Preservation of Strong Stability Associated with Analytic Semigroups

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1 Introduction

We begin by recalling the elementary example,

$$y' + \lambda y = 0,\ t > 0; \quad y(0) = y_0, \tag{1}$$

that was discussed in the Introduction. The solution $y(t) = e^{-\lambda t} y_0$ satisfies the *stability estimate*,

$$|y(t)| = e^{-\lambda t}|y_0| \le |y_0|, \quad t \ge 0, \tag{2}$$

which is *uniform* with respect to $\lambda > 0$. Recall that we showed that this stability estimate is preserved by the implicit Euler method. However, the explicit Euler method is only stable provided that $\Delta t < 2/\lambda$, so its stability is not uniform in λ. One consequence is that it is less efficient to compute with the explicit Euler method over long time intervals.

Note that in this example, we also have a stability estimate for the derivative. Namely,

$$|y'(t)| = \lambda e^{-\lambda t}|y_0| \le C t^{-1} |y_0|, \quad t > 0, \tag{3}$$

where $C = \max_{s \ge 0}(se^{-s}) = e^{-1}$. We call this property *strong stability*. Note that it is also uniform with respect to λ.

When we generalize (1) to a system of equations $y' + Ay = 0$, where A is a square matrix and y is a vector, the situation becomes quite a bit more complicated. Now the eigenvalues of A play the role of λ in (1). A system of equations $y' + Ay = 0$ is dissipative, if the eigenvalues of A have positive real parts, which is guaranteed,

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for example, if A is symmetric and positive definite. In this case, we seek stability bounds on *norms* of y and its derivatives analogous to the bounds that hold for (1). The uniformity of such bounds with respect to the eigenvalues of A is crucial because we often expect the eigenvalues to vary considerably in size.

In this chapter we generalize (1) even further to parabolic partial differential equations written as

$$u' + Au = 0,\ t > 0; \quad u(0) = u_0, \tag{4}$$

where A is a differential operator with respect to the spatial variable. The prototypical example is the heat equation with A being minus the Laplacian. We can write the solution formally as $u(t) = \exp(-tA)u_0$, where $\exp(-tA)$ is the solution operator. We are interested in the strong stability mentioned above, which, in this case, means that strong norms (involving space and time derivatives) of the solution can be bounded in terms of a weak norm (involving no derivatives) of the initial value. It turns out that (4) has this property precisely when the solution operator $\exp(-tA)$ is a so called *analytic semigroup*.

The strong stability, alias analytic semigroup property, is preserved by certain numerical methods, which makes it possible to prove error bounds with optimal and low regularity requirements. In this chapter we review the theory of analytic semigroups and discuss how it can be used in the error analysis of numerical methods that preserve this structure.

2 The heat equation

We consider the homogeneous initial-boundary value problem for the heat equation,

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= 0, && x \in \Omega,\ t > 0, \\ u(x,t) &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned} \tag{5}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $u = u(x,t)$, $u_t = \partial u/\partial t$, $\Delta u = \sum_{i=1}^d \partial^2 u/\partial x_i^2$. For simplicity, we assume in this presentation that Ω is a convex polygon, so that we have access to the elliptic regularity theory and so that finite element meshes can be fitted exactly to the domain.

We use the Hilbert space $L_2(\Omega)$, with its standard norm and inner product

$$\|v\| = \Big(\int_\Omega |v|^2\, dx \Big)^{1/2}, \quad (v,w) = \int_\Omega vw\, dx. \tag{6}$$

The norms in the Sobolev spaces $H^m(\Omega)$, $m \geq 0$, are denoted by

$$\|v\|_{H^m} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|^2 \right)^{1/2}. \tag{7}$$

The space $H_0^1(\Omega)$ consists of the functions in $H^1(\Omega)$ that vanish on $\partial\Omega$.

