1} X_0 X_1 + 2^{s_1+s_2} X_2 X_3 \\ N_3 &= X_0^2 + 2^{s_1} X_0 X_1 + 2^{2s_1} X_1^2 + 2^{s_1+s_2}(X_2^2 + X_2 X_3 + X_3^2) \end{aligned}$$

respectively, and it is easy to determine $[\mathbb{Z}_2^* : N(\mathcal{O}_2^*)]$ by inspection. □

(5.6) Lemma. *We keep the notation of* (5.3)*.*

If p is odd, then $-1 \in N(\mathcal{O}_p^*)$ iff $e(\mathcal{O})_p \neq 0$ or $p \equiv 1$ (mod 4).

If $p = 2$, then $-1 \in N(\mathcal{O}_2^*)$ iff $[\mathbb{Z}_2^* : N(\mathcal{O}_2^*)] = 1$ or $r_1 = 0$, $r_2 = 1$ and $\delta \equiv 3$ (mod 4).

Moreover if $\mathcal{O}$ is indefinite, then $-1 \in N(\mathcal{O}^*)$ iff $-1 \in N(\mathcal{O}_p^*)$ for all primes p.

Proof. The local results follow by inspection of the norm forms as in the proof of (5.3).

We get from (4.4) that if $\mathcal{O}$ is indefinite, then

$$h(\mathcal{O}) = h(\mathcal{M}) \frac{\prod_p [\mathbb{Z}_p^* : N(\mathcal{O}_p^*)]}{[\mathbb{Z}^* : N(\mathcal{O}^*)]}. \tag{5.7}$$

But $h(\mathcal{O}) = h(\widetilde{\mathcal{O}})$, if $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are in the same genus. Hence it follows from (5.7), that $[\mathbb{Z}^* : N(\mathcal{O}^*)] = [\mathbb{Z}^* : N(\widetilde{\mathcal{O}}^*)]$. But if $-1 \in N(\mathcal{O}_p^*)$ for all primes p and $\mathcal{O}$ is indefinite, then $-1 \in N(\widetilde{\mathcal{O}})$ for some $\widetilde{\mathcal{O}}$ in the genus of $\mathcal{O}$ and hence in all, since $[\mathbb{Z}^* : N(\mathcal{O}^*)] = [\mathbb{Z}^* : N(\widetilde{\mathcal{O}}^*)]$. □

With (4.4), (5.3) and (5.6) at hand, it is possible to determine the class number of any order in an indefinite rational quaternion algebra, since the class number of a maximal order $\mathcal{M}$ in this case satisfies $h(\mathcal{M}) = 1$, see [6]. We record here only the special case which will be needed below.

(5.8) Proposition. *Let $\mathcal{O}$ be an order in an indefinite rational quaternion algebra $\mathfrak{A}$ and let $\mathcal{M}$ be a maximal order containing $\mathcal{O}$. If $\mathcal{O}$ contains either $\mathbb{Z}[i]$ or $\mathbb{Z}[\epsilon]$, then $h(\mathcal{O}) = h(\mathcal{M}) = 1$.*

Proof. First suppose that $\mathcal{O} \supset \mathbb{Z}[i]$. It follows from (5.3) and Table 1, that $[\mathbb{Z}_p^* : N(\mathcal{O}_p^*)] > 1$ only if $p = 2$, $e(\mathcal{O}_2) = 0$ and $16 | d(\mathcal{O})$. In this case

$$f_{\mathcal{O}_2} \sim X^2 + Y^2 + \epsilon 2^r Z^2,$$

I. $\mathfrak{A} \cong M(2, \mathbb{Q}_p)$

(a) $\mathcal{O}$ maximal: $e^*(2,\mathcal{O}) = 1$

$e^*(3,\mathcal{O}) = 1$

(b) $e(\mathcal{O}) = -1:$ $e^*(2,\mathcal{O}) = \begin{cases} 2 & p \equiv 3 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$

$e^*(3,\mathcal{O}) = \begin{cases} 2 & p \equiv 2 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$

(c) $e(\mathcal{O}) = 0:$ $e^*(2,\mathcal{O}) = \begin{cases} 1 & p = 2, n = 2 \\ 2 & p = 2, n = 3 \\ 4 & p = 2, n = 4, f_{\mathcal{O}} \sim X^2 + Y^2 - 4Z^2 \\ 8 & p = 2, n > 4, f_{\mathcal{O}} \sim X^2 + Y^2 - 2^{n-2}Z^2 \\ 0 & \text{otherwise} \end{cases}$

$e^*(3,\mathcal{O}) = \begin{cases} 2 & p = 3, n = 2 \\ 6 & p = 3, n > 2, f_{\mathcal{O}} \sim X^2 + 3Y^2 - 3^{n-1}Z^2 \\ 0 & \text{otherwise} \end{cases}$

(d) $e(\mathcal{O}) = 1:$ $e^*(2,\mathcal{O}) = \begin{cases} 2 & p \equiv 1 \pmod 4 \\ 1 & p = 2, n = 1 \\ 0 & \text{otherwise} \end{cases}$

$e^*(3,\mathcal{O}) = \begin{cases} 2 & p \equiv 1 \pmod 3 \\ 1 & p = 3, n = 1 \\ 0 & \text{otherwise} \end{cases}$