We define the unbounded operator $A = -\Delta$ on $L_2(\Omega)$ with domain of definition $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then A is a closed, densely defined, and self-adjoint positive definite operator on $L_2(\Omega)$ with compact inverse. Using this notation we may write (5) as an initial-value problem in $L_2(\Omega)$, namely,

$$u' + Au = 0,\ t > 0; \quad u(0) = u_0, \tag{8}$$

where $u : \mathbf{R}^+ \to L_2(\Omega)$, $u' = du/dt$, and $u_0 \in L_2(\Omega)$. The solution of (8) is given by $u(t) = E(t)u_0$, with the solution operator, $E(t) = \exp(-tA)$, defined by

$$E(t)v = \sum_{j=1}^{\infty} e^{-t\lambda_j}(v, \varphi_j)\varphi_j, \quad v \in L_2(\Omega), \tag{9}$$

where $\lambda_j > 0$ and φ_j denote the eigenvalues and an orthonormal basis of eigenvectors of A.

Using Parseval's formula,

$$\|v\|^2 = \sum_{j=1}^{\infty} (v, \varphi_j)^2, \tag{10}$$

we easily obtain the *stability estimate*

$$\|E(t)v\| \le \|v\|, \quad t \ge 0, \tag{11}$$

In order to estimate the derivative of the solution, we use $\max_{s\ge 0}(s^2 e^{-2s}) = e^{-2} < 1$ to get

$$\|E'(t)v\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 e^{-2t\lambda_j}(v, \varphi_j)^2 = t^{-2} \sum_{j=1}^{\infty} (t\lambda_j)^2 e^{-2t\lambda_j}(v, \varphi_j)^2 \le t^{-2}\|v\|^2. \tag{12}$$

Hence, we obtain the *strong stability estimate*:

$$\|E'(t)v\| = \|AE(t)v\| \le t^{-1}\|v\|, \quad t > 0. \tag{13}$$

The term strong stability refers to the fact that the norms on the left sides are stronger than the one on the right side.

Since we assume that the domain Ω is a convex polygon, we may take advantage of the regularity of elliptic equations,

$$\|v\|_{H^2} \le C\|Av\|, \quad \forall v \in \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega), \tag{14}$$

and (13) to get a related estimate of the second order spatial derivatives:

$$\|E(t)v\|_{H^2} \le Ct^{-1}\|v\|, \quad t > 0. \tag{15}$$

The strong stability estimate (13) reflects the fact the solution operator, $E(t) = \exp(-tA)$, is an *analytic semigroup* on $L_2(\Omega)$. We proceed to discuss this concept in the following section.

3 Analytic semigroups

Let X be a Banach space with norm $\|\cdot\|$. A family $\{E(t)\}_{t\geq 0}$ of bounded linear operators on X is called a *semigroup of bounded linear operators*, if

- $E(0) = I$ (identity operator),
- $E(t+s) = E(t)E(s) \quad \forall\, s,t \geq 0$ (semigroup property).

The semigroup is called *strongly continuous*, if

$$\lim_{t\to 0^+} E(t)x = x \quad \forall x \in X. \tag{16}$$

The *infinitesimal generator* of the semigroup is the linear operator G defined by

$$Gx = \lim_{t\to 0^+} \frac{E(t)x - x}{t}, \tag{17}$$

its domain of definition $\mathcal{D}(G)$ being the space of all $x \in X$ for which the limit exists. One can prove that G is a closed and densely defined operator on X and that the function $u(t) = E(t)u_0$ solves the initial-value problem

$$u' = Gu,\ t > 0; \quad u(0) = u_0, \tag{18}$$

provided that $u_0 \in \mathcal{D}(G)$. We therefore often write $E(t) = \exp(tG)$. We refer to [12, Chapter 1] for the proofs of these statements. In Section 2 the semigroup $E(t) = \exp(-tA)$ on $L_2(\Omega)$ is generated by $-A = \Delta$.

On the other hand we might want to start with a linear operator A and seek conditions under which $G = -A$ generates a semigroup $E(t) = \exp(-tA)$ such that $u(t) = E(t)u_0$ solves

$$u' + Au = 0,\ t > 0; \quad u(0) = u_0. \tag{19}$$

Theorem 1 below provides such a condition.