II. $\mathfrak{A} \cong \mathbb{H}_p$

(a) $e(\mathcal{O}) = -1$ $e^*(2,\mathcal{O}) = \begin{cases} 2 & p \equiv 3 \pmod 4 \\ 1 & p = 2, n = 1 \\ 0 & \text{otherwise} \end{cases}$

$e^*(3,\mathcal{O}) = \begin{cases} 2 & p \equiv 2 \pmod 3 \\ 1 & p = 3, n = 1 \\ 0 & \text{otherwise} \end{cases}$

(b) $e(\mathcal{O}) = 0:$ $e^*(2,\mathcal{O}) = \begin{cases} 3 & p = 2, n = 2 \\ 2 & p = 2, n = 3 \\ 4 & p = 2, n = 4, f_{\mathcal{O}} \sim X^2 + Y^2 + 4Z^2 \\ 8 & p = 2, n > 4, f_{\mathcal{O}} \sim X^2 + Y^2 + 2^{n-2}Z^2 \\ 0 & \text{otherwise} \end{cases}$

$e^*(3,\mathcal{O}) = \begin{cases} 4 & p = 3, n = 2 \\ 6 & p = 3, n > 2, f_{\mathcal{O}} \sim X^2 + 3Y^2 + 3^{n-1}Z^2 \\ 0 & \text{otherwise} \end{cases}$

TABLE 1. The embedding numbers in orders $\mathcal{O}$ with discriminant $d(\mathcal{O}) = p^n$ in quaternion algebras over $\mathbb{Q}_p$. Observe that $e(\mathcal{O}) = -1$ and $n = 1$, if $\mathcal{O}$ is a maximal order in $\mathbb{H}_p$.

where $r \geq 2$. It follows from (5.3) that $[\mathbb{Z}_p^* : N(\mathcal{O}_p^*)] = 2$, and from (5.6) that $-1 \notin N(\mathcal{O}_2)$ and hence $[\mathbb{Z}^* : N(\mathcal{O}^*)] = 2$. We conclude from (4.4), that $h(\mathcal{O}) = h(\mathcal{M})$.

Now suppose that $\mathcal{O} \supset \mathbb{Z}[\epsilon]$. In this case $[\mathbb{Z}_p^* : N(\mathcal{O}_p^*)] > 1$ only if $p = 3$ and $e(\mathcal{O}_3) = 0$. Once again, it follows from (5.6), that $-1 \notin N(\mathcal{O}_3)$ and hence $h(\mathcal{O}) = h(\mathcal{M})$.

The fact that $h(\mathcal{M}) = 1$ when $\mathfrak{A}$ is indefinite was proved in [6]. □

Finally, we can use (3.3) in order to determine the number of elliptic points. Since $h(\mathbb{Z}[i]) = h(\mathbb{Z}[\epsilon]) = 1$, we get:

(5.9) Proposition. *Let $\mathcal{O}$ be an order in an indefinite rational quaternion algebra. Then the number of elliptic points of order m in $\mathcal{O}^1\backslash\mathcal{H}$ is*

$$e(m, \mathcal{O}) = \frac{1}{\kappa(\mathcal{O})} \prod_{p|d(\mathcal{O})} e^*(m, \mathcal{O}_p),$$

where $e^*(m, \mathcal{O}_p)$ are the local embedding numbers given in Table 1 and

$$\kappa(\mathcal{O}) = \begin{cases} 2 & \text{if } e(\mathcal{O}_3) = 0 \text{ or } (e(\mathcal{O}_2) = 0 \text{ and } 16|d(\mathcal{O})), \\ 1 & \text{otherwise.} \end{cases}$$

The only thing that is left to get a completely explicit formula for the genus is a formula for the area of a fundamental domain. But from the results in Section 2 and (5.3), we get the following:

(5.10) Proposition. *Let $\mathcal{O}$ be an order in an indefinite rational quaternion algebra $\mathfrak{A}$ and $\mathcal{M}$ a maximal order containing $\mathcal{O}$. Then the area of a fundamental domain $\mathcal{F}$ of $\mathcal{O}^1\backslash\mathcal{H}$ is*

$$\begin{aligned} \mu(\mathcal{F}) &= \frac{\pi}{3} \prod_{p|d(\mathfrak{A})} (p-1) \prod_{p|d(\mathcal{O})} [\mathcal{M}_p^1 : \mathcal{O}_p^1] \\ &= \frac{\pi}{3} d(\mathcal{O}) \prod_{p|d(\mathcal{O})} [\mathbb{Z}_p^* : N(\mathcal{O}_p^*)]^{-1} \cdot \frac{1}{p}(p^2-1)(p-e(\mathcal{O}_p))^{-1}, \end{aligned}$$

where $[\mathbb{Z}_p^* : N(\mathcal{O}_p^*)]$ is given in (5.3).

As an application of these formulas, we give in Table 2 a complete list of all orders $\mathcal{O}$ with $g = g(\mathcal{O}^1) \leq 2$. The first step is to find all maximal orders satisfying this condition. This is easily done, and we get the 12 different maximal orders in Table 2. Next we make the trivial observation, that $\mathcal{O} \subseteq \widetilde{\mathcal{O}}$ implies $g(\mathcal{O}^1) \geq g(\widetilde{\mathcal{O}}^1)$. Since there is a very limited set of isomorphism classes of maximal suborders in a given order, it is easy to take successively such suborders until $g \geq 3$.