The semigroup $E(t)$ is called *analytic* or *holomorphic*, if it can be extended to a complex analytic function $E(z)$ for z in a sector in the complex plane containing the positive t-axis. The following theorem, which we quote from [12, Chapter 2.5], gives two characterizations of an analytic semigroup; one in terms of the resolvent of its generator and the other one in terms of its derivative.

Recall that the *resolvent set* $\rho(A)$ of the linear operator A is the set of complex numbers λ such that $\lambda I - A$ is invertible and the *resolvent* $(\lambda I - A)^{-1}$ is a bounded linear operator on X.

Theorem 1. *Let $E(t)$ be a strongly continuous semigroup with generator $G = -A$. Assume that $E(t)$ is uniformly bounded, i.e., for some C,*

$$\|E(t)\| \leq C, \quad t \geq 0, \tag{20}$$

and $0 \in \rho(A)$. Then the following are equivalent:

(a) *The semigroup $E(t)$ can be extended to a complex analytic function $E(z)$ in a sector $\{z \in \mathbf{C} : |\arg(z)| \leq \delta\}$.*

(b) *There is an angle* $\varphi \in (0, \pi/2)$ *and a positive number* M *such that* $\rho(A) \supset \Sigma_\varphi = \{\lambda \in \mathbf{C} : \varphi \le |\arg(\lambda)| \le \pi\} \cup \{0\}$ *and*

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma_\varphi,\ \lambda \ne 0. \tag{21}$$

(c) $E(t)$ *is differentiable for* $t > 0$ *and there is a constant* C *such that*

$$\|AE(t)\| = \|E'(t)\| \le Ct^{-1}, \quad t > 0. \tag{22}$$

Here $\|\cdot\|$ is the usual norm of bounded linear operators on X.

The differentiability in property (c) implies that the function $u(t) = E(t)u_0$ solves (18) for all $u_0 \in X$, not just for $u_0 \in \mathcal{D}(G) = \mathcal{D}(A)$.

The proof of the implication (b) $\Rightarrow$ (c) is based on the formula

$$E(t) = \frac{1}{2\pi i} \int_{\Gamma_\varphi} e^{-t\lambda} (\lambda I - A)^{-1} \, d\lambda, \tag{23}$$

where the contour $\Gamma_\varphi = \{\lambda \in \mathbf{C} : |\arg(\lambda)| = \varphi\}$ is the boundary of the sector in (21) oriented so that the imaginary part decreases along Γ_φ. The resolvent estimate (21) may then be used to show that the integral in (23) is convergent and defines a uniformly bounded analytic semigroup.

Instead of starting with an analytic semigroup with generator $G = -A$ defined by (17), we may therefore start with a densely defined operator A with property (b) and define $E(t)$ by (23). This will then be a uniformly bounded analytic semigroup with generator $G = -A$.

In Section 2 we showed that the semigroup discussed there has the property (c), see (13). This shows that it is analytic on $X = L_2(\Omega)$ and that the resolvent estimate in (b) holds for some angle $\varphi \in (0, \pi/2)$. On the other hand, it is not hard to prove property (b) with arbitrary $\varphi \in (0, \pi/2)$ for a self-adjoint, positive definite, linear operator A.

More generally, one can prove that a strongly elliptic partial differential operator of second order in a smooth bounded domain Ω together with the homogeneous Dirichlet boundary condition generates an analytic semigroup in $L_p(\Omega)$ for $1 < p < \infty$. The approach is to prove the resolvent estimate (b) with respect to the L_p-norm, see [12, Chapter 7.3].

For such differential operators it is also possible to prove the resolvent estimate (b), and hence the strong stability (c), with respect to the L_∞-norm, see [16]. For example, the semigroup (9) satisfies

$$\begin{aligned} &\|E(t)v\|_{L_\infty} \le \|v\|_{L_\infty}, \quad t \ge 0 \quad \text{(maximum principle)}, \\ &\|E'(t)v\|_{L_\infty} \le Ct^{-1}\|v\|_{L_\infty}, \quad t > 0. \end{aligned} \tag{24}$$

However, the resulting semigroup is not analytic (not even strongly continuous) in $L_\infty(\Omega)$, because the domain of definition of the generator is not dense in $L_\infty(\Omega)$. Instead one has to work in the Banach space $X = \{u \in C(\bar{\Omega}) : \ u = 0 \text{ on } \partial\Omega\}$.

Unlike the case $X = L_p(\Omega)$, this space includes a boundary condition. This is often not desirable, because when the semigroup is used to solve the non-homogeneous equation $u' + Au = f$, the boundary condition is imposed on f. There is also a theory of semigroups generated by operators with non-dense domain, [13].

Since our main interest here is the preservation of the strong stability under discretization, we shall not dwell on this, but proceed to the next section, where we study spatial discretization by the finite element method.

4 Spatial discretization

We now return to the heat equation as described in Section 2 and consider the non-homogeneous version of (8):

$$u' + Au = f,\ t > 0; \quad u(0) = u_0. \tag{25}$$

If the function $f : \mathbf{R}^+ \to L_2(\Omega)$ is continuous, then the solution is given by

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(s)\,ds, \tag{26}$$

where $E(t) = \exp(-tA)$, defined in (9), is the analytic semigroup generated by $-A$.

The weak formulation of (25) is: find $u(t) \in H_0^1(\Omega)$ such that

$$\begin{aligned} (u', v) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega),\ t > 0, \\ u(0) = u_0, \end{aligned} \tag{27}$$

where $a(u, v) = (\nabla u, \nabla v) = (-\Delta u, v) = (Au, v)$ is the bilinear form associated with the operator A.

Let $\{V_h\}_{0<h<1}$ be a family of finite dimensional subspaces of H_0^1, where each V_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to a triangulation $\mathcal{T}_h$ of Ω with maximal mesh size h. We refer to, e.g., [4] and [5] for results about the finite element method for elliptic problems. The general reference for parabolic finite element problems is [17].

The approximate solution $u_h(t) \in V_h$ of (25) is defined by

$$\begin{aligned} (u_h', \chi) + a(u_h, \chi) = (f, \chi), \quad \forall \chi \in V_h,\ t > 0, \\ (u_h(0), \chi) = (u_0, \chi), \quad \forall \chi \in V_h. \end{aligned} \tag{28}$$

Introducing the linear operator $A_h : V_h \to V_h$ and the orthogonal projection $P_h : L_2(\Omega) \to V_h$, defined by

$$(A_h\psi, \chi) = a(\psi, \chi), \quad (P_h g, \chi) = (g, \chi) \quad \forall \psi, \chi \in V_h,\ g \in L_2(\Omega), \tag{29}$$

we may write (28) as

$$u_h' + A_h u_h = P_h f,\ t > 0; \quad u_h(0) = P_h u_0. \tag{30}$$

The operator A_h is self-adjoint positive definite (uniformly with respect to h) and the solution of (30) is given by

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h f(s)\,ds, \tag{31}$$

with the solution operator, $E_h(t) = \exp(-tA_h)$, defined by

$$E_h(t)v_h = \sum_{j=1}^{N_h} e^{-t\lambda_{h,j}}(v_h, \varphi_{h,j})\varphi_{h,j}, \quad v_h \in V_h, \tag{32}$$

where $\lambda_{h,j} > 0$ and $\varphi_{h,j}$ denote the eigenvalues and an orthonormal basis of eigenvectors of A_h. Due to the similarity of this formula with (9), we immediately obtain stability estimates analogous to (11), (13):

$$\|E_h(t)v_h\| \le \|v_h\|, \quad t \ge 0, \tag{33}$$

$$\|E_h'(t)v_h\| = \|A_h E_h(t)v_h\| \le t^{-1}\|v_h\|, \quad t > 0. \tag{34}$$

Note in particular that these hold uniformly with respect to h. We may therefore say that the strong stability associated with the analytic semigroup $E(t)$ is preserved under the spatial discretization.