(5.11) Remark. 1. The third column in Table 2 gives a quadratic form f, such that $\mathcal{O} \cong C_0(f)$. The fourth and fifth give the number of elliptic points of orders 2 and 3 and the sixth gives the genus.

2. All orders in Table 2 have type number equal to 1. This is easily deduced using (4.3). It is true for all orders containing elliptic elements, since $h(\mathbb{Z}[i]) = h(\mathbb{Z}[\epsilon]) = 1$. For the four Bass orders without elliptic elements,

$d(\mathfrak{A})$	$d(\mathcal{O})$	$f_{\mathcal{O}}$	e_2	e_3	g	Remarks
6	$2 \cdot 3$	$X^2+Y^2-2Z^2+XY$	2	2	0	maximal
	$2^2 \cdot 3$	$X^2+Y^2-3Z^2$	6	0	0	$e(\mathcal{O}_2)=0$
	$2 \cdot 3^2$	$X^2+Y^2-6Z^2+XY$	0	4	0	$e(\mathcal{O}_3)=0$
	$2^3 \cdot 3$	$X^2+Y^2-8Z^2+XY$	0	2	1	$e(\mathcal{O}_2)=-1$
	$2^3 \cdot 3$	$X^2+Y^2-6Z^2$	4	0	1	$e(\mathcal{O}_2)=0$
	$2 \cdot 3 \cdot 5$	$X^2+Y^2-7Z^2+XZ+YZ$	4	0	1	$e(\mathcal{O}_5)=1$
	$2^2 \cdot 3^2$	$X^2-3Y^2-3Z^2$	0	0	2	$e(\mathcal{O}_2)=e(\mathcal{O}_3)=0$
	$2 \cdot 3 \cdot 7$	$X^2+Y^2-14Z^2+XY$	0	4	1	$e(\mathcal{O}_7)=1$
	$2^4 \cdot 3$	$X^2+Y^2-12Z^2$	4	0	1	$e(\mathcal{O}_2)=0$
	$2^4 \cdot 3$	$2(X^2+Y^2-2Z^2+XY)$	0	0	2	$e(\mathcal{O}_2)=0,$ non-Bass
	$2 \cdot 3^3$	$X^2+Y^2-18Z^2+XY$	0	6	0	$e(\mathcal{O}_3)=0$
	$2 \cdot 3^3$	$X^2+Y^2-13Z^2+XZ+YZ$	2	0	2	$e(\mathcal{O}_3)=-1$
	$2^2 \cdot 3 \cdot 5$	$X^2+Y^2-15Z^2$	12	0	1	$e(\mathcal{O}_2)=0,\ e(\mathcal{O}_5)=1$
	$2^3 \cdot 3^2$	$X^2+Y^2-24Z^2+XY$	0	4	1	$e(\mathcal{O}_2)=-1, e(\mathcal{O}_3)=0$
	$2 \cdot 3 \cdot 13$	$X^2+Y^2-26Z^2+XY$	4	4	1	$e(\mathcal{O}_{13})=1$
	$2^5 \cdot 3$	$X^2+Y^2-24Z^2$	8	0	1	$e(\mathcal{O}_2)=0$
	$2 \cdot 3^2 \cdot 7$	$X^2+Y^2-42Z^2+XY$	0	8	1	$e(\mathcal{O}_3)=0,\ e(\mathcal{O}_7)=1$
	$2 \cdot 3^4$	$X^2+Y^2-54Z^2+XY$	0	6	2	$e(\mathcal{O}_3)=0$
10	$2 \cdot 5$	$X^2+Y^2-3Z^2+XY+YZ$	0	4	0	maximal
	$2^2 \cdot 5$	$X^2-3Y^2-2Z^2+2YZ$	0	0	2	$e(\mathcal{O}_2)=0$
	$2 \cdot 3 \cdot 5$	$X^2+Y^2-10Z^2+XY$	0	4	1	$e(\mathcal{O}_3)=1$
	$2^3 \cdot 5$	$X^2+Y^2-13Z^2+XY+YZ$	0	4	1	$e(\mathcal{O}_2)=-1$
	$2 \cdot 5^2$	$X^2-Y^2-15Z^2+XY+5YZ$	0	0	2	$e(\mathcal{O}_5)=0$
	$2 \cdot 5 \cdot 7$	$X^2+Y^2-23Z^2+XY+YZ$	0	8	1	$e(\mathcal{O}_7)=1$
	$2 \cdot 3^2 \cdot 5$	$X^2+Y^2-30Z^2+XY$	0	4	1	$e(\mathcal{O}_3)=0$
	$2 \cdot 3^3 \cdot 5$	$X^2+Y^2-90Z^2+XY$	0	12	1	$e(\mathcal{O}_3)=0$
14	$2 \cdot 7$	$X^2+Y^2-3Z^2+XZ+YZ$	2	0	1	maximal
	$2^2 \cdot 7$	$X^2+Y^2-7Z^2$	6	0	1	$e(\mathcal{O}_2)=0$
15	$3 \cdot 5$	$X^2+Y^2-5Z^2+XY$	0	2	1	maximal
	$3^2 \cdot 5$	$X^2+Y^2-15Z^2+XY$	0	4	1	$e(\mathcal{O}_3)=0$
	$2^2 \cdot 3 \cdot 5$	$X^2+Y^2-20Z^2+XY$	0	4	1	$e(\mathcal{O}_2)=-1$
	$2^2 \cdot 3^2 \cdot 5$	$X^2+Y^2-60Z^2+XY$	0	8	1	$e(\mathcal{O}_2)=-1,\ e(\mathcal{O}_3)=0$
21	$3 \cdot 7$	$X^2+Y^2-5Z^2+XZ$	4	0	1	maximal
22	$2 \cdot 11$	$X^2+Y^2-7Z^2+XY+YZ$	2	4	0	maximal
	$2^2 \cdot 11$	$X^2+Y^2-11Z^2$	6	0	2	$e(\mathcal{O}_2)=0$
26	$2 \cdot 13$	$X^2+13Y^2+3Z^2+XZ+13YZ$	0	0	2	maximal
33	$3 \cdot 11$	$X^2+Y^2-11Z^2+XY$	4	2	1	maximal
34	$2 \cdot 17$	$X^2+Y^2-11Z^2+XY+YZ$	0	4	1	maximal
38	$2 \cdot 19$	$X^2+Y^2-9Z^2+XZ+YZ$	2	0	2	maximal
46	$2 \cdot 23$	$X^2+Y^2-15Z^2+XY+YZ$	2	4	1	maximal
58	$2 \cdot 29$	$X^2+Y^2-19Z^2+XY+YZ$	0	4	2	maximal