We next demonstrate how the strong stability estimate (34) may be used in the error analysis. We need the Ritz projection $R_h : H_0^1(\Omega) \to V_h$ defined by

$$a(R_h v, \chi) = a(v, \chi), \quad \forall \chi \in V_h. \tag{35}$$

A simple calculation shows that

$$R_h = A_h^{-1}P_h A. \tag{36}$$

In fact, $R_h v$ is the finite element solution of an elliptic equation whose exact solution is v. Under the usual regularity assumptions on the triangulation and using the elliptic regularity estimate (14), we obtain the following error bound from the theory of finite elements for elliptic problems (see [4], [5])

$$\|R_h v - v\| \le Ch^2\|v\|_{H^2}, \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{37}$$

Let u and u_h denote the solutions of (25) and (30), respectively. We divide the error into two parts:

$$u_h(t) - u(t) = \big(u_h(t) - P_h u(t)\big) + \big(P_h u(t) - u(t)\big). \tag{38}$$

For the last term it follows from the optimality of the orthogonal projection and (37) that

$$\|P_h u(t) - u(t)\| \le \|R_h u(t) - u(t)\| \le Ch^2\|u(t)\|_{H^2}, \tag{39}$$

and it remains to estimate the first part $\theta(t) = u_h(t) - P_h u(t)$. A simple calculation using (30), (25), and (36) yields

$$\begin{aligned} &\theta' + A_h\theta = A_h P_h(A_h^{-1}P_h A - I)u = A_h P_h(R_h - I)u = A_h P_h \rho, \\ &\theta(0) = 0, \end{aligned} \tag{40}$$

where we introduced the notation $\rho = R_h u - u$. Hence, by the solution formula (32), we get

$$\theta(t) = \int_0^t A_h E_h(t-s) P_h \rho(s)\, ds, \tag{41}$$

and it should now be clear that (34) is useful.

We will derive several error estimates. Applying (34) directly, we note that $\|A_h E_h(t-s)\| \le (t-s)^{-1}$ has a non-integrable singularity. In order to handle this we assume that our family of finite element meshes is such that we have an inverse estimate

$$\|A_h v_h\| \le C h^{-\beta} \|v_h\|, \quad \forall v_h \in V_h, \tag{42}$$

for some positive number β. For example, if the mesh family is quasi-uniform then this holds with $\beta = 2$. Hence, using also (33) and (37), we obtain

$$\begin{aligned} \|\theta(t)\| &\le \Big(\int_0^{t-h^\beta} (t-s)^{-1}\, ds + C h^{-\beta} \int_{t-h^\beta}^t ds \Big) \sup_{s \le t} \|\rho(s)\| \\ &\le C h^2 \big(1 + \log(t/h^\beta)\big) \sup_{s\le t} \|u(s)\|_{H^2}, \quad t > h^\beta. \end{aligned} \tag{43}$$

Another way to avoid the singularity is to note that $A_h E_h(t-s) = D_s E_h(t-s)$ and integrate by parts:

$$\begin{aligned} \theta(t) = &\int_0^{t/2} A_h E_h(t-s) P_h \rho(s)\, ds \\ &+ P_h \rho(t) - E_h(t/2) P_h \rho(t/2) - \int_{t/2}^t E_h(t-s) P_h \rho'(s)\, ds. \end{aligned} \tag{44}$$

Hence, by (33), (34), and (37),

$$\begin{aligned} \|\theta(t)\| &\le \Big(\int_0^{t/2} (t-s)^{-1}\, ds + 2 \Big) \sup_{s\le t} \|\rho(s)\| + \int_{t/2}^t s^{-1}\, ds \sup_{s \le t} \big(s \|\rho'(s)\| \big) \\ &\le C h^2 \sup_{s \le t} \Big(\|u(s)\|_{H^2} + s \|u'(s)\|_{H^2} \Big). \end{aligned} \tag{45}$$

A third variant is obtained by integrating the first term on the right side of (44) by parts. With $\tilde\rho(t) = \int_0^t \rho(s)\, ds$ we get

$$\begin{aligned} \theta(t) = &A_h E_h(t/2) P_h \tilde\rho(t/2) - \int_0^{t/2} A_h^2 E_h(t-s) P_h \tilde\rho(s)\, ds \\ &+ P_h \rho(t) - E_h(t/2) P_h \rho(t/2) - \int_{t/2}^t E_h(t-s) P_h \rho'(s)\, ds, \end{aligned} \tag{46}$$

so that

$$
\begin{aligned}
\|\theta(t)\| \leq & \Big(2t^{-1} + \int_0^{t/2} (t-s)^{-2}\, ds\Big) \sup_{s\leq t} \|\tilde\rho(s)\| \\
& + 2t^{-1} \sup_{s\leq t} \big(s\|\rho(s)\|\big) + \int_{t/2}^t s^{-2}\, ds \sup_{s\leq t} \big(s^2\|\rho'(s)\|\big) \\
& \leq Ch^2 t^{-1} \sup_{s\leq t} \Big(\|\tilde u(s)\|_{H^2} + s\|u(s)\|_{H^2} + s^2\|u'(s)\|_{H^2}\Big).
\end{aligned}
\tag{47}
$$

Adding the contribution from (39) we conclude the following.

Theorem 2. *Let u and u_h denote the solutions of* (25) *and* (30)*, respectively. Then, with $\tilde u(t) = \int_0^t u(s)\, ds$,*

$$
\|u_h(t) - u(t)\| \leq Ch^2\big(1 + \log(t/h^\beta)\big) \sup_{s\leq t} \|u(s)\|_{H^2}, \quad t > h^\beta, \tag{48}
$$

$$
\|u_h(t) - u(t)\| \leq Ch^2 \sup_{s\leq t} \Big(\|u(s)\|_{H^2} + s\|u'(s)\|_{H^2}\Big), \quad t \geq 0, \tag{49}
$$

$$
\|u_h(t) - u(t)\| \leq Ch^2 t^{-1} \sup_{s\leq t} \Big(\|\tilde u(s)\|_{H^2} + s\|u(s)\|_{H^2} + s^2\|u'(s)\|_{H^2}\Big), \quad t > 0. \tag{50}
$$

The first error estimate, (48), requires the least regularity of the three. Except for the slowly growing logarithmic factor, it is also uniform in time, while the other bounds may accumulate slightly because of the factor s that appears in front of some terms.

In order to compare the last two error estimates we apply them to the solution of the homogeneous equation (8), i.e., $u(t) = E(t)u_0$. Then, according to (15),

$$
\|u(t)\|_{H^2} \leq Ct^{-1}\|u_0\|, \tag{51}
$$

and similarly

$$
\|u'(t)\|_{H^2} \leq Ct^{-2}\|u_0\|, \tag{52}
$$

and

$$
\|\tilde u(t)\|_{H^2} \leq C\|A\tilde u(t)\| = C\Big\|\int_0^t Au(s)\, ds\Big\| = C\Big\|\int_0^t u'(s)\, ds\Big\| \leq C\|u_0\|. \tag{53}
$$

Therefore (50) becomes

$$
\|E_h(t)P_h u_0 - E(t)u_0\| \leq Ch^2 t^{-1}\|u_0\|, \quad t > 0, \tag{54}
$$

which means that the error converges with optimal order even if the initial value is only in $L_2(\Omega)$. On the other hand, if $u_0 \in \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, then $\|AE(t)u_0\| \leq \|Au_0\|$, so that

$$
\|u(t)\|_{H^2} + t\|u'(t)\|_{H^2} \leq C\|u_0\|_{H^2}, \tag{55}
$$

and (49) becomes

$$\|E_h(t)P_h u_0 - E(t)u_0\| \le Ch^2\|u_0\|_{H^2}, \quad t \ge 0. \tag{56}$$

Thus, the error bound is uniform as $t \to 0$ if the initial value is smooth.

In Theorem 2 we used the strong stability (34) in order to prove three error estimates with low and optimal regularity requirement. Using only the stability estimate (33) we obtain instead

$$\|u_h(t) - u(t)\| \le Ch^2\Big(\sup_{s\le t}\|u(s)\|_{H^2} + \int_0^t \|u'(s)\|_{H^2}\,ds\Big), \quad t \ge 0, \tag{57}$$

which is slightly weaker than (49), e.g., it does not imply (56). It is proved by integration by parts as in (44) but with $t/2$ replaced by t.