TABLE 2. A complete list of all orders $\mathcal{O}$ in rational quaternion algebras such that the genus g of $\mathcal{O}^1\backslash\mathcal{H}$ satisfies $g \leq 2$.

one checks that they from top to bottom contain $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ and $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ respectively. All these orders have class number equal to 1. The only non-Bass order $\mathcal{O}$ has type number 1, since $f_{\mathcal{O}} = 2 \cdot f_{\mathcal{M}}$ for a maximal order $\mathcal{M}$ and $t(\mathcal{M}) = 1$.

3. In two cases in Table 2, we have $\mathcal{O} \supsetneq \widetilde{\mathcal{O}}$ but $\mathcal{O}^1 = \widetilde{\mathcal{O}}^1$. First the two orders with $d(\mathcal{O}) = 24$, $d(\widetilde{\mathcal{O}}) = 48$ and $e(\mathcal{O}_2) = e(\widetilde{\mathcal{O}}_2) = 0$ in $\mathfrak{A}$, with $d(\mathfrak{A}) = 6$. The same is true for the orders with $d(\mathcal{O}) = 30$ and $d(\widetilde{\mathcal{O}}) = 90$ in $\mathfrak{A}$, with $d(\mathfrak{A}) = 10$. □

6. Quadratic case

We conclude this paper with another concrete example. In this section let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension of $\mathbb{Q}$, where $d = d(K) > 0$ is the discriminant of K. Furthermore, we will assume that $\mathcal{O}$ is a maximal order in a quaternion algebra over K and $E = R^{*^2}$. Hence, we can identify Γ with $\mathcal{O}^1$ as Fuchsian groups. First we compile the necessary results to get an algorithmic formula for the genus g of $\mathcal{O}^1 \backslash \mathcal{H}$. We have implemented this using PARI, and we conclude with a complete list of all such orders with $g \leq 2$.

The calculation of the fundamental area only involves $\zeta_K(-1)$ and $N\mathfrak{p}$ for all $\mathfrak{p}|\mathfrak{A}$. Let $\mathcal{F}$ be a fundamental domain of $\mathcal{O}^1 \backslash \mathcal{H}$. An elegant formula for $\zeta_K(-1)$ for quadratic real fields due to Zagier [20] gives us

$$\mu(\mathcal{F}) = \frac{\pi}{120} \sum_{\substack{x^2 \equiv d \pmod 4 \\ |x| < \sqrt{d}}} \sigma\left(\frac{d - x^2}{4}\right) \prod_{\mathfrak{p} | d(\mathfrak{A})} (N\mathfrak{p} - 1),$$

where $\sigma(r)$ is the sum of divisors of r.

Next we consider elliptic elements. In this case, we only get elliptic points of orders 2 and 3 except in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$. In these fields, we get elements of orders 4, 6 and 5 respectively. These correspond to the minimal equations

$$x^2 - \sqrt{2}x + 1 = 0,\ x^2 - \sqrt{3}x + 1 = 0 \text{ and } x^2 - \frac{1+\sqrt{5}}{2}x + 1 = 0.$$

Let $S_1 \subseteq S \subseteq S_0$ be as in Section 3. First we will determine the different possibilities for $\mathfrak{a}$, where $S_1 = R + \mathfrak{a} \cdot S_0$. The three cases with elliptic elements of order higher than 3 all have $S_0 = S_1$, so $\mathfrak{a} = R$.

Now assume that $S_1 = R[i]$. Then the discriminant of S_1 is $16d^2$. Hence $[S_0 : S_1] \in \{1, 2, 4\}$. If $[S_0 : S_1] = 2$, then $(2) = \mathfrak{p}^2$ in R and $\mathfrak{a} = \mathfrak{p}$. If $[S_0 : S_1] = 4$, then $\mathfrak{a} = (2)$ and (2) can be split, ramified or unramified in R.