The previous discussion is based on the use of the spectral theorem for the self-adjoint operator $A = -\Delta$ in the Hilbert space $X = L_2(\Omega)$. When the operator A is not self-adjoint or the space X is a Banach space, we base the argument on the contour integral (23) and the resolvent estimate (21). We demonstrate this by analyzing the finite element approximation of (5) in $L_\infty(\Omega)$. We assume from now on that the mesh family is quasi-uniform.

In [14] it was shown, in the case of two spatial variables, that the semigroup (32) satisfies

$$\|E_h(t)v_h\|_{L_\infty} + t\|E_h'(t)v_h\|_{L_\infty} \le C\log(1/h)\|v_h\|_{L_\infty}, \quad t > 0. \tag{58}$$

The same bound, but with an additional factor $\log(1/h)$, was proved in [18] in the case of one spatial variable. This means that the strong stability in the maximum norm (24) is almost preserved by the finite element discretization. Combining this with the maximum norm analog of (37), see [4],

$$\|R_h v - v\|_{L_\infty} \le Ch^2\log(1/h)\|v\|_{W^2_\infty}, \quad \forall v \in H^1_0(\Omega)\cap W^2_\infty(\Omega), \tag{59}$$

we may prove error estimates in the maximum norm analogous to Theorem 2 and (54), (56), (57), with (non-important) logarithmic factors.

However, recent work in [15], [19] has removed the logarithm from (58) and we now know that

$$\|E_h(t)v_h\|_{L_\infty} + t\|E_h'(t)v_h\|_{L_\infty} \le C\|v_h\|_{L_\infty}, \quad t > 0, \tag{60}$$

holds in any number of spatial variables. The logarithm in (59) cannot be removed, [10], so the resulting error estimates still contain logarithmic factors.

In view of Theorem 1 we know that (60) implies that the resolvent estimate

$$\|(\lambda I - A_h)^{-1}\|_{L_\infty\to L_\infty} \le \frac{M}{|\lambda|}, \quad \forall \lambda \in \Sigma_\varphi,\ \lambda \ne 0, \tag{61}$$

holds for some angle $\varphi \in (0, \pi/2)$. However, the angle in (21) increases towards $\pi/2$, meaning that the estimate deteriorates, as the constant C in (22) increases.

We do not have precise information about the constant in (60), and therefore no information about the angle in (61). Note, by the way, that (58) would allow $\varphi \to \pi/2$ as $h \to 0$, so here is an example where the removal the logarithm makes a difference. Anyway, this is not satisfactory, since we know that the corresponding resolvent estimate for the differential operator $A = -\Delta$ holds for any angle. In the following section I will explain why it may be important to know the angle.

Therefore it seems advantageous to begin by proving the resolvent estimate and then deduce the strong stability. This approach was taken in, e.g., [20], [7], [1], [2]. We now know that the resolvent estimate (61) holds with arbitrary angle. This was first proved in [7] for the case of one spatial variable, and recently for the general case in [2].

5 Temporal discretization

In this subsection we show that the above program can be carried through also for certain completely discrete schemes. We begin by looking at the backward Euler method. We replace the time derivative in (30) by a backward difference quotient $\partial_t U_n = (U_n - U_{n-1})/k$, where k is a time step and U_n is the approximation of $u_n = u(t_n)$ and $t_n = nk$. The discrete solution $U_n \in V_h$ thus satisfies

$$\partial_t U_n + A_h U_n = P_h f_n,\ t_n > 0; \quad U_0 = P_h u_0. \tag{62}$$

The solution of (62) is given by

$$U_n = E_{kh}^n P_h u_0 + k \sum_{j=1}^{n} E_{kh}^{n-j-1} P_h f_j, \tag{63}$$

where the solution operator, $E_{kh}^n = (I + kA_h)^{-n}$, may be written

$$E_{kh}^n v_h = r(kA_h)^n = \sum_{j=1}^{N_h} r(k\lambda_{h,j})^n (v_h, \varphi_{h,j})\varphi_{h,j}, \quad v_h \in V_h, \tag{64}$$