Next we assume that $S_1 = R[\epsilon]$. The discriminant of S_1 is $9d^2$, so $[S_0 : S_1] \in \{1, 3\}$. If $[S_0 : S_1] = 3$, then $(3) = \mathfrak{p}^2$ in R and $\mathfrak{a} = \mathfrak{p}$. We conclude, that it suffices to determine the splitting of 2 or 3 in $L = K(S)$ in order to calculate the factor

$$N\mathfrak{f}\prod_{\mathfrak{p}|\mathfrak{f}}(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}}).$$

Now we consider the product of the local embedding numbers $e^*(S_\mathfrak{p},\mathcal{O}_\mathfrak{p})$. In this case, they only depend on the splitting of $\mathfrak{p}$ in L and $(\mathfrak{p},\mathfrak{f})$. We have by [2], (3.1):

(6.1) Lemma. *Let $\mathcal{O}$ be the maximal order in $\mathfrak{A}$, and S an order in a quadratic subfield L of $\mathfrak{A}$. If $S=R+\mathfrak{f}\cdot S_0$, where S_0 is the maximal order in L, then we have for all $\mathfrak{p}|d(\mathfrak{A})$ that*

$$e^*(S_\mathfrak{p},\mathcal{O}_\mathfrak{p})=\begin{cases}2 & \mathfrak{p} \textit{ unramified in } L \textit{ and } (\mathfrak{p},\mathfrak{f})=1,\\ 1 & \mathfrak{p} \textit{ ramified in } L \textit{ and } (\mathfrak{p},\mathfrak{f})=1,\\ 0 & \textit{otherwise.}\end{cases}$$

Finally, we consider the first factor in the formula of $l(S)$. We have to calculate class numbers and unit groups of biquadratic fields. This can be done using the classical results of Brauer and Hasse on abelian extensions. It turns out that using these results we end up with simply the class number of an imaginary quadratic field. What we need is the following result from [8], §26:

(6.2) Lemma. *Let $K_1=\mathbb{Q}(\sqrt{-d_1})$, $K_2=\mathbb{Q}(\sqrt{-d_2})$, $K_0=\mathbb{Q}(\sqrt{d_1d_2})$ be quadratic extensions of $\mathbb{Q}$ and $L=\mathbb{Q}(\sqrt{-d_1},\sqrt{-d_2})$, where $d_1,d_2>0$ and $K_1\neq K_2$. We denote the algebraic integers in L and K_0 by R and R_0 respectively. Let $\tilde{c}=[\widetilde{R}^*:R_0^*]$, where $\widetilde{R}^*$ are the units in R except roots of unity of order greater than 2. Then the class numbers of these fields satisfy*

$$h(L)=c\cdot h(K_0)\cdot h(K_1)\cdot h(K_2),$$

where $c=\tilde{c}$ if $L=\mathbb{Q}(\sqrt{-1},\sqrt{-2})$ and $c=\frac{1}{2}\tilde{c}$ otherwise.

(6.3) Proposition. *Let S_0 be the maximal order in $K(i)$, $K=\mathbb{Q}(\sqrt{d})$, and S a suborder of S_0. Suppose that S contains i but no root of unity of higher order than 4. If R_d is the maximal order in $\mathbb{Q}(\sqrt{-d})$, then*

$$\frac{[N(\mathcal{O}^*):N(\Gamma)]\cdot h(S_0)}{2[S_0^*:\Gamma\cap S]\cdot h(\mathcal{O})}=\frac{h(R_d)}{\delta},$$

where $\delta=6$ if $d=12$ and $\delta=2$ otherwise.

Proof. It follows from [6], Satz 1 that $h(\mathcal{O})=h(R)$ in this case.

Since $N(\mathcal{O}^*)$ consists of all elements in R^* which are positive at the real place at which $\mathfrak{A}$ is ramified, we get

$$[N(\mathcal{O}^*):N(\Gamma)]=[N(\mathcal{O}^*):{R^*}^2]=2.$$

Let c be as in (6.2). It is trivial to check, that $[S_0^*:\Gamma\cap S]=\delta c$. If we put everything together and use (6.2), we get

$$\frac{[N(\mathcal{O}^*):N(\Gamma)]\cdot h(S_0)}{2[S_0^*:\Gamma\cap S]\cdot h(\mathcal{O})}=\frac{h(S_0)}{\delta c\cdot h(R)}=\frac{h(R_d)\cdot h(\mathbb{Z}[i])}{\delta}=\frac{h(R_d)}{\delta}.\qquad\square$$

The proof of the corresponding result for the elliptic points of order 3 is completely analogous and we get:

(6.4) Proposition. *Let ϵ be a third root of unity, S_0 the maximal order in $K(\epsilon)$ and S a suborder of S_0. Suppose that S contains ϵ but no root of unity of higher order. If R_{3d} is the maximal order in $\mathbb{Q}(\sqrt{-3d})$, then*

$$\frac{[N(\mathcal{O}^*):N(\Gamma)]\cdot h(S_0)}{2[S_0^*:\Gamma\cap S]\cdot h(\mathcal{O})}=\frac{h(R_{3d})}{\delta},$$

where $\delta=4$ if $d=12$ and $\delta=2$ otherwise.

Finally we remark, that the factor in the cases of elliptic points of order higher than 3 is equal to $\frac{1}{2}$ if $d=12$ and 1 otherwise. This can be checked using (6.2), but it also follows from the fact that the genus is a non-negative integer.

If we combine all the results in this section, we get a formula which only involves the calculation of the class number of imaginary quadratic fields and determination of the splitting of primes in $K(i)$ and $K(\epsilon)$. We have implemented this using PARI, and we conclude the paper with a complete list of all (maximal) orders with $g\leq 2$.

(6.5) Proposition. *The list in Table 3 is complete.*

Proof. It is easy to check, that if there is an algebra over K with $g\leq 2$ for a maximal order, then there is an algebra over K ramified at only one prime with $g\leq 2$. We will first assume that $d(\mathfrak{A})=\mathfrak{p}$ is prime. Once we have a complete list of all algebras with prime discriminant, then it is not difficult to find the few algebras with non-prime discriminant.

We recall, that in our case the genus satisfies

$$g=\frac{\zeta_K(-1)}{2}(N\mathfrak{p}-1)+1-\frac{e(3,\mathcal{O})}{3}-\frac{e(2,\mathcal{O})}{4},$$

if $d>12$.

It is a direct computation using the trivial estimate $\sigma(r)\geq r$ and Zagier's formula [20] to show that

$$\zeta_K(-1)>\frac{1}{120}\left(dm-\frac{4m(m+1)(m-1)}{3}\right), \tag{6.6}$$

where $m=\left[\frac{\sqrt{d}}{2}\right]$. This lower bound grows like $d^{3/2}$.

In order to get a lower bound on the genus, we also need upper bounds on $e(n,\mathcal{O})$. First we consider $e(2,\mathcal{O})$. We have

$$\sum_{\mathfrak{f}|\mathfrak{a}}N\mathfrak{f}\prod_{\mathfrak{p}|\mathfrak{f}}(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}})\prod_{\mathfrak{p}}e^*(S_\mathfrak{p},\mathcal{O}_\mathfrak{p})\leq c_2(\mathfrak{p}):=\begin{cases}8 & \text{if } N\mathfrak{p}=2,\\ 32 & \text{otherwise,}\end{cases} \tag{6.7}$$

with the "worst" case being $\mathfrak{a}=(2)$, $(2)=\mathfrak{p}_2\overline{\mathfrak{p}}_2$ split in K and $\mathfrak{p}_2$, $\overline{\mathfrak{p}}_2$ and $\mathfrak{p}$ unramified in $K(i)$. The reason for the smaller bound when $N\mathfrak{p}=2$ is that $e^*(S_\mathfrak{p},\mathcal{O}_\mathfrak{p})=0$ unless $(\mathfrak{p},\mathfrak{f})=1$.

$d(K)$	$d(\mathfrak{A})$	e_2	e_3	e_4	e_5	e_6	g
5	(2)	1	0	0	2	0	0
	(3)	0	1	0	2	0	0
	$\mathfrak{p}_5$	0	2	0	1	0	0
	(7)	0	0	0	2	0	1
	$\mathfrak{p}_{11}$	2	2	0	0	0	0
	$\mathfrak{p}_{19}$	2	0	0	2	0	0
	$\mathfrak{p}_{29}$	0	2	0	2	0	0
	$\mathfrak{p}_{31}$	2	0	0	0	0	1
	$\mathfrak{p}_{41}$	0	2	0	0	0	1
	$\mathfrak{p}_{59}$	2	2	0	2	0	0
	$\mathfrak{p}_{61}$	0	0	0	0	0	2
	$\mathfrak{p}_{71}$	2	2	0	0	0	1
	$\mathfrak{p}_{79}$	2	0	0	2	0	1
	$\mathfrak{p}_{89}$	0	2	0	2	0	1
	$\mathfrak{p}_{101}$	0	2	0	0	0	2
	$\mathfrak{p}_{109}$	0	0	0	2	0	2
	$\mathfrak{p}_{131}$	2	2	0	0	0	2
	$\mathfrak{p}_{139}$	2	0	0	2	0	2
	$\mathfrak{p}_{149}$	0	2	0	2	0	2
	$\mathfrak{p}_{179}$	2	2	0	2	0	2
	$6\mathfrak{p}_5$	0	0	0	4	0	1
8	$\mathfrak{p}_2$	0	2	1	0	0	0
	(3)	0	1	0	0	0	1
	(5)	0	0	0	0	0	2
	$\mathfrak{p}_7$	2	0	2	0	0	0
	$\mathfrak{p}_{17}$	0	2	0	0	0	1
	$\mathfrak{p}_{23}$	2	2	2	0	0	0
	$\mathfrak{p}_{31}$	2	0	2	0	0	1
	$\mathfrak{p}_{41}$	0	2	0	0	0	2
	$\mathfrak{p}_{47}$	2	2	2	0	0	1
	$\mathfrak{p}_{71}$	2	2	2	0	0	2
	$7\mathfrak{p}_2$	0	0	4	0	0	1
12	$\mathfrak{p}_2$	0	2	0	0	1	0
	$\mathfrak{p}_3$	3	0	0	0	1	0
	$\mathfrak{p}_{11}$	3	2	0	0	1	0
	$\mathfrak{p}_{13}$	0	0	0	0	0	2
	$\mathfrak{p}_{23}$	3	2	0	0	1	1
	$\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}$	0	0	0	0	4	1