where $r(s) = (1+s)^{-1}$ and where $\lambda_{h,j} > 0$ and $\varphi_{h,j}$ denote the eigenvalues and an orthonormal basis of eigenvectors of A_h. Since $r(s) \le 1$ and $|sr(s)^n| \le Cn^{-1}$ for $s \ge 0$, see [17, Lemma 7.3], we easily obtain stability estimates analogous to (11), (13):

$$\|E_{kh}^n v_h\| \le \|v_h\|, \quad t_n \ge 0, \tag{65}$$

$$\|\partial_t E_{kh}^n v_h\| = \|A_h E_{kh}^n v_h\| \le C t_n^{-1} \|v_h\|, \quad t_n > 0. \tag{66}$$

Note in particular that these hold uniformly with respect to h and k. We may therefore say that the strong stability associated with the analytic semigroup $E(t)$ is preserved under this spatial and temporal discretization. More generally, this holds for rational functions that are *strongly A-stable* (see [17, Lemma 7.3]):

$$|r(s)| \le 1, \quad \forall s \ge 0; \quad r(\infty) = 0. \tag{67}$$

Imitating the proof of Theorem 2 we obtain, instead of (41),

$$\theta_n = k \sum_{j=1}^{n} A_h E_{kh}^{n-j-1} \Big(P_h \rho_j + A_h^{-1} P_h \omega_j \Big), \tag{68}$$

where $\omega_n = \partial_t U_n - u'(t_n)$. Using summation by parts as in (44) we may prove, e.g.,

$$\begin{aligned} \|U_n - u(t_n)\| &\le C h^2 \sup_{s \le t} \Big(\|u(s)\|_{H^2} + s \|u'(s)\|_{H^2} \Big) \\ &\quad + C k \sup_{s \le t} \Big(\|A^{-1} u''(s)\| + s \|u''(s)\| \Big), \quad t_n \ge 0, \end{aligned} \tag{69}$$

which is analogous to (49).

Leaving the Hilbert space setting, we now consider a rational approximation of the semigroup $E(t)$ of the form

$$E_{kh}^n v_h = r(kA_h)^n, \tag{70}$$

where r is an $A(\theta)$-*stable* rational function, i.e., for some angle $\theta \in [0, \pi/2]$,

$$|r(\lambda)| \le 1, \quad \text{for } |\arg(\lambda)| \le \theta. \tag{71}$$

The operator is defined by the Dunford-Taylor integral,

$$r(kA_h) = r(\infty) I + \frac{1}{2\pi i} \int_{\Gamma_\varphi} r(k\lambda)(\lambda I - A_h)^{-1} \, d\lambda, \tag{72}$$

where the contour $\Gamma_\varphi = \{\lambda \in \mathbf{C} : |\arg(\lambda)| = \varphi\}$ is the boundary of the sector in the maximum-norm resolvent estimate (61) oriented so that the imaginary part decreases along Γ_φ. Recall from the previous section that the angle φ can be chosen arbitrarily and, in particular, does not depend on h. If $\theta \ge \varphi$, then we can use (71) together with (61) to prove the stability estimate, see, e.g., [17, Theorem 8.2], [11], [6],

$$\|E_{kh}^n v_h\|_{L_\infty} \le C \|v_h\|_{L_\infty}, \quad t_n \ge 0. \tag{73}$$

Thus, we need the analyticity of the discrete semigroup $E_h(t)$, in the form of the resolvent estimate, already in order to prove the standard stability. In fact, (73) is not true in general if the semigroup is only strongly continuous, [3], [6].

Looking for strong stability, we need to assume that the rational function r is *strongly* $A(\theta)$-*stable*, i.e., we assume in addition that $r(\infty) = 0$. Then one can prove

$$\|A_h E_{kh}^n v_h\|_{L_\infty} \le C t_n^{-1} \|v_h\|_{L_\infty}, \quad t_n > 0, \tag{74}$$

see [9]. This opens the possibility of proving error estimates analogous to those of Theorem 2.

For a similar result for a particular class of time discretizations but with variable timesteps, see [8].

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