$d(K)$	$d(\mathfrak{A})$	e_2	e_3	g
13	(2)	1	0	1
	$\mathfrak{p}_3$	2	2	0
	$\mathfrak{p}_{13}$	0	0	2
	$\mathfrak{p}_{17}$	0	4	1
	$\mathfrak{p}_{23}$	2	4	1
	$\mathfrak{p}_{29}$	0	4	2
	(6)	4	0	1
17	$\mathfrak{p}_2$	2	2	0
	(3)	0	1	2
	(6)	0	4	1
	$2\mathfrak{p}_{17}$	0	8	1
21	(2)	2	0	1
	$\mathfrak{p}_3$	4	1	0
	$\mathfrak{p}_5$	0	5	0
	$\mathfrak{p}_7$	4	0	1
	$\mathfrak{p}_{17}$	0	5	2
24	$\mathfrak{p}_2$	1	3	0
	$\mathfrak{p}_3$	6	0	0
	$\mathfrak{p}_5$	0	3	1
	$5\mathfrak{p}_2$	0	12	1
28	$\mathfrak{p}_2$	0	4	0
	$\mathfrak{p}_3$	4	2	0
	$\mathfrak{p}_7$	4	0	2
	$3\mathfrak{p}_2$	0	4	1
29	(2)	3	0	1
	(3)	0	3	2
	$\mathfrak{p}_5$	0	6	0
	$\mathfrak{p}_7$	6	0	1
33	$\mathfrak{p}_2$	2	3	0
	$\mathfrak{p}_3$	4	0	1
	$2\mathfrak{p}_3$	4	0	1
	$2\mathfrak{p}_{11}$	4	12	1
37	(2)	1	0	2
	$\mathfrak{p}_3$	2	4	0
	$\mathfrak{p}_{11}$	2	8	2
40	$\mathfrak{p}_2$	1	4	0
	$\mathfrak{p}_3$	6	2	0
	$\mathfrak{p}_5$	0	4	2
	$3\mathfrak{p}_2$	4	4	1
41	$\mathfrak{p}_2$	4	2	0
	$2\mathfrak{p}_5$	0	8	1

$d(K)$	$d(\mathfrak{A})$	e_2	e_3	g
44	$\mathfrak{p}_2$	1	4	0
	$\mathfrak{p}_5$	0	4	2
	$\mathfrak{p}_7$	10	0	2
53	(2)	3	0	2
	$\mathfrak{p}_{11}$	6	10	2
56	$\mathfrak{p}_2$	2	4	0
57	$\mathfrak{p}_2$	2	5	0
	$\mathfrak{p}_3$	4	1	2
	$2\mathfrak{p}_3$	4	4	1
60	$\mathfrak{p}_2$	0	6	0
	$\mathfrak{p}_3$	8	0	1
61	$\mathfrak{p}_3$	6	4	0
	$\mathfrak{p}_5$	0	8	2
65	$\mathfrak{p}_2$	4	4	0
	$2\mathfrak{p}_5$	0	16	1
69	$\mathfrak{p}_3$	8	0	1
	$\mathfrak{p}_5$	0	9	2
73	$\mathfrak{p}_2$	2	4	1
	$2\mathfrak{p}_3$	4	8	1
76	$\mathfrak{p}_2$	1	4	1
	$\mathfrak{p}_3$	10	2	1
85	$\mathfrak{p}_3$	4	6	1
88	$\mathfrak{p}_2$	1	8	0
	$\mathfrak{p}_3$	6	4	2
89	$\mathfrak{p}_2$	6	2	1
92	$\mathfrak{p}_2$	0	8	0
93	$\mathfrak{p}_3$	4	3	2
97	$\mathfrak{p}_2$	2	4	2
104	$\mathfrak{p}_2$	3	4	1
105	$\mathfrak{p}_2$	4	6	1
109	$\mathfrak{p}_3$	6	6	2
113	$\mathfrak{p}_2$	4	6	1
120	$\mathfrak{p}_2$	2	10	0
129	$\mathfrak{p}_2$	6	5	2
137	$\mathfrak{p}_2$	4	6	2
140	$\mathfrak{p}_2$	2	8	1
152	$\mathfrak{p}_2$	3	8	1
156	$\mathfrak{p}_2$	0	10	2
168	$\mathfrak{p}_2$	2	12	1
172	$\mathfrak{p}_2$	1	12	2

TABLE 3. A complete list of all maximal orders $\mathcal{O}$ in quaternion algebras $\mathfrak{A}$ over quadratic fields, such that the genus g of $\mathcal{O}^1\backslash\mathcal{H}$ satisfies $g \leq 2$.

Similarly we get for $e(3, \mathcal{O})$, that

$$\sum_{\mathfrak{f}|\mathfrak{a}} N\mathfrak{f} \prod_{\mathfrak{p}|\mathfrak{f}} (1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}}) \prod_{\mathfrak{p}} e^*(S_\mathfrak{p}, \mathcal{O}_\mathfrak{p}) \leq c_3(\mathfrak{p}) := \begin{cases} 2 & \text{if } N\mathfrak{p} = 3, \\ 10 & \text{otherwise.} \end{cases} \tag{6.8}$$

From this, (6.3) and (6.4), we conclude that

$$g \geq g_1(d,\mathfrak{p}) := \frac{\zeta_K(-1)}{2}(N\mathfrak{p}-1)+1-\frac{c_3(\mathfrak{p})}{6}h(R_{3d})-\frac{c_2(\mathfrak{p})}{8}h(R_d). \tag{6.9}$$

In order to get an efficient lower bound on g, we need a good enough upper bound on $h(R_d)$. A proof of the following bound can be found in [12]. If $-d_1$ is the discriminant of $\mathbb{Q}(\sqrt{-d_1})$ and $d_1 > 4$, then

$$h\left(\mathbb{Q}(\sqrt{-d_1})\right) \leq \frac{1}{\pi}\sqrt{d_1}\,(\ln d_1 + 2)\,. \tag{6.10}$$

Since the absolute value of the discriminant of R_d is less than or equal to $4d$, and the absolute value of the discriminant of R_{3d} is less than or equal to $3d$, we conclude from (6.6), (6.9) and (6.10) that

$$\begin{aligned} g \geq g_2(d,\mathfrak{p}) \;&:=\; \tfrac{1}{240}\left(dm - \tfrac{4m(m+1)(m-1)}{3}\right)(N\mathfrak{p}-1)+1 \\ &\quad -\tfrac{1}{\pi}\sqrt{d}\left(\tfrac{c_3(\mathfrak{p})\sqrt{3}}{6}(\ln 3d+2)+\tfrac{c_2(\mathfrak{p})}{4}(\ln 4d+2)\right). \end{aligned} \tag{6.11}$$

The bound $g_2(d,\mathfrak{p})$ is strictly increasing as a function of $d > 1$ as soon as $g_2(d,\mathfrak{p}) > 1$. In particular, $g_2(d,\mathfrak{p}) > 2$ implies that $g_2(d',\mathfrak{p}) > 2$ if $d' > d$.

Fix a rational prime p. In order to find all algebras with $d(\mathfrak{A}) = \mathfrak{p}$ and $N\mathfrak{p} = p$ or $N\mathfrak{p} = p^2$, we first determine the smallest discriminant d_1 such that $g_2(d_1,\mathfrak{p}) > 2$. From d_1, we step downwards until we find the largest discriminant d_2 such that $g_1(d_2,\mathfrak{p}) \leq 2$. Then we have to calculate the genus corresponding to all discriminants up to d_2. In Table 4, we give the bounds d_1 and d_2 for primes less than 150.

p	d_1	d_2	p	d_1	d_2	p	d_1	d_2	p	d_1	d_2	p	d_1	d_2
2	14329	2573	19	1469	197	47	520	101	79	293	29	109	201	29
3	12597	2141	23	1177	173	53	456	77	83	273	29	113	197	29
5	7649	1109	29	901	101	59	401	53	89	257	29	127	168	29
7	4917	701	31	840	101	61	389	53	97	229	29	131	161	29
11	2812	437	37	681	101	67	348	53	101	217	29	137	152	29
13	2305	341	41	609	101	71	328	53	103	213	29	139	149	29
17	1676	269	43	577	101	73	321	53	107	204	29	149	145	13

TABLE 4. The number d_1 is the least discriminant such that $g_2(d_1,\mathfrak{p}) > 2$, and d_2 the largest such that $g_1(d_2,\mathfrak{p}) \leq 2$.

For primes $p \geq 149$ we have $d_2 \leq 13$, for primes $p \geq 211$ we have $d_2 \leq 8$, for primes $p \geq 251$ we have $d_2 \leq 5$ and finally for primes $p \geq 853$ the bounds (6.9) and (6.11) excludes the possibility that $g \leq 2$. Hence we get a small number of cases to check using a computer. We remark that for the 3 fields with discriminant less than 13, we also have to take the elliptic points of higher order into account. Determining the few with non-prime discriminant is easily done by computer search using the list of all with prime discriminant. □

(6.12) Remark. In the columns for $d(\mathfrak{A})$, $\mathfrak{p}_q$ indicates that $(q) = \mathfrak{p}_q\overline{\mathfrak{p}}_q$ is split or ramified in K. If (q) is split, then the entry in Table 3 corresponds to the two different algebras ramified in $\mathfrak{p}_q$ and $\overline{\mathfrak{p}}_q$ respectively. Also observe that there are 15 entries (corresponding to 18 algebras) in the list, which are ramified at 3 primes. The observant reader has of course noticed, that all algebras in Table 3 which are ramified at 3 primes have genus equal to 1. In fact, in all cases of algebras ramified at more than one prime that we have checked, the genus is odd. This includes several thousands of algebras ramified at 3 and 5 primes. It has a natural explanation for orders without elliptic elements, since then

$$\frac{\zeta_K(-1)}{2} \prod_{\mathfrak{p}|d(\mathfrak{A})} (N\mathfrak{p}-1)$$

tend to be an even integer. However, we have no general argument why this should be true. We conclude with the remark, that the smallest genus of a maximal order in an algebra ramified at 5 primes is 81. □

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